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Strong Laws of Large Numbers for Fuzzy Set-Valued Random Variables in G_α Space

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Abstract

In this paper, we shall present the strong laws of large numbers for fuzzy set-valued random variables in the sense of d_H^∞. The results are based on the result of single-valued random variables obtained by Taylor [1] and set-valued random variables obtained by Li Guan [2].

Keywords

Laws of Large Numbers, Fuzzy Set-Valued Random Variable, Hausdorff Metric

1. Introduction

With the development of set-valued stochastic theory, it has become a new branch of probability theory. And limits theory is one of the most important theories in probability and statistics. Many scholars have done a lot of research in this aspect. For example, Artstein and Vitale in [3] had proved the strong law of large numbers for independent and identically distributed random variables by embedding theory. Hiai in [4] had extended it to separable Banach space. Taylor and Inoue had proved the strong law of large numbers for independent random variable in the Banach space in [5]. Many other scholars also had done lots of works in the laws of large numbers for set-valued random variables. In [2], Li proved the strong laws of large numbers for set-valued random variables in G_α space in the sense of d_H metric.

As we know, the fuzzy set is an extension of the set. And the concept of fuzzy set-valued random variables is a natural generalization of that of set-valued random variables, so it is necessary to discuss convergence theorems of fuzzy set-valued random sequence. The limits of theories for fuzzy set-valued random sequences are also been discussed by many researchers. Colubi *et al.* [6], Feng [7] and Molchanov [8] proved the strong laws of large numbers for fuzzy set-valued random variables; Puri and Ralescu [9], Li and Ogura [10] proved conver-

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gence theorems for fuzzy set-valued martingales. Li and Ogura [11] proved the SLLN of [12] in the sense of d_H^∞ by using the "sandwich" method. Guan and Li [13] proved the SLLN for weighted sums of fuzzy set-valued random variables in the sense of d_H^∞ which used the same method. In this paper, what we concerned are the convergence theorems of fuzzy set-valued sequence in G_α space in the sense of d_H^∞.

The purpose of this paper is to prove the strong laws of large numbers for fuzzy set-valued random variables in G_α space, which is both the extension of the result in [1] for single-valued random sequence and also the extension in [2] for set-valued random sequence.

This paper is organized as follows. In Section 2, we shall briefly introduce some concepts and basic results of set-valued and fuzzy set-valued random variables. In Section 3, I shall prove the strong laws of large numbers for fuzzy set-valued random variables in G_α space, which is in the sense of Hausdorff metric d_H^∞.

2. Preliminaries on Set-Valued Random Variables

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space, $(\mathfrak{X}, \|\cdot\|)$ is a real separable Banach space, $\mathbf{K}(\mathfrak{X})$ is the family of all nonempty closed subsets of $\mathfrak{X}$, and $\mathbf{K}_b(\mathfrak{X})(\mathbf{K}_k(\mathfrak{X}))$ is the family of all non-empty bounded closed(compact) subsets of $\mathfrak{X}$, and $\mathbf{K}_{kc}(\mathfrak{X})$ is the family of all non-empty compact convex subsets of $\mathfrak{X}$.

Let A and B be two nonempty subsets of $\mathfrak{X}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication by

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\lambda A = \{\lambda a : a \in A\}$$

The Hausdorff metric on $\mathbf{K}(\mathfrak{X})$ is defined by

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

for $A, B \in \mathbf{K}(\mathfrak{X})$. For an A in $\mathbf{K}(\mathfrak{X})$, let $\|A\|_{\mathbf{K}} = d_H(\{0\}, A)$.

The metric space $(\mathbf{K}_b(\mathfrak{X}), d_H)$ is complete, and $\mathbf{K}_{bc}(\mathfrak{X})$ is a closed subset of $(\mathbf{K}_b(\mathfrak{X}), d_H)$ (cf. [14], Theorems 1.1.2 and 1.1.3). For more general hyperspaces, more topological properties of hyperspaces, readers may refer to the books [15] and [14].

For each $A \in \mathbf{K}(\mathfrak{X})$, define the support function by

$$s(x^*, A) = \sup_{a\in A}\langle x^*, a\rangle, \quad x^* \in \mathfrak{X}^*,$$

where $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$.

Let $\mathbf{S}^*$ denote the unit sphere of $\mathfrak{X}^*$, $C(\mathbf{S}^*)$ the all continuous functions of $\mathbf{S}^*$, and the norm is defined as $\|v\|_C = \sup_{x^*\in\mathbf{S}^*}$.

The following is the equivalent definition of Hausdorff metric.

For each $A, B \in \mathbf{K}_{bc}(\mathfrak{X})$,

$$d_H(A, B) = \sup\left\{\left|s(x^*, A) - s(x^*, A)\right| : x^* \in S^*\right\}.$$

A set-valued mapping $F : \Omega \to \boldsymbol{K}(\mathfrak{X})$ is called *a set-valued random variable* (*or a random set, or a multi-function*) if, for each open subset O of $\mathfrak{X}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \varnothing\} \in \mathcal{A}$.

For each set-valued random variable F, the expectation of F, denoted by $E[F]$, is defined by

$$E[F] = \left\{\int_\Omega f\mathrm{d}\mu : f \in S_F\right\},$$

where $\int_\Omega f\mathrm{d}\mu$ is the usual Bochner integral in $L^1[\Omega, \mathfrak{X}]$, the family of integrable $\mathfrak{X}$-valued random variables, and $S_F = \left\{f \in L^1[\Omega; \mathfrak{X}] : f(\omega) \in F(\omega), a.e.(\mu)\right\}$.

Let $\mathbf{F}_k(\mathfrak{X})$ denote the family of all functions $v:\mathfrak{X}\to[0,1]$ which satisfy the following conditions:

1) The level set $v_1=\{x\in\mathfrak{X}:v(x)=1\}\neq\varnothing$.

2) Each v is upper semicontinuous, *i.e.* for each $\alpha\in(0,1]$, the α level set $v_\alpha=\{x\in\mathfrak{X}:v(x)\geq\alpha\}$ is a closed subset of $\mathfrak{X}$.

3) The support set $v_{0+}=cl\{x\in\mathfrak{X}:v(x)>0\}$ is compact.

A function v in $\mathbf{F}_k(\mathfrak{X})$ is called convex if it satisfies

$$v(\lambda x+(1-\lambda)y)\geq\min\{v(x),v(y)\},$$

for any $x,y\in\mathfrak{X},\lambda\in(0,1]$.

Let $\mathbf{F}_{kc}(\mathfrak{X})$ be the subset of all convex fuzzy sets in $\mathbf{F}_k(\mathfrak{X})$.

It is known that v is convex in the above sense if and only if, for any $\alpha\in(0,1]$, the level set v_α is a convex subset of $\mathfrak{X}$ (cf. Theorem 3.2.1 of [16]). For any $v\in\mathbf{F}_k(\mathfrak{X})$, the closed convex hull $\overline{co}v\in\mathbf{F}_{kc}(\mathfrak{X})$ of v is defined by the relation $(\overline{co}v)_\alpha=\overline{co}v_\alpha$ for all $\alpha\in(0,1]$.

For any two fuzzy sets v^1,v^2, define

$$(v^1+v^2)(x)=\sup\{\alpha\in(0,1]:x\in v^1_\alpha+v^2_\alpha\},$$

for any $x\in\mathfrak{X}$.

Similarly for a fuzzy set v and a real number λ, define

$$(\lambda v)(x)=\sup\{\alpha\in(0,1]:x\in\lambda v_\alpha\},$$

for any $x\in\mathfrak{X}$.

The following two metrics in $\mathbf{F}_k(\mathfrak{X})$ which are extensions of the Hausdorff metric d_H are often used (cf. [17] and [18], or [14]): for $v^1,v^2\in\mathbf{F}_k(\mathfrak{X})$,

$$d_H^\infty(v^1,v^2)=\sup_{\alpha\in(0,1]}d_H(v^1_\alpha,v^2_\alpha),$$

$$d_H^1(v^1,v^2)=\int_0^1 d_H(v^1_\alpha,v^2_\alpha)\,\mathrm{d}\alpha.$$

Denote $\|v\|_F=:d_H^\infty(v,I_0)=\sup_{\alpha>0}\|v_\alpha\|_K$, where I_0 is the fuzzy set taking value one at 0 and zero for all $x\neq 0$. The space $(\mathbf{F}_k(\mathfrak{X}),d_H^\infty)$ is a complete metric space (cf. [18], or [14]: Theorem 5.1.6) but not separable (cf. [17], or [14]: Remark 5.1.7).

It is well known that $v_\alpha=\bigcap_{\beta<\alpha}v_\beta$, for every $\alpha\in(0,1]$. Due to the completeness of $(\mathbf{F}_k(\mathfrak{X}),d_H^\infty)$, every Cauchy sequence $\{v^n:n\in\mathbb{N}\}$ has a limit v in $\mathbf{F}_k(\mathfrak{X})$.

A fuzzy set-valued random variable (or *a fuzzy random set*, *or a fuzzy random variable* in literature) is a mapping $X:\Omega\to\mathbf{F}_k(\mathfrak{X})$, such that $X_\alpha(\omega)=\{x\in\mathfrak{X}:X(\omega)(x)\geq\alpha\}$ is a set-valued random variable for every $\alpha\in(0,1]$ (cf. [18] or [14]).

The *expectation* of any fuzzy set-valued random variable X, denoted by $E[X]$, is an element in $\mathbf{F}_k(\mathfrak{X})$ such that, for every $\alpha\in(0,1]$,

$$(E[X])_\alpha=E[X_\alpha],$$

where the expectation of right hand is Aumann integral. From the existence theorem (cf. [19]), we can get an equivalent definition: for any $x\in\mathfrak{X}$,

$$E(X)(x)=\sup\{\alpha\in[0,1]:x\in E[X_\alpha]\}.$$

Note that $E[X]$ is always convex when $(\Omega,\mathcal{A},\mu)$ is nonatomic.

3. Main Results

In this section, we will give the limit theorems for fuzzy set-valued random variables in G_α space. I will firstly introduce the definition of G_α space. The following Definition 3.1 and Lemma 3.2 are from Taylor's book [8], which will be used later.

Definition 3.1. A Banach space $\mathfrak{X}$ is said to satisfy the condition G_α for some $\alpha \in (0,1]$. If there exists a mapping $G : \mathfrak{X} \to \mathfrak{X}^*$, such that

1) $\|G(x)\| = \|x\|^\alpha$;

2) $G(x)x = \|x\|^{1+\alpha}$;

3) $\|G(x) - G(y)\| = A\|x - y\|^\alpha$, for all $x, y \in \mathfrak{X}$ and some positive constant *A*.

Note that *Hilbert* spaces are G_1 with constant $A = 1$ and identity mapping *G*.

Lemma 3.2. Let $\mathfrak{X}$ be a *Banach* space which satisfies the condition of G_α, $\{V_1, V_2, \cdots, V_n\}$ be independent random elements in $\mathfrak{X}$, such that $E[V_k] = 0$ and $E\left[\|V_k\|^{1+\alpha}\right] < +\infty$ for each $k = 1, 2, \cdots, n$. Then

$$E\left[\|V_1 + \cdots + V_n\|^{1+\alpha}\right] \le A \sum_{k=1}^{n} E\left[\|V_k\|^{1+\alpha}\right]$$

where *A* is the positive constant in 3) of definition 3.1.

In order to obtain the main results, we firstly need to prove Lemma 3.5. The following lemma are from [14] (cf. p89, Lemma 3.1.4), which will be used to prove Lemma 3.5.

Lemma 3.3. Let $\{C_n : n \in N\}$ be a sequence in $\mathbf{K}_k(\mathfrak{X})$. If

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^{n} \overline{co} C_k, C\right) = 0,$$

for some $C \in \mathbf{K}_{kc}(\mathfrak{X})$, then

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^{n} C_k, C\right) = 0.$$

Lemma 3.4. (cf. [13]) For any $\nu \in \mathbf{F}_k(\mathfrak{X})$, there exists a finite $0 = t_0 < t_1 < \cdots < t_M = 1$, such that

$$d_H\left(\nu_{t_k}, \nu_{t_{k-1}^+}\right) \le \varepsilon, \text{ for all } k = 1, \cdots, M.$$

Now we prove that the result of Lemma 3.3 is also true for fuzzy sets.

Lemma 3.5. Let $\{\nu^n : n \in \mathbb{N}\}$ be a sequence in $\mathbf{F}_k(\mathfrak{X})$. If

$$\lim_{n\to\infty} d_H^\infty\left(\frac{1}{n}\sum_{k=1}^{n} \overline{co} \nu^k, \nu\right) = 0, \tag{3.1}$$

for some $\nu \in \mathbf{F}_{kc}(\mathfrak{X})$, then

$$\lim_{n\to\infty} d_H^\infty\left(\frac{1}{n}\sum_{k=1}^{n} \nu^k, \nu\right) = 0.$$

Proof. By (3.1), we can have

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^{n} \overline{co} \nu_\alpha^k, \nu_\alpha\right) = 0,$$

and

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^{n} \overline{co} \nu_{\alpha+}^k, \nu_{\alpha+}\right) = 0,$$

for $\alpha \in (0,1]$. Then by Lemma 3.3, for $\alpha \in (0,1]$, we have

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_\alpha^k,\nu_\alpha\right)=0,$$

and

$$\lim_{n\to\infty} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha+}^k,\nu_{\alpha+}\right)=0.$$

By Lemma 3.4, take an $\varepsilon>0$, there exists a finite $0=\alpha_0<\alpha_1<\cdots<\alpha_M=1$, such that

$$d_H\left(\nu_{\alpha_j},\nu_{\alpha_{j-1}+}\right)\le\varepsilon, \text{ for all } 1,\cdots,M.$$

Then for $\alpha_{j-1}<\alpha<\alpha_j$,

$$\begin{aligned} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_\alpha^k,\nu_\alpha\right) &\le d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha_j}^k,\nu_{\alpha_{j-1}+}\right)+d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha_{j-1}+}^k,\nu_{\alpha_j}\right)\\ &\le d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha_j}^k,\nu_{\alpha_j}\right)+d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha_{j-1}+}^k,\nu_{\alpha_{j-1}+}\right)+2d_H\left(\nu_{\alpha_j},\nu_{\alpha_{j-1}+}\right)\end{aligned}$$

Consequently,

$$\sup_{\alpha\in(0,1]} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_\alpha^k,\nu_\alpha\right)\le\max_{1\le j\le M} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha_j}^k,\nu_{\alpha_j}\right)+\max_{1\le j\le M} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_{\alpha_{j-1}+}^k,\nu_{\alpha_{j-1}+}\right)+2\varepsilon.$$

Since the first two terms on the right hand converge to 0 in probability one, we have

$$\limsup_{n\to\infty}\sup_{\alpha\in(0,1]} d_H\left(\frac{1}{n}\sum_{k=1}^{n}\nu_\alpha^k,\nu_\alpha\right)\le 2\varepsilon,$$

but ε is arbitrary and the result follows. □

Theorem 3.6. Let $\mathfrak{X}$ be a *Banach* space which satisfies the condition of G_α, let $\{X^n:n\ge1\}$ be independent fuzzy set-valued random variables in $\mathbf{F}_k(\mathfrak{X})$, such that $E\left[X^n\right]=I_0$ for any *n*. If

$$\sum_{j=1}^{\infty}E\left[\phi_0\left(\left\|X^j\right\|_{\mathbf{F}}\right)\right]<+\infty,$$

where $\phi_0(t)=t^{1+\alpha}$ for $0\le t\le1$ and $\phi_0(t)=t$ for $t\ge1$, then $\sum_{j=1}^{\infty}X^j$ converges with probability 1 in the sense of d_H^∞.

Proof. Define

$$U^j=X^jI_{\left\{\left\|X^j\right\|_{\mathbf{F}}\le1\right\}},\quad W^j=X^jI_{\left\{\left\|X^j\right\|_{\mathbf{F}}>1\right\}}.$$

Note that $X^j=W^j+U^j$ for each *j*, and both $\{W^j:j\ge1\}$ and $\{U^j:j\ge1\}$ are independent sequence of fuzzy set-valued random variables. When $\left\|X^j\right\|_{\mathbf{F}}>1$, we have $\left\|W^j\right\|_{\mathbf{F}}=\left\|X^j\right\|_{\mathbf{F}}$, and $\phi_0\left(\left\|X^j\right\|_{\mathbf{F}}\right)=\left\|X^j\right\|_{\mathbf{F}}$. Then, for any m,n

$$E\left[\left\|\sum_{j=n}^{m}W^j\right\|_{\mathbf{F}}\right]\le\sum_{j=n}^{m}E\left[\left\|W^j\right\|_{\mathbf{F}}\right]=\sum_{j=n}^{m}E\left[\phi_0\left(\left\|X^j\right\|_{\mathbf{F}}\right)\right].$$

And from $\sum_{j=1}^{\infty}E\left[\phi_0\left\|X^j\right\|_{\mathbf{F}}\right]<\infty$, we know that $\left\{E\left[\left\|\sum_{j=1}^{m}W^j\right\|_{\mathbf{F}}\right]:m\ge1\right\}$ is a *Cauchy* sequence. So, we have

$$E\left[\left\|\sum_{j=1}^{m} W^{j}\right\|_{\mathbf{F}}\right] \text{ converges as } m \to \infty.$$

Since convergence in the mean implied convergence in probability, Ito and Nisios result in [9] for independent random elements (cf. Section 4.5) provides that

$$\left\|\sum_{j=1}^{m} W^{j}\right\|_{\mathbf{F}} \text{ converges in probability } 1.$$

So, for any n, $m \geq 1$, $m > n$, by triangle inequality we have

$$\begin{aligned} d_H^{\infty}\left(\sum_{j=1}^{n} W^{j}, \sum_{j=1}^{m} W^{j}\right) &= d_H^{\infty}\left(\sum_{j=1}^{n} W^{j}, \sum_{j=1}^{n} W^{j} + \sum_{j=n+1}^{m} W^{j}\right) \\ &\leq d_H^{\infty}\left(\sum_{j=1}^{n} W^{j}, \sum_{j=1}^{n} W^{j}\right) + d_H^{\infty}\left(I_0 \sum_{j=n+1}^{m} W^{j}\right) \\ &= d_H^{\infty}\left(I_0, \sum_{j=n+1}^{m} W^{j}\right) \\ &= \left\|\sum_{j=n+1}^{m} W^{j}\right\|_{\mathbf{F}} \to 0, \, a.e. \text{ as } n, m \to \infty. \end{aligned}$$

It means $\left\{\sum_{j=1}^{n} W_j : n \geq 1\right\}$ is a *Cauchy* sequence in the sense of d_H^{∞}. By the completeness of $\left(\mathbf{F}_k\left(\mathfrak{X}\right), d_H^{\infty}\right)$, we have $\sum_{j=1}^{n} W_j$ converges almost everywhere in the sense of d_H^{∞}.

Next we shall prove that $\sum_{j=1}^{n} U^{j}$ converges in the sense of d_H^{∞}. Firstly, we assume that $\left\{U^{j}\right\}$ are all convex fuzzy set-valued random variables. Then by the equivalent definition of Hausdorff metric, we have

$$\begin{aligned} E\left[\left\|\sum_{j=n}^{m} U^{j}\right\|_{\mathbf{F}}^{1+\alpha}\right] &= E\left[\sup_{\beta \in (0,1]} \left\|\sum_{j=n}^{m} U_{\beta}^{j}\right\|_{\mathbf{K}}^{1+\alpha}\right] \\ &= E\left[\sup_{\beta \in (0,1]} d_H^{1+\alpha}\left(\sum_{j=n}^{m} U_{\beta}^{j}, \{0\}\right)\right] \\ &= E\left[\sup_{\beta \in (0,1]} \sup_{x^* \in S^*} \left|s\left(x^*, \sum_{j=n}^{m} U_{\beta}^{j}\right)\right|^{1+\alpha}\right] \end{aligned}$$

For any fixed n, m, there exists a sequence $x_k^* \in S^*$, such that

$$\lim_{k \to \infty} \left|s\left(x_k^*, \sum_{j=n}^{m} U_{\beta}^{j}\right)\right| = \sup_{x^* \in S^*} \left|s\left(x^*, \sum_{j=n}^{m} U_{\beta}^{j}\right)\right|.$$

That means there exist a sequence $x_k^* \in S^*$, such that

$$E\left[\left\|\sum_{j=n}^{m} U^{j}\right\|_{\mathbf{F}}^{1+\alpha}\right] = E\left[\sup_{\beta \in (0,1]} \lim_{k \to \infty} \left|s\left(x_k^*, \sum_{j=n}^{m} U_{\beta}^{j}\right)\right|^{1+\alpha}\right].$$

Then by Cr inequality, dominated convergence theorem and Lemma 3.2, we have

$$
\begin{aligned}
E\left[\left\|\sum_{j=n}^{m} U^{j}\right\|_{\mathbf{F}}^{1+\alpha}\right] &= E\left[\sup_{\beta\in(0,1]}\lim_{k\to\infty}\left|s\left(x_k^*,\sum_{j=n}^{m}U_\beta^j\right)\right|\right]^{1+\alpha} \\
&\le E\left[\sup_{\beta\in(0,1]}\lim_{k\to\infty}\sum_{j=n}^{m}\left|s\left(x_k^*,U_\beta^j\right)\right|\right]^{1+\alpha} \\
&\le E\left[\lim_{k\to\infty}\sup_{\beta\in(0,1]}\sum_{j=n}^{m}\left|s\left(x_k^*,U_\beta^j\right)\right|\right]^{1+\alpha} \\
&\le \lim_{k\to\infty}E\left[\sup_{\beta\in(0,1]}\sum_{j=n}^{m}\left|s\left(x_k^*,U_\beta^j\right)-E\left[s\left(x_k^*,U_\beta^j\right)\right]+E\left[s\left(x_k^*,U_\beta^j\right)\right]\right|\right]^{1+\alpha} \\
&= \lim_{k\to\infty}E\left[\lim_{i\to\infty}\sum_{j=n}^{m}\left|s\left(x_k^*,U_{\beta_i}^j\right)-E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]+E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]\right|\right]^{1+\alpha} \\
&= \lim_{k\to\infty}\lim_{i\to\infty}E\left[\sum_{j=n}^{m}\left|s\left(x_k^*,U_{\beta_i}^j\right)-E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]+E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]\right|\right]^{1+\alpha} \\
&\le \lim_{k\to\infty}\lim_{i\to\infty}E\left[\sum_{j=n}^{m}\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|-E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]+\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|\right]\right]^{1+\alpha} \\
&\le \lim_{k\to\infty}\lim_{i\to\infty}2^{1+\alpha}\left\{E\left\{\sum_{j=n}^{m}\left|s\left(x_k^*,U_{\beta_i}^j\right)-E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]\right|\right\}^{1+\alpha}+\left(\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta}^j\right)\right|\right]\right)^{1+\alpha}\right\} \\
&\le \lim_{k\to\infty}\lim_{i\to\infty}2^{1+\alpha}\left\{A\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)-E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]\right|^{1+\alpha}\right]+\left(\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|\right]\right)^{1+\alpha}\right\} \\
&\le \lim_{k\to\infty}\lim_{i\to\infty}2^{1+\alpha}\left\{2^{1+\alpha}A\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|^{1+\alpha}\right]+2^{1+\alpha}A\sum_{j=n}^{m}\left|E\left[s\left(x_k^*,U_{\beta_i}^j\right)\right]\right|^{1+\alpha}+\left(\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,W_{\beta_i}^j\right)\right|\right]\right)^{1+\alpha}\right\} \\
&\le \lim_{k\to\infty}\lim_{i\to\infty}2^{1+\alpha}\left\{2^{1+\alpha}A\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|^{1+\alpha}\right]+2^{1+\alpha}A\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|^{1+\alpha}\right]+\left(\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,W_{\beta_i}^j\right)\right|\right]\right)^{1+\alpha}\right\} \\
&= \lim_{k\to\infty}\lim_{i\to\infty}2^{1+\alpha}\left\{2^{2+\alpha}A\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,U_{\beta_i}^j\right)\right|^{1+\alpha}\right]+\left(\sum_{j=n}^{m}E\left[\left|s\left(x_k^*,W_{\beta_i}^j\right)\right|\right]\right)^{1+\alpha}\right\} \\
&\le \lim_{i\to\infty}2^{1+\alpha}\left\{2^{2+\alpha}A\sum_{j=n}^{m}E\left[\sup_{x^*\in S^*}\left|s\left(x^*,U_{\beta_i}^j\right)\right|^{1+\alpha}\right]+\left(\sum_{j=n}^{m}E\left[\sup_{x^*\in S^*}\left|s\left(x^*,W_{\beta_i}^j\right)\right|\right]\right)^{1+\alpha}\right\} \\
&= \lim_{i\to\infty}2^{1+\alpha}\left\{2^{2+\alpha}A\sum_{j=n}^{m}E\left[\left\|U_{\beta_i}^j\right\|_{\mathbf{K}}^{1+\alpha}\right]+\left(\sum_{j=n}^{m}E\left[\left\|W_{\beta_i}^j\right\|_{\mathbf{K}}\right]\right)^{1+\alpha}\right\} \\
&= 2^{1+\alpha}\left\{2^{2+\alpha}A\sum_{j=n}^{m}E\left[\left\|U^j\right\|_{\mathbf{F}}^{1+\alpha}\right]+\left(\sum_{j=n}^{m}E\left[\left\|W^j\right\|_{\mathbf{F}}\right]\right)^{1+\alpha}\right\} \\
&\le 2^{1+\alpha}\left\{2^{2+\alpha}A\sum_{j=n}^{m}E\left[\phi_0\left(\left\|X^j\right\|_{\mathbf{F}}\right)\right]+\left(\sum_{j=n}^{m}E\left[\phi_0\left(\left\|X^j\right\|_{\mathbf{F}}\right)\right]\right)^{1+\alpha}\right\}
\end{aligned}
$$

for each n and m.

Then, we know $\left\{E\left[\left\|\sum_{j=1}^{m} U^j\right\|_{\mathbf{F}}^{1+\alpha}\right]\right\}$ is a *Cauchy* sequence. Hence, $\left\{E\left[\left\|\sum_{j=1}^{m} U^j\right\|_{\mathbf{F}}\right]\right\}$ is a Cauchy sequence. Thus by the similar way as above to prove $\sum_{j=1}^{\infty} W^j$ converges with probability 1 in the sense of d_H^∞. We also can prove that

$$\sum_{j=1}^{\infty} U^j \text{ converges}$$

with probability 1 in the sense of d_H^∞. In fact, for each $n \le m$,

$$\begin{aligned} d_H^\infty\left(\sum_{j=1}^{n} U^j, \sum_{j=1}^{m} U^j\right) &= d_H^\infty\left(\sum_{j=1}^{n} U^j, \sum_{j=1}^{n} U^j + \sum_{j=n+1}^{m} U^j\right) \\ &\le \left\|\sum_{j=n+1}^{m} U^j\right\|_{\mathbf{F}} \\ &\to 0, a.e. \quad \text{as } n, m \to \infty. \end{aligned}$$

So, we can prove that

$$\sum_{j=1}^{\infty} X^j \quad \textit{converges}$$

with probability 1 in the sense of d_H^∞. If $\{U^j\}$ are not convex, we can prove $\sum_{j=1}^{n} \overline{co} U^j$ converges with probability 1 in the sense of d_H^∞ as above, and by the Lemma 3.5, we can prove that $\sum_{j=1}^{n} U^j$ converges with probability 1 in the sense of d_H^∞. Then the result was proved. □

From Theorem 3.6, we can easily obtain the following corollary.

Corollary 3.7. Let $\mathfrak{X}$ be a separable Banach space which is G_α for some $0 < \alpha \le 1$. Let $\{X^n : n \ge 1\}$ be a sequence of independent fuzzy set-valued random variables in $\mathbf{F}_k(\mathfrak{X})$, such that $E[X^n] = I_0$ for each *n*. If $\phi_n : \mathbb{R}^+ \to \mathbb{R}^+, n = 1, 2, \cdots,$ are continuous and such that $\frac{\phi_n(t)}{t}$ and $\frac{t^{1+\alpha}}{\phi_n(t)}$ are non-decreasing, then for each $\alpha_n \subset \mathbb{R}^+$ the convergence of

$$\sum_{n=1}^{\infty} \frac{E\left[\phi_n\left(\left\|X^n\right\|_F\right)\right]}{\phi_n(\alpha_n)}$$

implies that

$$\sum_{n=1}^{\infty} \frac{X^n}{\alpha_n}$$

converges with probability one in the sense of d_H^∞.

Proof. Let

$$U^j = \frac{X^j}{\alpha_j} I_{\left\{\left\|X^j\right\|_{\mathbf{F}} \le \alpha_j\right\}} \quad \text{and} \quad W^j = \frac{X^j}{\alpha_j} I_{\left\{\left\|X^j\right\|_{\mathbf{F}} > \alpha_j\right\}}.$$

If $\left\|X^n\right\|_{\mathbf{F}} > \alpha_n$, by the non-decreasing property of $\frac{\phi_n(t)}{t}$, we have

$$\frac{\phi_n(\alpha_n)}{\alpha_n} \le \frac{\phi_n\left(\left\|X^n\right\|_{\mathbf{F}}\right)}{\left\|X^n\right\|_{\mathbf{F}}}.$$

That is

$$\frac{\left\|X^n\right\|_{\mathbf{F}}}{\alpha_n} \le \frac{\phi_n\left(\left\|X^n\right\|_{\mathbf{F}}\right)}{\phi_n\left(\alpha_n\right)}. \tag{4.1}$$

If $\left\|X^n\right\|_{\mathbf{F}} \le \alpha_n$, by the non-decreasing property of $\frac{t^{1+\alpha}}{\phi_n(t)}$, we have

$$\frac{\left\|X^n\right\|_{\mathbf{F}}^{1+\alpha}}{\phi_n\left(\left\|X^n\right\|_{\mathbf{F}}\right)} \le \frac{\alpha_n^{1+\alpha}}{\phi_n\left(\alpha_n\right)}.$$

That is

$$\frac{\left\|X^n\right\|_{\mathbf{F}}^{1+\alpha}}{\alpha_n^{1+\alpha}} \le \frac{\phi_n\left(\left\|X^n\right\|_{\mathbf{F}}\right)}{\phi_n\left(\alpha_n\right)}. \tag{4.2}$$

Then as the similar proof of Theorem 3.6, we can prove both $\sum_{j=1}^{\infty} U^j$ and $\sum_{j=1}^{\infty} W^j$ converges with probability one in the sense of d_H^{∞}, and the result was obtained. □

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