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Proofs of the Density Theorem and Fatou's Radial Limit Theorem Using the Poisson Integral

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Abstract

Using only the Poisson integral and elementary convergence theorems, we prove the well-known Density theorem and Fatou's radial limit theorem.

Keywords

Poisson Integral, Density, Radial Limits, Harmonic Functions

Subject Areas: **Function Theory**

1. Introduction

In most texts, [1] (p. 261), [2] (p. 187), [3] (p. 129), to name a few, the Density Theorem is proven using the fact that if f is Lebesgue integrable and if $\sigma = \int f$, then $\sigma' = f$ a.e. This result in turn is proven using the Vitali Covering Theorem. The same procedure is also used in the proof of Fatou's radial limit theorem (see [4], p. 129, and [5], Vol. II, p. 362-364). We circumvent this reasoning in an attempt to make the Density and Fatou theorems more accessible to the reader. Our major reference will be Royden's book *Real Analysis*, and we will restrict ourselves to its first four chapters. These include a general introduction to set theory, the real number system, Lebesgue measurable functions, and the Lebesgue integral. Our major analytical tool will be the Poisson integral and we will use some of its well-known fundamental properties.

We first introduce the background material that will be used in the course of this note. Our setting is the unit disk D and its boundary C. We shall say that *a sequence* $\{I_k\}$ *of arcs in* C *converges to* $e^{i\theta} \in C$ *and write* $I_k \to e^{i\theta}$, $e^{i\theta} \in C$, if $e^{i\theta} \in I_k$ for each k and $\lim_{k\to\infty} diam(I_k) = 0$. Let A be any subset of C and let m^* and m denote respectively the outer Lebesgue measure and the Lebesgue measure on C. For any measurable set E in C we define $\sigma_A(E) = m^*(A \cap E)$. We shall say that the derivative of σ_A at $e^{i\theta}$ exists if there exists a number $\sigma'_A(e^{i\theta})$ such that for any sequence $\{I_k\}$ of arcs converging to $e^{i\theta}$,

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$$\lim_{k\to\infty}\frac{\sigma_A(I_k)}{m(I_k)}=\sigma'_A(\mathrm{e}^{i\theta}).$$

If $\sigma'_A(\mathrm{e}^{i\theta})=1$, $\mathrm{e}^{i\theta}$ is called a *point of density of A* and if $\sigma'_A(\mathrm{e}^{i\theta})=0$, $\mathrm{e}^{i\theta}$ is called *a point of dispersion of A*. The Density Theorem states that if A is any set (measurable or not) in C, then $\mathrm{e}^{i\theta}$ is a point of density for A for almost all $\mathrm{e}^{i\theta}$ in A. We shall prove that this result in the case A is measurable.

Let $u(z)$, $z\in D$, be defined by the Poisson integral,

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,\varphi)\chi_A(\mathrm{e}^{i\varphi})\mathrm{d}\varphi \tag{1}$$

where $\chi_A(\mathrm{e}^{i\varphi})$ is the characteristic function on the measurable set $A\subset C$, and

$$P(z,\varphi)=\frac{(1-r^2)}{1+r^2-2r\cos(\varphi-\theta)},\quad z=r\mathrm{e}^{i\theta},$$

is the Poisson kernel. When A is a finite union of open arcs, then χ_A is a bounded piecewise continuous function on C. So, if ξ is a point of continuity of χ_A, then a straight-forward examination of the integral in (1) shows that the limit of $u(z)$, as z approaches ξ exists and equals $\chi_A(\xi)$ (see [6], Vol. II, p. 156; [7], p. 206; and [4], p. 130). The kernel has two nice properties:

i) $P(z,\varphi)\ge 0$ for all $z\in D$, $\varphi\in[0,2\pi)$, and

ii) $(1/2\pi)\int_0^{2\pi}P(z,\varphi)\mathrm{d}\varphi=1$ (see [8], p. 165-167 and [9], p. 305-307).

It follows that $u(z)\le 1$ for all $z\in D$. We will use these elementary results in Sections 2 and 3.

2. The Density Theorem

Theorem: *Let A be measurable. Then almost every point of A is a point of density of A.*

Proof: Let B be those points of A that are not points of density of A; that is,

$$B=\left\{\mathrm{e}^{i\theta}\in A:\operatorname{liminf}_{\delta\to 0}\frac{m(A\cap I_\delta)}{m(I_\delta)}<1\right\}$$

where I_δ, $\delta>0$, denotes an arc containing $\mathrm{e}^{i\theta}$ having length δ. For each $k=1,2,\cdots$, let $\xi_k=1/k$. We will first show that B can be rewritten as

$$\left\{\mathrm{e}^{i\theta}\in A:\operatorname{liminf}_{k\to\infty}\frac{m\left(A\cap I_{\xi_k}(\mathrm{e}^{i\theta})\right)}{2\xi_k}<1\right\}$$

where $I_{\xi_k}(\mathrm{e}^{i\theta})$ is an arc centered at $\mathrm{e}^{i\theta}$ with length $2\xi_k$ such that $\xi_k\to 0$ as $k\to\infty$. This reformulation of B is crucial to our proof. Clearly, any point in this set is in B. Now suppose $\mathrm{e}^{i\theta}\in B$. Then there exists a positive number ε and a sequence $\{\delta_n\}$ such that $\delta_n\to 0$ as $n\to\infty$ and

$$\frac{m\left(A\cap I_{\delta_n}\right)}{m\left(I_{\delta_n}\right)}<1-\varepsilon.$$

For each n let s_n denote the length of the largest component of $I_{\delta_n}-\{\mathrm{e}^{i\theta}\}$ and choose k_n such that $\xi_{k_n+1}\le s_n<\xi_{k_n}$. Then

$$\frac{m\left(A\cap I_{\xi_{k_n}}(\mathrm{e}^{i\theta})\right)}{2\xi_{k_n}}=\frac{m\left(A\cap I_{\delta_n}\right)+m\left(A\cap\left(I_{\xi_{k_n}}(\mathrm{e}^{i\theta})-I_{\delta_n}\right)\right)}{2\xi_{k_n}}$$

$$<\frac{(1-\varepsilon)\delta_n}{2\xi_{k_n}}+\left(1-\frac{\delta_n}{2\xi_{k_n}}\right)=1-\varepsilon\frac{\delta_n}{2\xi_{k_n}}<1-\frac{\varepsilon}{2}\left(\frac{\xi_{k_n+1}}{\xi_{k_n}}\right).$$

Since the last expression approaches $1 - \varepsilon/2$ as $k_n \to \infty$, it follows that

$$\liminf_{k\to\infty} \frac{m\left(A \cap I_{\xi_k}\left(e^{i\theta}\right)\right)}{2\xi_k} < 1.$$

Using this reformulation of B, the facts that for each k, $m\left(A \cap I_{\xi_k}\left(e^{i\theta}\right)\right)\big/2\xi_k$ is a continuous function of θ and consequently measurable, theorem 20 in [10] (p. 56), and that A is a measurable set, it readily follows that B is a measurable set. Hence, it is the union of an F-σ set and a set N of measure zero; that is, $B = \left(\bigcup F_n\right) \cup N$ where each F_n is a closed subset of B. If we show each F_n has measure zero, then the theorem is proved. We pick a F_n and denote it by F in order to avoid layered subscript notation. Now $F^C = \bigcup_{k=1}^{\infty} O_k$, where O_k are pairwise disjoint open arcs. For each $n = 1, 2, \cdots$, let $D_n = C - \bigcup_{k=1}^{n} O_k$, χ_n be the characteristic function of D_n, and set $u_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\varphi)\chi_n\left(e^{i\varphi}\right)d\varphi$ for $|z| < 1$. Note that for all n, $D_n \supset D_{n+1} \supset F$, and $\bigcap_n D_n = F$. We know from our introductory remarks that except for a finite number of $e^{i\theta}$ on C, $\lim_{r\to 1} u_n\left(re^{i\theta}\right) = \chi_n\left(e^{i\theta}\right)$. Let S_n denote this exceptional set. If we set $S = \bigcup_{n=1}^{\infty} S_n$, then for all $e^{i\theta} \in C - S$ and for all n,

$$\lim_{r\to 1} u_n\left(re^{i\theta}\right) = \chi_n\left(e^{i\theta}\right). \tag{2}$$

Also, for all $e^{i\theta} \in C$,

$$\lim_{n\to\infty} \chi_n\left(e^{i\theta}\right) = \chi_F\left(e^{i\theta}\right). \tag{3}$$

We now show that $m(F) = 0$ using an indirect argument. Before we formally proceed we indicate the direction our proof will take: We define the Poisson integral of the characteristic function on F and using Equations (2) and (3), along with the assumption that F has positive measure, find a subset of F where the radial limit of this function is 1. We then use the reformulation of B to show that this cannot happen.

So, suppose that $m(F) > 0$. Momentarily fix θ. For each $k = 1, 2, \cdots$, define $r_k = 1 - \xi_k$. Then $r_k \to 1$ as $k \to \infty$. Since for each k, $P\left(r_k e^{i\theta}, \varphi\right)$ is a nonnegative and integrable function of φ, we have ([10], p. 73) that for each ξ_k there exists $\delta_k > 0$ such that if E is any set with $m(E) < \delta_k$, then

$$\frac{1}{2\pi}\int_E P\left(r_k e^{i\theta}, \varphi\right)d\varphi < \xi_k. \tag{4}$$

For each $k = 1, 2, \cdots$, let $\eta_k = \min\left(m(F)\big/2^{k+1}, \delta_k\right)$. Using Egorov's Theorem ([10], p. 59), we have for each k an open set M_k such that

$$m\left(M_k^C\right) < \eta_k \text{ and } \chi_n\left(e^{i\varphi}\right) \to \chi_F\left(e^{i\varphi}\right) \text{ uniformly on } M_k. \tag{5}$$

Note that $m\left(\bigcup_k M_k^C\right) \le \sum_k m\left(M_k^C\right) < \sum_k \frac{m(F)}{2^{k+1}} = \frac{m(F)}{2}$, and since S is at most countable, $F - \left(\left(\bigcup_k M_k^C\right) \cup S\right)$ has positive measure. Set $u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\varphi)\chi_F\left(e^{i\varphi}\right)d\varphi$.

We now claim that if $e^{i\theta}$ is in $F - \left(\left(\bigcup_k M_k^C\right) \cup S\right)$, then $\lim_{k\to\infty} u\left(r_k e^{i\theta}\right) = 1$. Let ε be an arbitrary positive number and choose K such that for $k > K$, $\xi_k < \varepsilon/4$. Now for $k > K$,

$$\begin{aligned} 1 - u\left(r_k e^{i\theta}\right) &= \frac{1}{2\pi}\int_0^{2\pi}\left(1 - \chi_F\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta}, \varphi\right)d\varphi \\ &= \frac{1}{2\pi}\int_{M_k}\left(1 - \chi_F\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta}, \varphi\right)d\varphi \\ &\quad + \frac{1}{2\pi}\int_{M_k^C}\left(1 - \chi_F\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta}, \varphi\right)d\varphi. \end{aligned}$$

Since χ_n converges uniformly to χ_F on M_k we choose $N = N(\varepsilon)$ such that $\left|\chi_N\left(e^{i\varphi}\right) - \chi_F\left(e^{i\varphi}\right)\right| < \varepsilon/4$ on M_k.

Thus,

$$\begin{aligned}\frac{1}{2\pi}\int_{M_k}\left(1-\chi_F\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta},\varphi\right)d\varphi &\le \frac{1}{2\pi}\int_{M_k}\left(1-\chi_N\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta},\varphi\right)d\varphi \\ &\quad +\frac{1}{2\pi}\int_{M_k}\left(\chi_N\left(e^{i\varphi}\right)-\chi_F\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta},\varphi\right)d\varphi \\ &\le \left(1-u_N\left(r_k e^{i\theta}\right)\right)+\left(\varepsilon/4\right)\left[\frac{1}{2\pi}\int_0^{2\pi}P\left(r_k e^{i\theta},\varphi\right)d\varphi\right] \\ &\le \left(1-u_N\left(r_k e^{i\theta}\right)\right)+\left(\varepsilon/4\right).\end{aligned}$$

Using (4) with the fact that $m\left(M_k^C\cap F^C\right)<\eta_k<\delta_k$ one also has

$$\frac{1}{2\pi}\int_{M_k^C}\left(1-\chi_F\left(e^{i\varphi}\right)\right)P\left(r_k e^{i\theta},\varphi\right)d\varphi=\frac{1}{2\pi}\int_{M_k^C\cap F^C}P\left(r_k e^{i\theta},\varphi\right)d\varphi<\xi_k<\varepsilon/4.$$

Hence, $0\le 1-u\left(r_k e^{i\theta}\right)\le 1-u_N\left(r_k e^{i\theta}\right)+\varepsilon/2$ for all $k > K$. Because $e^{i\theta}\in F-\left(\left(\bigcup_k M_k^C\right)\cup S\right)$, Equation (2) holds. In addition, we have for all n that $\chi_n = \chi_F$ on F. Consequently, for k sufficiently large,

$$0\le 1-u\left(r_k e^{i\theta}\right)\le\varepsilon,$$

and since ε was arbitrary, our claim is established.

However, using the reformulation of B we know that since $e^{i\theta}\in F$ and $F\subset B$, there exists a positive number η and a subsequence $\left\{I_{\xi_{k_n}}\left(e^{i\theta}\right)\right\}$ of arcs with the property that

$$m\left(F\cap I_{\xi_{k_n}}\left(e^{i\theta}\right)\right)<\left(1-\eta\right)m\left(I_{\xi_{k_n}}\left(e^{i\theta}\right)\right).$$

Since $r_{k_n}=1-\xi_{k_n}$, we have

$$\begin{aligned}1-u\left(r_{k_n}e^{i\theta}\right)&=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(1-\chi_F\left(e^{i\varphi}\right)\right)\left(1-r_{k_n}^2\right)}{1+r_{k_n}^2-2r_{k_n}\cos\left(\varphi-\theta\right)}d\varphi\ge\frac{1}{2\pi}\int_{I_{\xi_{k_n}}\left(e^{i\theta}\right)}\left(\frac{\left(1-\chi_F\left(e^{i\varphi}\right)\right)\left(1-r_{k_n}^2\right)}{1+r_{k_n}^2-2r_{k_n}\cos\left(\varphi-\theta\right)}\right)d\varphi \\ &\ge\frac{1}{2\pi}\frac{\left(1-r_{k_n}\right)}{\left(1-r_{k_n}\right)^2+4r_{k_n}\sin^2\left(\frac{\xi_{k_n}}{2}\right)}\int_{I_{\xi_{k_n}}\left(e^{i\theta}\right)}\left(1-\chi_F\left(e^{i\varphi}\right)\right)d\varphi \\ &\ge\frac{\xi_{k_n}m\left(F^C\cap I_{\xi_{k_n}}\right)}{2\pi\left[\xi_{k_n}^2+4\left(1-\xi_{k_n}\right)\sin^2\left(\frac{\xi_{k_n}}{2}\right)\right]}>\frac{\xi_{k_n}\left(2\eta\xi_{k_n}\right)}{2\pi\left[\xi_{k_n}^2+4\left(1-\xi_{k_n}\right)\sin^2\left(\frac{\xi_{k_n}}{2}\right)\right]} \\ &>\frac{\eta\xi_{k_n}^2}{\pi\xi_{k_n}{}^2\left[1+\left(\left(1-\xi_{k_n}\right)\sin^2\left(\frac{\xi_{k_n}}{2}\right)\right)\Big/\left(\frac{\xi_{k_n}}{2}\right)^2\right]}.\end{aligned}$$

Since the last expression approaches $\eta/2\pi$ as $\xi_{k_n}\to 0$ or as $r_{k_n}\to 1$, we have that the

$$\limsup_{k_n\to\infty}u\left(r_{k_n}e^{i\theta}\right)<1,$$

and this contradicts our previous claim. Thus, the measure of F cannot be positive and so $m(F) = 0$.

3. Density and the Radial

In this section we establish relationships between the density of A at a point of C and the radial limit of the Poisson integral of the characteristic function on A at this point. The proofs of the first two theorems employ well-known procedures and inequalities. Theorem 3 highlights the last result in the proof of the Density Theorem. We then use these relationships and prove in Corollary 5 Fatou's radial limit theorem.

Theorem 1. *If* $e^{i\theta}$ *is a point of density of A and u(z) represents the Poisson integral of* χ_A, *then the radial limit of u(z) at* $e^{i\theta}$ *is* 1.

Proof: Without loss of generality we can assume $e^{i\theta} = e^{i0} = 1$ and we express the Poisson integral of χ_A over the interval $[-\pi, \pi]$ instead of $[0, 2\pi]$. We must show that $\lim_{r\to 1} u(r) = 1$.

Since 1 is a point of density of the measurable set A, it follows that 1 is a point of dispersion of A^C, the complement of A with respect to C. Letting $\varepsilon(\delta) = m\left(A^C \cap I_\delta(1)\right)/2\delta$, where $I_\delta(1)$ denotes the arc on C centered at 1with length 2δ, we know that $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Let $z = r$. Then

$$1 - u(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(1-\chi_A\left(e^{i\varphi}\right)\right)\left(1-r^2\right)}{1+r^2-2rcos(\varphi)}\,d\varphi. \tag{6}$$

There are two possibilities that can happen with the function $\varepsilon(\delta)$, $\delta > 0$. Either there exists a δ_0 such that $\varepsilon(\delta_0) = 0$; or for all δ, $\varepsilon(\delta) \neq 0$.

In the first case it follows that $m\left(A^C \cap I_\delta(1)\right) = 0$ for all $\delta < \delta_0$. We now rewrite (6) as

$$\frac{1}{2\pi}\int_{-\delta_0}^{\delta_0}\left(1-\chi_A\left(e^{i\varphi}\right)\right)P(r,\varphi)\,d\varphi + \frac{1}{2\pi}\int_{\delta_0}^{\pi}\left(1-\chi_A\left(e^{i\varphi}\right)\right)P(r,\varphi)\,d\varphi + \frac{1}{2\pi}\int_{-\pi}^{-\delta_0}\left(1-\chi_A\left(e^{i\varphi}\right)\right)P(r,\varphi)\,d\varphi.$$

In the first integral we know that $m\left(A^C \cap I_{\delta_0}(1)\right) = 0$; and for $e^{i\varphi} \in A \cap I_{\delta_0}(1)$, $1-\chi_A\left(e^{i\varphi}\right) = 0$. Hence, the integral is zero. For the second integral we have that $\delta_0 \le \varphi \le \pi$ and so $\delta_0/2 \le \varphi/2 \le \pi/2$. Now

$$1 + r^2 - 2r\cos(\varphi) = (1-r)^2 + 4r\sin^2\left(\frac{\varphi}{2}\right) > 4r\sin^2\left(\frac{\varphi}{2}\right).$$

On $[0, \pi/2]$ we have that $\sin(x) \ge (2/\pi)x$. So if $r > 1/2$ we get $4r\sin^2\left(\frac{\varphi}{2}\right) > \frac{2\varphi^2}{\pi^2}$, and hence

$$\frac{1}{2\pi}\int_{\delta_0}^{\pi}\left(1-\chi_A\left(e^{i\varphi}\right)\right)P(r,\varphi)\,d\varphi \le \frac{1}{\pi}\int_{\delta_0}^{\pi}\frac{(1-r)\pi^2}{2\varphi^2}\,d\varphi = \frac{(1-r)\pi}{2}\left[\frac{1}{\delta_0} - \frac{1}{\pi}\right] \to 0 \text{ as } r \to 1.$$

The third integral is handled like the second, and so our theorem follows.

In the second case it follows that if δ is sufficiently small, $m\left(A^C \cap I_\delta(1)\right)$ is a continuous increasing function of δ. So there exists a δ_0 such that the expression $(\delta/2)\,m\left(A^C \cap I_\delta\right) = \delta^2\varepsilon(\delta)$ is continuous and increasing on $(0, \delta_0)$. Consequently, the expression $\delta\sqrt{\varepsilon(\delta)}$ is a continuous increasing function of δ on $(0, \delta_0)$ with the property that $\delta\sqrt{\varepsilon(\delta)} \to 0$ as $\delta \to 0$. Using this result we know that as $r \to 1$, $1 - r \to 0$ and so it can be represented by $\delta_r\sqrt{\varepsilon(\delta_r)}$ for some δ_r. Furthermore, $1 - r \to 0$ iff $\delta_r \to 0$. On the interval $[-\delta_r, \delta_r]$ we use the inequality $1 + r^2 - 2r\cos(\varphi) = (1-r)^2 + 4r\sin^2\left(\frac{\varphi}{2}\right) > (1-r)^2$ to get

$$\frac{1}{2\pi}\int_{-\delta_r}^{\delta_r}\left(1-\chi_A\left(e^{i\varphi}\right)\right)P(r,\varphi)\,d\varphi \le \frac{1}{2\pi}\int_{A^C\cap I_{\delta_r}}\frac{1-r^2}{(1-r)^2}\,d\varphi \le \frac{1}{\pi(1-r)}\int_{A^C\cap I_{\delta_r}}d\varphi \le \frac{1}{\pi}\frac{m\left(A^C \cap I_{\delta_r}\right)}{\delta_r\sqrt{\varepsilon(\delta_r)}}$$

$$= \frac{1}{\pi}\frac{\varepsilon(\delta_r)(2\delta_r)}{\delta_r\sqrt{\varepsilon(\delta_r)}} = \sqrt{\varepsilon(\delta_r)}\left(\frac{2}{\pi}\right) \to 0 \text{ as } \delta_r \to 0.$$

We now rewrite (6) as was done in the first case but using δ_r instead of δ_0. The first integral is handled above. The second integral is handled exactly as before. We get

$$\frac{1}{2\pi}\int_{\delta_r}^{\pi}\left(1-\chi_A\left(e^{i\varphi}\right)\right)P(r,\varphi)\,d\varphi = \frac{(1-r)\pi}{2}\left[\frac{1}{\delta_r} - \frac{1}{\pi}\right] = \frac{\left(\delta_r\sqrt{\varepsilon(\delta_r)}\right)\pi}{2}\left[\frac{1}{\delta_r} - \frac{1}{\pi}\right]$$

$$= \sqrt{\varepsilon(\delta_r)}\left[\frac{\pi - \delta_r}{2}\right] \to 0 \text{ as } \delta_r \to 0.$$

The third integral is handled like above. Hence, our theorem follows. Using a similar argument we can obtain the second result of this section.

Theorem 2. *If* $e^{i\theta}$ *is a point of dispersion of A, then the radial limit of u(z) at* $e^{i\theta}$ *is 0.*

Theorem 3. *If* $e^{i\theta}$ *is neither a point of density of A nor a point of dispersion of A, then the radial limit, if it exists, cannot be* 1 *or* 0.

Proof: From the hypothesis we know there exists an $\varepsilon > 0$ and two sequences $\{I_{\delta_n}\},\{I_{\rho_n}\}$ of arcs, where δ_n and ρ_n denote the length of I_{δ_n} and I_{ρ_n} respectively, such that $e^{i\theta} \in I_{\delta_n}$ for all n, $e^{i\theta} \in I_{\rho_n}$ for all n,

$$m\left(A \cap I_{\delta_n}\right) < (1-\varepsilon)\, m\left(I_{\delta_n}\right) \text{ for all } n,$$

$$m\left(A \cap I_{\rho_n}\right) > \varepsilon m\left(I_{\rho_n}\right) \text{ for all } n,$$

and $\delta_n \to 0$ and $\rho_n \to 0$ as $n \to \infty$. Since the arcs are not centered at $e^{i\theta}$ we define ξ_n to be the maximum of the lengths of the components of $I_{\delta_n} - \{e^{i\theta}\}$ and define $I_{\xi_n}\left(e^{i\theta}\right)$ to be the arc centered at $e^{i\theta}$ with length $2\xi_n$. Using I_{ρ_n} we define τ_n similarly. Let $r_n = 1-\xi_n$ and $s_n = 1-\tau_n$. Then,

$$1-u\left(r_n e^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1-\chi_A\left(e^{i\varphi}\right)\right)\left(1-r_n^2\right)}{1+r_n^2-2r_n\cos(\varphi-\theta)}\,d\varphi \ge \frac{1}{2\pi}\int_{I_{\xi_n}\left(e^{i\theta}\right)} \frac{\left(1-\chi_A\left(e^{i\varphi}\right)\right)\left(1-r_n^2\right)}{1+r_n^2-2r_n\cos(\xi_n)}\,d\varphi$$

$$\ge \frac{1}{2\pi}\frac{(1-r_n)}{(1-r_n)^2+4r_n\sin^2\left(\frac{\xi_n}{2}\right)}\int_{I_{\xi_n}\left(e^{i\theta}\right)}\left(1-\chi_A\left(e^{i\varphi}\right)\right)d\varphi$$

$$\ge \frac{\xi_n m\left(A^C \cap I_{\delta_n}\right)}{2\pi\left[\xi_n^2+4(1-\xi_n)\sin^2\left(\frac{\xi_n}{2}\right)\right]} > \frac{\xi_n\left(\varepsilon\delta_n\right)}{2\pi\left[\xi_n^2+4(1-\xi_n)\sin^2\left(\frac{\xi_n}{2}\right)\right]}$$

$$> \frac{\varepsilon\xi_n^2}{2\pi\xi_n^2\left[1+\left((1-\xi_n)\sin^2\left(\frac{\xi_n}{2}\right)\right)\Big/\left(\frac{\xi_n}{2}\right)^2\right]}.$$

Since the last expression approaches $\varepsilon/4\pi$ as $\xi_n \to 0$ or as $r_n \to 1$, we have that the

$$\limsup_{n\to\infty} u\left(r_n e^{i\theta}\right) < 1-\frac{\varepsilon}{8\pi}. \tag{7}$$

A similar argument using $s_n = 1-\tau_n$ shows that the $\liminf_{n\to\infty} u\left(s_n e^{i\theta}\right) > \varepsilon/8\pi$. It follows that if the radial limit of $u(z)$ at $e^{i\theta}$ exists, then it cannot be 1 or 0.

Corollary 1: *Let A be any measurable set in C. Let* $u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\varphi)\,\chi_A\left(e^{i\varphi}\right)d\varphi$. *Then for almost every* $e^{i\theta} \in C$, $\lim_{r\to 1} u\left(re^{i\theta}\right) = \chi_A\left(e^{i\theta}\right)$.

Proof: Using the Density Theorem on A and Theorem 1, we have for almost every $e^{i\theta}$ in A, $\lim_{r\to 1} u\left(re^{i\theta}\right) = 1 = \chi_A\left(e^{i\theta}\right)$. Since A^C is also measurable we have that almost every point of A^C is a point of density of A^C. Hence, almost every point of A^C is a point of dispersion of A. By Theorem 2 we have that for almost every $e^{i\theta}$ in A^C, $\lim_{r\to 1} u\left(re^{i\theta}\right) = 0 = \chi_A\left(e^{i\theta}\right)$.

Corollary 2: *Let f be a simple function defined on C. Let* $u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\varphi)\, f\left(e^{i\varphi}\right)d\varphi$. *Then for almost every* $e^{i\theta} \in C$, $\lim_{r\to 1} u\left(re^{i\theta}\right) = f\left(e^{i\theta}\right)$.

Proof: If f is simple, then $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ where the A_i are pair-wise disjoint and measurable and the a_i are distinct and nonzero. Using Corollary 1, our result follows.

Corollary 3: *Let f be a bounded measurable function defined on C. Let* $u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,\varphi)f(\mathrm{e}^{i\varphi})\mathrm{d}\varphi$. *Then for almost every* $\mathrm{e}^{i\theta}\in C$, $\lim_{r\to 1}u(r\mathrm{e}^{i\theta})=f(\mathrm{e}^{i\theta})$.

Proof: Since f is bounded and measurable, there exists simple functions ψ_n, $\psi_n\le f$, such that $\lim_{n\to\infty}\psi_n(\mathrm{e}^{i\varphi})=f(\mathrm{e}^{i\varphi})$ uniformly on C. By adding and subtracting the appropriate terms and using the triangle inequality one can show that

$$\left|u(r\mathrm{e}^{i\theta})-f(\mathrm{e}^{i\theta})\right|\le\frac{1}{2\pi}\int_0^{2\pi}P(r\mathrm{e}^{i\theta},\varphi)\left|f(\mathrm{e}^{i\varphi})-\psi_n(\mathrm{e}^{i\varphi})\right|\mathrm{d}\varphi$$
$$+\left|\frac{1}{2\pi}\int_0^{2\pi}P(r\mathrm{e}^{i\theta},\varphi)\psi_n(\mathrm{e}^{i\varphi})\mathrm{d}\varphi-\psi_n(\mathrm{e}^{i\theta})\right|+\left|f(\mathrm{e}^{i\theta})-\psi_n(\mathrm{e}^{i\theta})\right|.$$

We analyze each term on the right hand side of this inequality. Since the kernel $P(z,\varphi)$ is non-negative and its definite integral is 2π (see Section 1), and since ψ_n converges uniformly to f on C, the first term can be made arbitrarily small when n is sufficiently large. From Corollary 2 we get that the second term can be made arbitrarily small as r approaches 1. The last term approaches 0 for n sufficiently large since ψ_n converges uniformly to f on C. Consequently, our result follows.

Corollary 4: *Let f be a nonnegative integrable function defined on C. Let* $u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,\varphi)f(\mathrm{e}^{i\varphi})\mathrm{d}\varphi$. *Then for almost every* $\mathrm{e}^{i\theta}\in C$, $\lim_{r\to 1}u(r\mathrm{e}^{i\theta})=f(\mathrm{e}^{i\theta})$.

Proof: Let $S=\{\mathrm{e}^{i\theta}:f(\mathrm{e}^{i\theta})<+\infty\}$. Since f is integrable we know that $m(S)=2\pi$. For each $n=1,2,\cdots$, define $f_n(\mathrm{e}^{i\varphi})=\min(f(\mathrm{e}^{i\varphi}),n)$. Each f_n is bounded, nonnegative, and measurable on S. The functions also satisfy the following on S: $f_n(\mathrm{e}^{i\varphi})\le f_{n+1}(\mathrm{e}^{i\varphi})\le f(\mathrm{e}^{i\varphi})$ and $\lim_{n\to\infty}f_n(\mathrm{e}^{i\varphi})=f(\mathrm{e}^{i\varphi})$. Since $P(z,\varphi)$ is non-negative we have that $P(z,\varphi)f_n(\mathrm{e}^{i\varphi})\le P(z,\varphi)f_{n+1}(\mathrm{e}^{i\varphi})\le P(z,\varphi)f(\mathrm{e}^{i\varphi})$ and hence

$\lim_{n\to\infty}P(z,\varphi)f_n(\mathrm{e}^{i\varphi})=P(z,\varphi)f(\mathrm{e}^{i\varphi})$ on S. By the Monotone Convergence Theorem ([10], p. 72) we know that

$$\int_C P(z,\varphi)f_n(\mathrm{e}^{i\varphi})\mathrm{d}\varphi\to\int_C P(z,\varphi)f(\mathrm{e}^{i\varphi})\mathrm{d}\varphi\ \text{ as } n\to\infty.$$

Once again, by adding and subtracting the appropriate terms and using the triangle inequality one can show that

$$\left|u(r\mathrm{e}^{i\theta})-f(\mathrm{e}^{i\theta})\right|\le\frac{1}{2\pi}\left|\int_0^{2\pi}P(r\mathrm{e}^{i\theta},\varphi)f(\mathrm{e}^{i\varphi})\mathrm{d}\varphi-\int_0^{2\pi}P(r\mathrm{e}^{i\theta},\varphi)f_n(\mathrm{e}^{i\varphi})\mathrm{d}\varphi\right|$$
$$+\left|\frac{1}{2\pi}\int_0^{2\pi}P(r\mathrm{e}^{i\theta},\varphi)f_n(\mathrm{e}^{i\varphi})\mathrm{d}\varphi-f_n(\mathrm{e}^{i\theta})\right|+\left|f(\mathrm{e}^{i\theta})-f_n(\mathrm{e}^{i\theta})\right|.$$

From our above remarks, the first term can be made arbitrarily small for n sufficiently large. Using Corollary 3, the second term can be made arbitrarily small as r approaches 1. The last term approaches 0 on S as n gets large. Consequently, our result follows.

Corollary 5: (*Fatou*) *Let f be integrable on C. Let* $u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,\varphi)f(\mathrm{e}^{i\varphi})\mathrm{d}\varphi$. *Then for almost every* $\mathrm{e}^{i\theta}\in C$, $\lim_{r\to 1}u(r\mathrm{e}^{i\theta})=f(\mathrm{e}^{i\theta})$.

Proof: We know that $f=f^+-f^-$ where $f^+(\mathrm{e}^{i\varphi})=\max(f(\mathrm{e}^{i\varphi}),0)$ and $f^-(\mathrm{e}^{i\varphi})=\max(-f(\mathrm{e}^{i\varphi}),0)$. Since f is integrable, both f^+ and f^- must be integrable. We can now use Corollary 4 on f^+ and f^- to get our result.

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