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On a Class of Supereulerian Digraphs

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Abstract

The 2-sum of two digraphs D_1 and D_2, denoted $D_1 \oplus_2 D_2$, is the digraph obtained from the disjoint union of D_1 and D_2 by identifying an arc in D_1 with an arc in D_2. A digraph *D* is supereulerian if *D* contains a spanning eulerian subdigraph. It has been noted that the 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian. We obtain several sufficient conditions on D_1 and D_2 for $D_1 \oplus_2 D_2$ to be supereulerian. In particular, we show that if D_1 and D_2 are symmetrically connected or partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.

Keywords

Supereulerian, Digraph 2-Sums, Arc-Strong-Connectivity, Hamiltonian-Connected Digraphs

1. Introduction

We consider finite graphs and digraphs, and undefined terms and notations will follow [1] for graphs and [2] for digraphs. Throughout this paper, the notation (u,v) denotes an arc oriented from *u* to *v*. A digraph *D* is **strict** if it contains no parallel arcs nor loops; and is **symmetric** if for any vertices *u*, $v \in V(D)$, if $(u,v) \in A(D)$, then $(v,u) \in A(D)$. If two arcs of *D* have a common vertex, we say that these two arcs are adjacent in *D*. A directed path in a digraph *D* from a vertex *u* to a vertex *v* is called a (u,v)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred as a dicycle or dipath. A dipath *P* is a hamiltonian dipath if $V(P) = V(D)$. A digraph *D* is hamiltonian if *D* contains a hamiltonian dicycle. An (x,y)-hamiltonian dipath is a hamiltonian dipath from *x* to *y*. A digraph *D* is hamiltonian-connected if *D* has an (x,y)-hamiltonian dipath for every choice of distinct vertices $x, y \in V(D)$.

As in [2], $\lambda(D)$ denotes the arc-strong-connectivity of *D*. A digraph *D* is strong if and only if $\lambda(D) \ge 1$. For $X, Y \subseteq V(D)$, we define

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$$(X,Y)_D = \{(x,y) \in A(D) : x \in X \text{ and } y \in Y\}; \text{ and } \partial_D^+(X) = (X, V(D)-X)_D.$$

For a subset $A' \subseteq A(D)$, the subdigraph **arc-induced** by A' is the digraph $D[A'] = (V', A')$, where V' is the set of vertices in *V* which are incident with at least one arc in A'.

Let

$$d_D^+(X) = \left|\partial_D^+(X)\right|, \text{ and } d_D^-(X) = \left|\partial_D^-(X)\right|.$$

When $X = \{v\}$, we write $d_D^+(v) = \left|\partial_D^+\{v\}\right|$ and $d_D^-(v) = \left|\partial_D^-\{v\}\right|$. Let $N_D^+(v) = \{u \in V(D) - v : (v,u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) - v : (u,v) \in A(D)\}$ denote the **out-neighbourhood** and **in-neighbourhood** of *v* in *D*, respectively. Vertices in $N_D^+(v)$, $N_D^-(v)$ are called the **out-neighbours**, **in-neighbours** of *v*. Thus for a digraph *D* and an integer $k \ge 0$,

$$\lambda(D) \ge k \text{ if and only if for any } W \text{ with } \varnothing \ne W \subset V(D), \left|\partial_D^+(W)\right| \ge k. \quad (1)$$

Boesch, Suffel, and Tindell [3] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs. They indicated that this problem would be very difficult. Pulleyblank [4] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Catlin [5] in 1992 presented the first survey on supereulerian graphs. Chen *et al*. [6] surveyed the reduction method associated with the supereulerian problem and their applications. An updated survey presenting the more recent developments can be found in [7].

It is natural to consider the supereulerian problem in digraphs. A digraph *D* is **eulerian** if it contains a closed ditrail *W* such that $A(W) = A(D)$, or, equivalently, if *D* is strong and for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph *D* is **supereulerian** if *D* contains a closed ditrail *W* such that $V(W) = V(D)$, or, equivalently, if *D* contains a spanning eulerian subdigraph. Some recent developments on supereulerian digraphs are given in [8]-[12].

A central problem is to determine or characterize supereulerian digraphs. In Section 2, the **2-sum** $D_1 \oplus_2 D_2$ of two digraphs D_1 and D_2 is defined, and some basic properties of 2-sums are discussed. We will observe that a 2-sum of two supereulerian (or even hamiltonian) digraphs may not be supereulerian. Thus it is natural to seek sufficient conditions on D_1 and D_2 for the 2-sum of D_1 and D_2 to be supereulerian. In the last section, we will present several sufficient conditions for supereulerian 2-sums of digraphs. In particular, we show that if D_1 and D_2 are either symmetrically connected or partially symmetric (to be defined in Section 3), then $D_1 \oplus_2 D_2$ is supereulerian.

2. The 2-Sums of Digraphs

The definition and some elementary properties of the 2-sums of digraphs are presented in this section. A digraph is nontrivial if it contains at least one arc. Throughout this section, all digraphs are assumed to be nontrivial.

Definition 2.1 *Let* D_1 *and* D_2 *be two vertex disjoint digraphs, and let* $a_1 = (v_{11}, v_{12}) \in A(D_1)$ *and* $a_2 = (v_{21}, v_{22}) \in A(D_2)$ *be two distinguished arcs. The* **2-sum** $D_1 \oplus_{a_1,a_2} D_2$ *of* D_1 *and* D_2 *with base arcs* a_1 *and* a_2 *is obtained from the union of* D_1 *and* $D_2 - a_2$ *by identifying* v_{11} *with* v_{21} *and* v_{12} *with* v_{22}*, respectively. When the arcs* a_1 *and* a_2 *are not emphasized or is understood from the context, we often use* $D_1 \oplus_2 D_2$ *for* $D_1 \oplus_{a_1,a_2} D_2$.

Lemma 1 *Let* D_1 *and* D_2 *be two vertex disjoint strong digraphs. Then*

$$\lambda(D_1 \oplus_2 D_2) \ge \min\{\lambda(D_1), \lambda(D_2)\}.$$

Proof. Let $k \ge 0$ be an integer such that $\min\{\lambda(D_1), \lambda(D_2)\} = k$, and let $\lambda(D_1 \oplus_2 D_2) = k'$. We shall show that $k' \ge k$. By (1), there exists a proper nonempty vertex subset $X \subset V(D_1 \oplus_2 D_2)$ such that $\left|\partial_{D_1 \oplus_2 D_2}^+(X)\right| = k'$. Let $S = \partial_{D_1 \oplus_2 D_2}^+(X)$. We argue by contradiction and assume that $k' < k$.

By Definition 2.1, we have $v_{11} = v_{21} \in V(D_2)$ and $v_{12} = v_{22} \in V(D_2)$ in $D_1 \oplus_2 D_2$. If $X \cap V(D_1) \ne \varnothing$ and $X \cap V(D_2) = \varnothing$, we obtain that $v_{11} = v_{21} \notin X$ and $v_{12} = v_{22} \notin X$, then $X \subset V(D_1)$ and $S = \partial_{D_1}^+(X)$. It follows by (1) that $k' = |S| \ge \lambda(D_1) \ge k$, contrary to the assumption that $k' < k$. Similarly, if $X \cap V(D_1) = \varnothing$ and $X \cap V(D_2) \ne \varnothing$, then $X \subset V(D_2)$ and $S = \partial_{D_2}^+(X)$, hence a contradiction to the assumption that $k' < k$ is obtained from $k' = |S| \ge \lambda(D_2) \ge k$.

Thus, we may assume that $X \cap V(D_1) \neq \varnothing$ and $X \cap V(D_2) \neq \varnothing$. Let $X' = X \cap V(D_1)$. Then X' is a proper nonempty subset of $V(D_1)$, and $\partial^+_{D_1}(X') \subseteq S$. It follows by (1) that $k' = |S| \geq |\partial^+_{D_1}(X')| \geq \lambda(D_1) \geq k$ contrary to the assumption that $k' < k$.

Example 2.1 *The converse of Lemma* 1 *may not always stand, as indicated by the example below, depicted in* **Figure 1**. *Let* $V(D_1) = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ *and* $V(D_2) = \{v_{21}, v_{22}, v_{23}, v_{24}\}$. *Let* $A(D_1) = \{(v_{11}, v_{12}), (v_{13}, v_{12}), (v_{14}, v_{13}), (v_{11}, v_{14}), (v_{11}, v_{13}), (v_{14}, v_{12})\}$ *and* $A(D_2) = \{(v_{21}, v_{22}), (v_{22}, v_{23}), (v_{23}, v_{24}), (v_{24}, v_{21}), (v_{23}, v_{21}), (v_{24}, v_{22})\}$. *Let* $a_1 = (v_{11}, v_{12})$ *and* $a_2 = (v_{21}, v_{22})$. *Then, it is routine to verify that* $\lambda(D_1 \oplus_{a_1,a_2} D_2) \geq 1$. *While* D_2 *is strong, the digraph* D_1 *contains a vertex* v_{11} *with* $d^-_{D_1}(v_{11}) = 0$, *and so* $\lambda(D_1) = 0$.

Lemma 2 *A digraph D is not supereulerian if for some integer* $m > 0$, $V(D)$ *has vertex disjoint subsets* $\{B, B_1, \cdots, B_m\}$ *satisfying both of the following*:

i) $N^-_D(B_i) \subseteq B, \forall i \in \{1, 2, \cdots, m\}$.

ii) $|\partial^-_D(B)| \leq m - 1$.

Proof. By contradiction, we assume that both i) and ii) hold and D is supereulerian. Let S be a spanning eulerian subdigraph of D, then $B \subset V(S) = V(D)$ and $A(S) \subset A(D)$. Since S is eulerian, for any subset $X \subset V(S)$, it follows that $|\partial^+_S(X)| = |\partial^-_S(X)|$. Thus, by ii), we conclude that

$$|\partial^+_D(B) \cap A(S)| = |\partial^-_D(B) \cap A(S)| \leq |\partial^-_D(B)| \leq m - 1. \tag{2}$$

By i) and by (2), there must be a B_j with $j \in \{1, 2, \cdots, m\}$ such that $\partial^-_D(B_j) \cap A(S) = \varnothing$, contrary to the assumption that $V(S) = V(D)$.

Lemma 2 can be applied to find examples of hamiltonian digraphs whose 2-sum is not supereulerian, as shown in Example 2.2 below.

Example 2.2 *Let* $n_1, n_2 \geq 3$ *be integers and* C_{n_1} *and* C_{n_2} *be two vertex disjoint dicycles with length* n_1 *and* n_2, *respectively. We claim that* $C_{n_1} \oplus_2 C_{n_2}$ *is not supereulerian. To justify this claim, we denote* $V(C_{n_1}) = \{v_{11}, v_{12}, \cdots, v_{1n_1}\}$, *and* $V(C_{n_2}) = \{v_{21}, v_{22}, \cdots, v_{2n_2}\}$. *Without loss of generality, we assume that* $a_1 = (v_{11}, v_{12})$ *and* $a_2 = (v_{21}, v_{22})$, *and* $C_{n_1} \oplus_2 C_{n_2} = C_{n_1} \oplus_{a_1,a_2} C_{n_2}$. *Let* B, B_1 *and* B_2 *be subgraphs of* $C_{n_1} \oplus_2 C_{n_2}$ *with* $V(B) = \{v_{12}\}$, $V(B_1) = \{v_{13}\}$ *and* $V(B_2) = \{v_{23}\}$, *respectively. By Lemma* 2, *we conclude that* $C_{n_1} \oplus_2 C_{n_2}$ *is not supereulerian* (*see* **Figure 2**).

3. Sufficient Conditions for Supereulerian 2-Sums of Digraphs

In this section, we will show several sufficient conditions on D_1 and D_2 to assure that the 2-sum $D_1 \oplus_2 D_2$

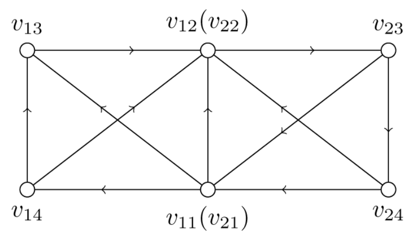

Figure 1. $\lambda(D_1 \oplus_2 D_2) = 1$ but $\min\{\lambda(D_1), \lambda(D_2)\} = 0$.

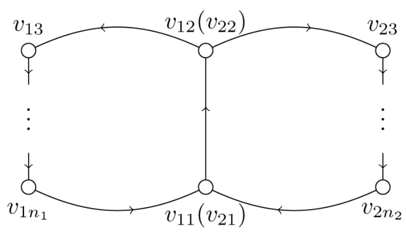

Figure 2. The 2-sum $C_{n_1} \oplus_2 C_{n_2}$ of C_{n_1} and C_{n_2}.

is supereulerian.

Proposition 1 *Let* D_1 *and* D_2 *be two vertex disjoint supereulerian digraphs with* $a_1=(v_{11},v_{12})\in A(D_1)$ *and* $a_2=(v_{21},v_{22})\in A(D_2)$, *and let* $D_1\oplus_2 D_2$ *denote* $D_1\oplus_{a_1,a_2} D_2$. *Each of the following holds.*

i) For some $i\in\{1,2\}$, if D_i has a spanning eulerian subdigraph S_i such that $a_i\notin A(S_i)$, then $D_1\oplus_2 D_2$ is supereulerian.

ii) If for some $i\in\{1,2\}$, D_i is hamiltonian-connected, then $D_1\oplus_2 D_2$ is supereulerian.

Proof. i) Since D_1 and D_2 are supereulerian digraphs, D_1 and D_2 are strongly connected, and so by Lemma 1, $D_1\oplus_2 D_2$ is also strongly connected. Without loss of generality, we assume that $i=1$ and D_1 has a spanning eulerian subdigraph S_1 such that $a_1\notin A(S_1)$. Since D_2 is supereulerian, we can pick a spanning eulerian subdigraph S_2' in D_2. Then $A(S_1)\cap A(S_2')=\varnothing$ and $V(S_1)\cap V(S_2')\neq\varnothing$. It follows that $D\left[A(S_1)\cup A(S_2')\right]$ is a spanning eulerian subdigraph in $D_1\oplus_2 D_2$.

ii) Without loss of generality, we assume that $i=1$ and D_1 is hamiltonian-connected, and so D_1 has a (v_{11},v_{12})-hamiltonian dipath T_1 and a (v_{12},v_{11})-hamiltonian dipath T_2. Since D_2 is supereulerian, D_2 contains a spanning eulerian subdigraph S_2'. Define

$$S=\begin{cases} D\left[A(T_1)\cup A\left(S_2'-\{(v_{21},v_{22})\}\right)\right] & \text{if } (v_{21},v_{22})\in A(S_2') \\ D\left[\left(A(T_2)\cup\{(v_{11},v_{12})\}\right)\cup A(S_2')\right] & \text{if } (v_{21},v_{22})\notin A(S_2') \end{cases}.$$

As in any case, S is strongly connected and every vertex $v\in V(S)$ satisfies $d_S^+(v)=d_S^-(v)$, and so S is eulerian. Since $V(S)=V(T_i)\cup V(S_2')=V(D_1)\cup V(D_2)$, for $i\in\{1,2\}$, we conclude that S is a spanning eulerian subdigraph of $D_1\oplus_2 D_2$, and so $D_1\oplus_2 D_2$ is supereulerian.

Theorem 2 [13] *If a strict digraph on* $n\geq 3$ *vertices has* $(n-1)^2+1$ *or more arcs, then it is hamiltonian-connected.*

Corollary 1 *Let* D_1 *be a strict digraph on* $n_1\geq 3$ *vertices and with* $|A(D_1)|\geq(n_1-1)^2+1$. *If* D_2 *is a supereulerian digraph, then* $D_1\oplus_2 D_2$ *is supereulerian.*

Proof. By Theorem 2, D_1 is hamiltonian-connected. Then by Proposition 1 (ii), $D_1\oplus_2 D_2$ is supereulerian.

Two classes of supereulerian digraphs seem to be of particular interests in studying supereulerian digraph 2-sums. We first present their definitions.

Definition 3.2 *Let D be a digraph such that either* $D=K_1$ *or* $A(D)\neq\varnothing$. *If for any* $u,v\in V(D)$, *D contains a symmetric dipath from u to v, then D is called a* ***symmetrically connected*** *digraph.*

Given a digraph *D*, define a relation ~ on $V(D)$ such that $u\sim v$ if and only if $u=v$ or *D* has a symmetrically connected subdigraph *H* with $u,v\in V(H)$. By definition, one can routinely verify that ~ is an equivalence relation. Each equivalence class induces a symmetrically connected component of *D*. Hence *D* is symmetrically connected if and only if *D* has only one symmetrically connected component. A symmetrically connected component of *D* is also called a maximal symmetrically connected subdigraph of *D*. When *D* has more than one symmetrically connected components, we have the following definition.

Definition 3.3 *Let D be a weakly connected digraph and* $\{H_1,H_2,\cdots,H_c\}$ *be the set of maximal symmetrically connected subdigraphs of D with* $c\geq 2$. *If for any proper nonempty subset* $\mathcal{J}\subset\{H_1,H_2,\cdots,H_c\}$,

$$\begin{aligned}&\text{there exist an } H_i\in\mathcal{J}\text{, a vertex } v\in V(H_i)\text{, and an } H_j\notin\mathcal{J}\text{ such that}\\ &N_D^+(v)\cap V(H_j)\neq\varnothing \text{ and } N_D^-(v)\cap V(H_j)\neq\varnothing,\end{aligned}\tag{3}$$

then *D* is **partially symmetric**.

It is known that both symmetrically connected digraphs and partially symmetric digraphs are supereulerian.

Theorem 3 ([14] and [15]) *Each of the following holds.*

i) Every symmetrically connected digraph is supereulerian.

ii) Every partially symmetric digraph is supereulerian.

A main result of this section is to show that the digraph 2-sums of symmetrically connected or partially symmetric digraphs are supereulerian.

Lemma 3 *Let* D_1 *and* D_2 *be two vertex disjoint digraphs with* $a_1=(v_{11},v_{12})\in A(D_1)$ *and* $a_2=(v_{21},v_{22})\in A(D_2)$, *and let* $D_1\oplus_2 D_2$ *denote* $D_1\oplus_{a_1,a_2} D_2$. *Each of the following holds.*

i) If D_1 and D_2 are symmetrically connected, then $D_1\oplus_2 D_2$ is symmetrically connected.

ii) If D_1 and D_2 are partially symmetric, then $D_1 \oplus_2 D_2$ is partially symmetric.

iii) If D_1 is symmetric and D_2 is partially symmetric, then $D_1 \oplus_2 D_2$ is partially symmetric.

Proof. i) For any vertices $x, y \in V(D_1 \oplus_2 D_2)$, we shall show that $D_1 \oplus_2 D_2$ always has a symmetric (x, y)-dipath. If for some $i \in \{1,2\}$, we have $x, y \in V(D_i)$, then as D_i is symmetrically connected, D_i contains a symmetric (x, y)-dipath *P*. Since D_i is a subdigraph of $D_1 \oplus_2 D_2$, *P* is also a symmetric (x, y)-dipath of $D_1 \oplus_2 D_2$. Hence we may assume that $x \in V(D_1)$ and $y \in V(D_2)$. Since D_1 and D_2 are symmetrically connected, D_1 contains a symmetric (x, v_{11})-dipath P_1 and D_2 contains a symmetric (v_{21}, y)-dipath P_2. By Definition 2.1, v_{11} and v_{21} represent the same vertex in $D_1 \oplus_2 D_2$, and so $D_1 \oplus_2 D_2[A(P_1) \cup A(P_2)]$ is a symmetric (x, y)-dipath in $D_1 \oplus_2 D_2$.

ii) Fix $i \in \{1,2\}$. Since D_i is partially symmetric, for some integer $c_i > 1$, let $\{H'_{i1}, H'_{i2}, \cdots, H'_{ic_i}\}$ be the set of all maximal symmetrically connected subdigraphs of D_i. Without loss of generality, we assume that $v_{11} \in V(H'_{11})$ and $v_{21} \in V(H'_{21})$; and for some s, t with $1 \le s \le c_1$ and $1 \le t \le c_2$, $v_{12} \in V(H'_{1s})$ and $v_{22} \in V(H'_{2t})$. (We allow the possibility that $s = 1$ and/or $t = 1$). Define, for $1 \le h \le c_1$ and $1 \le j \le c_2$,

$$H_{1h} = \begin{cases} H'_{1h} & \text{if } h \notin \{1, s\} \\ H'_{11} \cup H'_{21} & \text{if } h = 1 \\ H'_{1s} \cup H'_{2t} & \text{if } h = s \end{cases} \quad \text{and} \quad H_{2j} = \begin{cases} H'_{2j} & \text{if } j \notin \{1, t\} \\ H'_{11} \cup H'_{21} & \text{if } j = 1 \\ H'_{1s} \cup H'_{2t} & \text{if } j = t \end{cases}.$$

Then, $\mathcal{H} = \{H_{11}, H_{12}, \cdots, H_{1c_1}, H_{21}, H_{22}, \cdots, H_{2c_2}\}$ is the set of all maximal symmetrically connected subdigraphs of $D_1 \oplus_2 D_2$. Note that $H_{11} = H_{21}$ and $H_{1s} = H_{2t}$. We shall show by definition that $D_1 \oplus_2 D_2$ is partially symmetric. To do that, let $\mathcal{J}$ be a nonempty proper subset of $\mathcal{H}$. We shall show that (3) holds.

Since $\mathcal{H} = \{H_{11}, H_{12}, \cdots, H_{1c_1}, H_{21}, H_{22}, \cdots, H_{2c_2}\}$, we either have $\mathcal{J} \cap \{H_{11}, H_{12}, \cdots, H_{1c_1}\} \ne \emptyset$ or $\mathcal{J} \cap \{H_{21}, H_{22}, \cdots, H_{2c_2}\} \ne \emptyset$. By symmetry, we may assume that $\mathcal{J} \cap \{H_{11}, H_{12}, \cdots, H_{1c_1}\} \ne \emptyset$.

Suppose first that $\{H_{11}, H_{12}, \cdots, H_{1c_1}\} - \mathcal{J} \ne \emptyset$. Let $\mathcal{J}' = \{H'_{1h} \mid H_{1h} \in \mathcal{J}\}$. Then $\{H'_{11}, H'_{12}, \cdots, H'_{1c_1}\} - \mathcal{J}' \ne \emptyset$. Since D_1 is partially symmetric, there exist an $H'_{1h_0} \in \mathcal{J}'$, a vertex $v \in V(H'_{1h_0})$, and an $H'_{1j_0} \in \{H'_{11}, H'_{12}, \cdots, H'_{1c_1}\} - \mathcal{J}'$ such that

$$N^+_{D_1}(v) \cap V(H'_{1j_0}) \ne \emptyset \text{ and } N^-_{D_1}(v) \cap V(H'_{1j_0}) \ne \emptyset.$$

This implies that the vertex $v \in V(H_{1h_0})$, $H_{1h_0} \in \mathcal{J}$, and $H_{1j_0} \notin \mathcal{J}$ such that

$$N^+_{D_1 \oplus_2 D_2}(v) \cap V(H_{1j_0}) \ne \emptyset \text{ and } N^-_{D_1 \oplus_2 D_2}(v) \cap V(H_{1j_0}) \ne \emptyset.$$

Thus (3) holds in this case.

Hence we may assume that $\{H_{11}, H_{12}, \cdots, H_{1c_1}\} \subset \mathcal{J}$. Since $\mathcal{J}$ is a proper subset, we must have $\{H_{21}, H_{22}, \cdots, H_{2c_2}\} - \mathcal{J} \ne \emptyset$. Since $H_{21} = H_{11} \in \mathcal{J}$, we also have $\{H_{21}, H_{22}, \cdots, H_{2c_2}\} \cap \mathcal{J} \ne \emptyset$. With a similar argument, we conclude that (3) must also hold in this case.

iii) Let $H_0 = D_1$ and let $\{H'_1, H'_2, \cdots, H'_c\}$ be the set of all maximal symmetrically connected subdigraphs of D_2 with $v_{21} \in V(H'_1)$ and for some $j \in \{1, 2, \cdots, c\}$, $v_{22} \in V(H'_j)$. (We allow the possibility that $j = 1$). Define

$$H_i = \begin{cases} H'_1 \cup H_0 \cup H'_j & \text{if } i = 1 \text{ or } i = j \\ H'_i & \text{if } i \notin \{1, j\} \end{cases}.$$

Then $\mathcal{H} = \{H_1, H_2, \cdots, H_c\}$ is the set of all maximal symmetrically connected subdigraphs of $D_1 \oplus_2 D_2$. Note that $H_1 = H_j$ with this notation. Let $\mathcal{J}$ be a nonempty proper subset of $\mathcal{H}$. We shall show that (3) holds.

Let $J' = \{H'_i \mid H_i \in \mathcal{J}\}$. Since $\mathcal{J}$ is proper, $\mathcal{J}'$ is a nonempty proper subset of $\{H'_1, H'_2, \cdots, H'_c\}$. Since D_2 is partially symmetric, by Definition 3.2, there exist an $H'_{i_0} \in \mathcal{J}'$, a vertex $v \in V(H'_{i_0})$, and an $H'_{j_0} \in \{H'_1, H'_2, \cdots, H'_c\} - \mathcal{J}'$ such that

$$N^+_{D_1}(v) \cap V(H'_{j_0}) \ne \emptyset \text{ and } N^-_{D_1}(v) \cap V(H'_{j_0}) \ne \emptyset.$$

This implies that vertex $v \in V\left(H_{i_0}\right)$, $H_{i_0} \in \mathcal{J}$ and $H_{j_0} \notin \mathcal{J}$ such that

$$N^{+}_{D_1 \oplus_2 D_2}(v) \cap V\left(H_{j_0}\right) \neq \varnothing \text{ and } N^{-}_{D_1 \oplus_2 D_2}(v) \cap V\left(H_{j_0}\right) \neq \varnothing.$$

Thus (3) holds, and so by definition, $D_1 \oplus_2 D_2$ is partially symmetric.

Theorem 4 *Let* D_1 *and* D_2 *be two digraphs. Each of the following holds.*

i) If D_1 and D_2 are symmetrically connected, then $D_1 \oplus_2 D_2$ is supereulerian.

ii) If D_1 and D_2 are partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.

iii) If D_1 is symmetric and D_2 is partially symmetric, then $D_1 \oplus_2 D_2$ is supereulerian.

Proof. This follows from Theorem 3 and Lemma 3.

It is also natural to consider sufficient conditions on D_1 and D_2 for $D_1 \oplus_2 D_2$ to be hamiltonian.

Theorem 5 *If* D_1 *is hamiltonian and* D_2 *is hamiltonian-connected digraphs, then* $D_1 \oplus_2 D_2$ *is hamiltonian.*

Proof. Let $V(D_1) = \{v_{11}, v_{12}, \cdots, v_{1n_1}\}$ with $C = v_{11}v_{12}\cdots v_{1n_1}v_{11}$ be a hamiltonian dicycle of D_1 and $V(D_2) = \{v_{21}, v_{22}, \cdots, v_{2n_2}\}$. Let $a_1 = (v_{11}, v_{12}) \in A(D_1)$ and $a_2 = (v_{21}, v_{22}) \in A(D_2)$, and $D_1 \oplus_2 D_2 = D_1 \oplus_{a_1,a_2} D_2$. Since D_2 is hamiltonian-connected, D_2 contains a (v_{21}, v_{22})-hamiltonian dipath *P*. Thus $\left(C - \{a_1\}\right) \cup P$ is a hamiltonian dicycle in $D_1 \oplus_2 D_2$.

Theorem 6 (Thomassen [16]) *If a semicomplete digraph D is* 4*-strong, then D is hamiltonian-connected.*

By Theorem 5 and 6, we have the following corollary.

Corollary 2 *Let* D_1 *and* D_2 *be two* 4*-strong semicomplete digraphs, then* $D_1 \oplus_2 D_2$ *is hamiltonian.*

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