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Reconstruction of Three Dimensional Convex Bodies from the Curvatures of Their Shadows

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Abstract

In this article, we study necessary and sufficient conditions for a function, defined on the space of *flags* to be the projection curvature radius function for a convex body. This type of inverse problems has been studied by Christoffel, Minkwoski for the case of mean and Gauss curvatures. We suggest an algorithm of reconstruction of a convex body from its projection curvature radius function by finding a representation for the support function of the body. We lead the problem to a system of differential equations of second order on the sphere and solve it applying *a consistency method* suggested by the author of the article.

Keywords

Integral Geometry, Convex Body, Projection Curvature, Support Function

1. Introduction

The problem of reconstruction of a convex body from the mean and Gauss curvatures of the boundary of the body goes back to Christoffel and Minkwoski [1]. Let F be a function defined on 2-dimensional unit sphere S^2. The following problems have been studied by E. B. Christoffel: what are necessary and sufficient conditions for F to be the mean curvature radius function for a convex body. The corresponding problem for Gauss curvature is considered by H. Minkovski [1]. W. Blaschke [2] provides a formula for reconstruction of a convex body **B** from the mean curvatures of its boundary. The formula is written in terms of spherical harmonics.

A. D. Aleksandrov and A. V. Pogorelov generalize these problems for a class of symmetric functions $G(R_1, R_2)$ of principal radii of curvatures (see [3]-[5]).

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Let $\mathbf{B} \subset \mathbf{R}^n$ be a convex body with sufficiently smooth boundary and let $R_1(\omega), \cdots, R_{n-1}(\omega)$ signify the principal radii of curvature of the boundary of **B** at the point with outer normal direction $\omega \in \mathrm{S}^{n-1}$. In *n*-dimensional case, a Christoffel-Minkovski problem is posed and solved by Firay [6] and Berg [7] (see also [8]): what are necessary and sufficient conditions for a function *F*, defined on S^{n-1} to be function $\sum R_{i_1}(\omega) \cdots R_{i_p}(\omega)$ for a convex body, where $1 \le p \le n-1$ and the sum is extended over all increasing sequences $i_1, \cdots, i_p$ of indices chosen from the set $i = 1, \cdots, n-1$.

R. Gardner and P. Milanfar [9] provide an algorithm for reconstruction of an origin-symmetric convex body K from the volumes of its projections.

D. Ryabogin and A. Zvavich [10] reconstruct a convex body of revolution from the areas of its shadows by giving a precise formula for the support function.

In this paper, we consider a similar problem posed for the projection curvature radius function of convex bodies. We lead the problem to a system of differential equations of second order on the sphere and solve it applying *a consistency method* suggested by the author of the article. The solution of the system of differential equations is itself interesting.

Let $\mathbf{B} \subset \mathbf{R}^3$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of the boundary $\partial\mathbf{B}$. We need some notations.

S^2—the unit sphere in $\mathbf{R}^3$, $\mathrm{S}_\omega \subset \mathrm{S}^2$—the great circle with pole at $\omega \in \mathrm{S}^2$, $\mathbf{B}(\omega)$—projection of **B** onto the plane containing the origin in $\mathbf{R}^3$ and orthogonal to ω, $R(\omega^\perp, \varphi)$—curvature radius of $\partial\mathbf{B}(\omega)$ at the point with outer normal direction $\varphi \in \mathrm{S}_\omega$ and call projection curvature radius of **B**.

Let F be a positive continuously differentiable function defined on the space of "flags" $\mathcal{F} = \{(\omega, \varphi) : \omega \in \mathrm{S}^2, \varphi \in \mathrm{S}_\omega\}$. In this article, we consider:

Problem 1. What are necessary and sufficient conditions for *F* to be the projection curvature radius function $R(\omega^\perp, \varphi)$ for a convex body?

Problem 2. Reconstruction of that convex body by giving a precise formula for the support function.

Note that one can lead the problem of reconstruction of a convex body by projection curvatures using representation of the support function in terms of mean curvature radius function (see [7]). The approach of the present article is useful for practical point of view, because one can calculate curvatures of projections from the shadows of a convex body. Let's note that it is impossible to calculate mean radius of curvature from the limited number of shadows of a convex body. Also let's note that this is a different approach for such problems, because in the present article we lead the problem to a differential equation of spatial type on the sphere and solve it using a new method (so called consistency method).

The most useful analytic description of compact convex sets is by the support function (see [11]). The support function of **B** is defined as

$$H(x) = \sup_{y \in \mathbf{B}} \langle y, x \rangle, \; x \in \mathbf{R}^3.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbf{R}^3$. The support function of **B** is positively homogeneous and convex. Below, we consider the support function *H* of a convex body as a function on S^2 (because of the positive homogeneity of *H* the values on S^2 determine *H* completely).

$C^k(\mathrm{S}^2)$ denotes the space of *k* times continuously differentiable functions defined on S^2. A convex body **B** is *k*-smooth if its support function $H \in C^k(\mathrm{S}^2)$.

Given a function *H* defined on S^2, by $H_\omega(\varphi)$, $\varphi \in \mathrm{S}_\omega$ we denote the restriction of *H* onto the circle S_ω for $\omega \in \mathrm{S}^2$, and call the restriction function of *H*.

Below, we show (Theorem 1) that Problem 1. is equivalent to the problem of existence of a function *H* defined on S^2 such that $H_\omega(\cdot)$ satisfies the differential equation

$$H_\omega(\varphi) + \left[H_\omega(\varphi)\right]''_{\varphi\varphi} = F(\omega, \varphi), \text{ for } \varphi \in \mathrm{S}_\omega \tag{1}$$

for every $\omega \in \mathrm{S}^2$.

Definition 1. If for a given *F* there exists *H* defined on S^2 that satisfies Equation (1), then *H* is called a solution of Equation (1).

In Equation (1), $H_\omega(\varphi)$ is a function defined on the space of an ordered pair orthogonal unit vectors, say e_1, e_2, (in integral geometry such a pair is a flag and the concept of a flag was first systematically employed by

R.V. Ambartzumian in [12]).

There are two equivalent representations of an ordered pair orthogonal unit vectors e_1, e_2, dual each other:

$$(\omega,\varphi) \quad \text{and} \quad (\Omega,\Phi), \tag{2}$$

where $\omega \in \mathrm{S}^2$ is the spatial direction of the first vector e_1, and φ is the planar direction in S_ω coincides with the direction of e_2, while $\Omega \in \mathrm{S}^2$ is the spatial direction of the second vector e_2, and Φ is the planar direction in S_Ω coincides with the direction of e_1. The second representation we will write by capital letters.

Given a flag function $g(\omega,\varphi)$, we denote by g^* the image of g defined by

$$g^*(\Omega,\Phi) = g(\omega,\varphi), \tag{3}$$

where $(\omega,\varphi)^* = (\Omega,\Phi)$ (dual each other).

Let G be a function defined on $\mathcal{F}$. For every $\omega \in \mathrm{S}^2$, Equation (1) reduces to a differential equation on the circle S_ω.

Definition 2. If $G(\omega,\cdot)$ is a solution of that equation for every $\omega \in \mathrm{S}^2$, then G is called *a flag solution* of Equation (1).

Definition 3. If a flag solution $G(\omega,\varphi)$ satisfies

$$G^*(\Omega,\Phi) = G^*(\Omega) \tag{4}$$

(no dependence on the variable Φ), then G is called *a consistent flag solution*.

There is an important principle: *each consistent flag solution G of Equation* (1) *produces a solution of Equation* (1) *via the map*

$$G(\omega,\varphi) \to G^*(\Omega,\Phi) = G^*(\Omega) = H(\Omega), \tag{5}$$

and vice versa: the restriction functions of any solution of Equation (1) onto the great circles is a consistent flag solution.

Hence, the problem of finding a solution reduces to finding a consistent flag solution.

To solve the latter problem, the present paper applies *the consistency method* first used in [13]-[15] in an integral equations context.

We denote: $e[\Omega,\Phi]$—the plane containing the origin of $\mathbf{R}^3$, direction $\Omega \in \mathrm{S}^2$, Φ determines rotation of the plane around Ω, $\mathbf{B}[\Omega,\Phi]$—projection of $\mathbf{B} \in \mathcal{B}$ onto the plane $e[\Omega,\Phi]$, $R^*(\Omega,\Phi)$—curvature radius of $\partial\mathbf{B}[\Omega,\Phi]$ at the point with outer normal direction $\Omega \in \mathrm{S}^2$. It is easy to see that

$$R^*(\Omega,\Phi) = R(\omega^\perp,\varphi),$$

where (Ω,Φ) is dual to (ω,φ).

Note that in the Problem 1. uniqueness (up to a translation) follows from the classical uniqueness result on Christoffel problem, since

$$R_1(\Omega) + R_2(\Omega) = \frac{1}{\pi}\int_0^{2\pi} R^*(\Omega,\Phi)\,\mathrm{d}\Phi. \tag{6}$$

Equation (1) has the following geometrical interpretation.

It is known (see [11]) that 2 times continuously differentiable homogeneous function H defined on $\mathbf{R}^3$, is convex if and only if

$$H_\omega(\varphi) + \left[H_\omega(\varphi)\right]''_{\varphi\varphi} \geq 0 \text{ for every } \omega \in \mathrm{S}^2 \text{ and } \varphi \in S_\omega, \tag{7}$$

where $H_\omega(\cdot)$ is the restriction of H onto S_ω.

So in case $F > 0$, it follows from (7), that if H is a solution of Equation (1) then its homogeneous extension is convex.

It is known from convexity theory that if a homogeneous function H is convex then there is a unique convex body $\mathbf{B} \subset \mathbf{R}^3$ with support function H and $F(\omega,\varphi)$ is the projection curvature radius function of $\mathbf{B}$ (see [11]).

The support function of each parallel shifts (translation) of that body $\mathbf{B}$ will again be a solution of Equation (1). By uniqueness, every two solutions of Equation (1) differ by a summand $\langle a,\cdot\rangle$ defined on S^2, where

$a\in\mathbf{R}^3$. Thus we have the following theorem.

Theorem 1 *Let F be a positive function defined on* $\mathcal{F}$. *If Equation* (1) *has a solution H then there exists a convex body* **B** *with projection curvature radius function F, whose support function is H. Every solution of Equation* (1) *has the form* $H(\cdot)+\langle a,\cdot\rangle$, *where* $a\in\mathbf{R}^3$, *being the support function of the convex body* $\mathbf{B}+a$.

The converse statement is also true. The support function *H* of a 2-smooth convex body **B** satisfies Equation (1) for $F=R$, where *R* is the projection curvature radius function of **B** (see [16]).

The purpose of the present paper is to find a necessary and sufficient condition that ensures a positive answer to both Problems 1,2 and suggest an algorithm of construction of the body **B** by finding a representation of the support function in terms of projection curvature radius function. This happens to be a solution of Equation (1).

Throughout the paper (in particular, in Theorem 2 that follows) we use usual spherical coordinates ν,τ for points S^2 based on a choice of a North Pole and a reference point $\tau=0$ on the equator. The point with coordinates ν,τ we will denote by (ν,τ), the points $(0,\tau)$ lie on the equator. On S_ω we choose anticlockwise direction as positive. On the plane $\omega^\perp$ containing S_ω we consider the Cartesian x and y-axes where the direction of the y-axis $\vec{y}$ is taken to be the projection of the North Pole onto $\omega^\perp$. The direction of the x-axis $\boldsymbol{x}$ we take as the reference direction on S_ω and call it the East direction. Now we describe the main result.

Theorem 2 *Let* **B** *be a* 3-*smooth convex body with positive Gaussian curvature at every point of* $\partial\mathbf{B}$ *and R is the projection curvature radius function of* **B**. *Then for* $\Omega\in\mathrm{S}^2$ *chosen as the North pole*

$$\begin{aligned}H(\Omega)=&\frac{1}{4\pi}\int_0^{2\pi}\left[\int_0^{\frac{\pi}{2}}R\left((0,\tau)^\perp,\varphi\right)\cos\varphi\,\mathrm{d}\varphi\right]\mathrm{d}\tau\\&+\frac{1}{8\pi^2}\int_0^{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}R\left((0,\tau)^\perp,\varphi\right)\left((\pi+2\varphi)\cos\varphi-2\sin^3\varphi\right)\mathrm{d}\varphi\right]\mathrm{d}\tau\\&-\frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}}\frac{\sin\nu}{\cos^2\nu}\mathrm{d}\nu\int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}R\left((\nu,\tau)^\perp,\varphi\right)\sin^3\varphi\,\mathrm{d}\varphi\end{aligned}\tag{8}$$

is a solution of Equation (1) for $F=R$. On S_ω we measure φ from the East direction.

Remark, that the order of integration in the last integral of (8) cannot be changed.

Obviously Theorem 2 suggests a practical algorithm of reconstruction of convex body from projection curvature radius function *R* by calculation of support function *H*.

We turn to Problem 1. Let *R* be the projection curvature radius function of a convex body **B**. Then $F\equiv R$ necessarily satisfies the following conditions:

a) For every $\omega\in\mathrm{S}^2$ and any reference point on S_ω

$$\int_0^{2\pi}F(\omega,\varphi)\sin\varphi\,\mathrm{d}\varphi=\int_0^{2\pi}F(\omega,\varphi)\cos\varphi\,\mathrm{d}\varphi=0.\tag{9}$$

This follows from Equation (1), see also [16].

b) For every direction $\Omega\in\mathrm{S}^2$ chosen as the North pole

$$\int_0^{2\pi}\left[F^*\left((\nu,\tau),\boldsymbol{y}\right)\right]'_{\nu=0}\mathrm{d}\tau=0,\tag{10}$$

where the function F^* is the image of *F* (see (3)) and $\boldsymbol{y}$ is the direction of the y-axis on $(\nu,\tau)^\perp$ (Theorem 5).

Let *F* be a positive 2 times differentiable function defined on $\mathcal{F}$. Using (8), we construct a function $\bar{F}$ defined on S^2:

$$\begin{aligned}\bar{F}(\Omega)=&\frac{1}{4\pi}\int_0^{2\pi}\left[\int_0^{\frac{\pi}{2}}F\left((0,\tau),\varphi\right)\cos\varphi\,\mathrm{d}\varphi\right]\mathrm{d}\tau\\&+\frac{1}{8\pi^2}\int_0^{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}F\left((0,\tau),\varphi\right)\left((\pi+2\varphi)\cos\varphi-2\sin^3\varphi\right)\mathrm{d}\varphi\right]\mathrm{d}\tau\\&-\frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}}\frac{\sin\nu}{\cos^2\nu}\mathrm{d}\nu\int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}F\left((\nu,\tau),\varphi\right)\sin^3\varphi\,\mathrm{d}\varphi\end{aligned}\tag{11}$$

Note that the last integral converges if the condition (10) is satisfied.

Theorem 3 *A positive* 2 *times differentiable function F defined on* $\mathcal{F}$ *represents the projection curvature radius function of some convex body* **B** *if and only if F satisfies the conditions* (9), (10) *and the extension* (*to* $\mathbf{R}^3$) *of the function F defined by* (11) *is convex.*

2. The Consistency Condition

We fix $\omega \in S^2$ and try to solve Equation (1) as a differential equation of second order on the circle S_ω. We start with two results from [16].

a) For any smooth convex domain *D* in the plane

$$h(\varphi) = \int_0^{\varphi} R(\psi)\sin(\varphi-\psi)\,\mathrm{d}\psi, \tag{12}$$

where $h(\varphi)$ is the support function of *D* with respect to a point $s \in \partial D$. In (12) we measure φ from the normal direction at *s*, $R(\psi)$ is the curvature radius of ∂D at the point with normal direction ψ.

b) (12) is a solution of the following differential equation

$$R(\varphi) = h(\varphi) + h''(\varphi). \tag{13}$$

One can easy verify that (also it follows from (13) and (12))

$$G(\omega,\varphi) = \int_0^{\varphi} F(\omega,\psi)\sin(\varphi-\psi)\,\mathrm{d}\psi, \tag{14}$$

is a flag solution of Equation (1).

Theorem 4 *Every flag solution of Equation* (1) *has the form*

$$g(\omega,\varphi) = \int_0^{\varphi} F(\omega,\psi)\sin(\varphi-\psi)\,\mathrm{d}\psi + C(\omega)\cos\varphi + S(\omega)\sin\varphi \tag{15}$$

where C_n and S_n are some real coefficients.

Proof of Theorem 4. *Every continuous flag solution of Equation* (1) *is a sum of* $G + g_0$, *where* g_0 *is a flag solution of the corresponding homogeneous equation:*

$$H_\omega(\varphi) + \left[H_\omega(\varphi)\right]''_{\varphi\varphi} = 0,\ \ \varphi \in S_\omega, \tag{16}$$

for every $\omega \in S^2$. We look for the general flag solution of Equation (16) in the form of a Fourier series

$$g_0(\omega,\varphi) = \sum_{n=0,1,2,\cdots} \left[C_n(\omega)\cos n\varphi + S_n(\omega)\sin n\varphi\right]. \tag{17}$$

After substitution of (17) into (16) we obtain that $g_0(\omega,\varphi)$ satisfies (16) if and only if

$$g_0(\omega,\varphi) = C_1(\omega)\cos\varphi + S_1(\omega)\sin\varphi.$$

Now we try to find functions *C* and *S* in (15) from the condition that *g* satisfies (4). We write $g(\omega,\varphi)$ in dual coordinates *i.e.* $g(\omega,\varphi) = g^*(\Omega,\Phi)$ and require that $g^*(\Omega,\Phi)$ should not depend on Φ for every $\Omega \in S^2$, *i.e.* for every $\Omega \in S^2$

$$\left(g^*(\Omega,\Phi)\right)'_\Phi = \left(G(\omega,\varphi) + C(\omega)\cos\varphi + S(\omega)\sin\varphi\right)'_\Phi = 0, \tag{18}$$

where $G(\omega,\varphi)$ was defined in (14).

Here and below $(\cdot)'_\Phi$ denotes the derivative corresponding to right screw rotation around Ω. Differentiation with use of expressions (see [14])

$$\tau'_\Phi = \frac{\sin\varphi}{\cos\nu},\ \varphi'_\Phi = -\tan\nu\sin\varphi,\ \nu'_\Phi = -\cos\varphi, \tag{19}$$

after a natural grouping of the summands in (18), yields the Fourier series of $-\left(G(\omega,\varphi)\right)'_\Phi$. By uniqueness of the Fourier coefficients

$$\left(C(\omega)\right)'_{\nu}+\frac{\left(S(\omega)\right)'_{\tau}}{\cos\nu}+\tan\nu C(\omega)=\frac{1}{\pi}\int_0^{2\pi}A(\omega,\varphi)\cos 2\varphi \mathrm{d}\varphi$$

$$\left(C(\omega)\right)'_{\nu}-\frac{\left(S(\omega)\right)'_{\tau}}{\cos\nu}-\tan\nu C(\omega)=\frac{1}{2\pi}\int_0^{2\pi}A(\omega,\varphi)\mathrm{d}\varphi$$

$$\left(S(\omega)\right)'_{\nu}-\frac{\left(C(\omega)\right)'_{\tau}}{\cos\nu}+\tan\nu S(\omega)=\frac{1}{\pi}\int_0^{2\pi}A(\omega,\varphi)\sin 2\varphi \mathrm{d}\varphi, \tag{20}$$

where

$$A(\omega,\varphi)=\int_0^{\varphi}\left[F(\omega,\psi)'_{\Phi}\sin(\varphi-\psi)+F(\omega,\psi)\cos(\varphi-\psi)\varphi'_{\Phi}\right]\mathrm{d}\psi. \tag{21}$$

3. Averaging

Let H be a solution of Equation (1), *i.e.* restriction of H onto the great circles is a consistent flag solution of Equation (1). By Theorem 1 there exists a convex body $\mathbf{B}\in\mathcal{B}$ with projection curvature radius function $R=F$, whose support function is H.

To calculate $H(\Omega)$ for a $\Omega\in \mathrm{S}^2$ we take Ω for the North Pole of S^2. Returning to the Formula (15) for every $\omega=(0,\tau)\in \mathrm{S}_{\Omega}$ we have

$$H(\Omega)=\int_0^{\frac{\pi}{2}}R(\omega^{\perp},\psi)\sin\left(\frac{\pi}{2}-\psi\right)\mathrm{d}\psi+S(\omega), \tag{22}$$

We integrate both sides of (22) with respect to uniform angular measure $\mathrm{d}\tau$ over $[0,2\pi)$ to get

$$2\pi H(\Omega)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R\left((0,\tau)^{\perp},\psi\right)\cos\psi \mathrm{d}\psi \mathrm{d}\tau+\int_0^{2\pi}S\left((0,\tau)\right)\mathrm{d}\tau. \tag{23}$$

Now the problem is to calculate

$$\int_0^{2\pi}S\left((0,\tau)\right)\mathrm{d}\tau=\bar{S}(0). \tag{24}$$

We are going to integrate both sides of (20) and (21) with respect to $\mathrm{d}\tau$ over $[0,2\pi)$. For $\omega=(\nu,\tau)$, where $\nu\in\left[0,\frac{\pi}{2}\right)$ and $\tau\in(0,2\pi)$ we denote

$$\bar{S}(\nu)=\int_0^{2\pi}S\left((\nu,\tau)\right)\mathrm{d}\tau, \tag{25}$$

$$\pi A(\nu)=\int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}\left[\int_0^{\varphi}\left[R(\omega^{\perp},\psi)'_{\Phi}\sin(\varphi-\psi)+R(\omega^{\perp},\psi)\cos(\varphi-\psi)\varphi'_{\Phi}\right]\mathrm{d}\psi\right]\sin 2\varphi \mathrm{d}\varphi. \tag{26}$$

Integrating both sides of (20) and (21) and taking into account that

$$\int_0^{2\pi}\left(C(\nu,\tau)\right)'_{\tau}\mathrm{d}\tau=0$$

for $\nu\in[0,\pi/2)$ we get

$$\overline{S'}(\nu)+\tan\nu\bar{S}(\nu)=A(\nu), \tag{27}$$

i.e. a differential equation for the unknown coefficient $\bar{S}(\nu)$.

We have to find $\bar{S}(0)$ given by (24). It follows from (27) that

$$\left(\frac{\bar{S}(\nu)}{\cos\nu}\right)=\frac{A(\nu)}{\cos\nu}. \tag{28}$$

Integrating both sides of (5.1) with respect to $\mathrm{d}\nu$ over $[0,\pi/2)$ we obtain

$$\bar{S}(0)=\frac{\bar{S}(\nu)}{\cos\nu}\bigg|_{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{A(\nu)}{\cos\nu}\mathrm{d}\nu. \tag{29}$$

Now, we are going to calculate $\frac{\bar{S}(\nu)}{\cos\nu}\bigg|_{\frac{\pi}{2}}$.

It follows from (15) that

$$\begin{aligned}\pi\bar{S}(\nu)&=\int_0^{2\pi}\int_0^{2\pi}\left[H_\omega(\varphi)-\int_0^{\varphi}R(\omega^{\perp},\psi)\sin(\varphi-\psi)\mathrm{d}\psi\right]\sin\varphi\mathrm{d}\varphi\mathrm{d}\tau\\&=\int_0^{2\pi}\int_0^{2\pi}H_\omega(\varphi)\sin\varphi\mathrm{d}\varphi\mathrm{d}\tau-\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}R(\omega^{\perp},\psi)\big((2\pi-\psi)\cos\psi+\sin\psi\big)\mathrm{d}\psi\mathrm{d}\tau.\end{aligned} \tag{30}$$

Let $\varphi\in S_\omega$ be the direction that corresponds to $\varphi\in[0,2\pi)$, for $\omega=(\nu,\tau)$. As a point of S^2, let φ have spherical coordinates u,t with respect to Ω. By the sinus theorem of spherical geometry

$$\cos\nu\sin\varphi=\sin u. \tag{31}$$

From (31), we get

$$(u)'_{\nu=\frac{\pi}{2}}=-\sin\varphi. \tag{32}$$

Fixing τ and using (32) we write a Taylor formula at a neighborhood of the point $\nu=\pi/2$:

$$H_{(\nu,\tau)}(\varphi)=H\big((0,\varphi+\tau)\big)+H'_\nu\big((0,\varphi+\tau)\big)\sin\varphi\left(\frac{\pi}{2}-\nu\right)+o\left(\frac{\pi}{2}-\nu\right). \tag{33}$$

Similarly, for $\psi\in[0,2\pi)$ we get

$$\begin{aligned}R\left((\nu,\tau)^{\perp},\psi\right)&=R\left(\left(\frac{\pi}{2},\tau\right)^{\perp},\psi+\tau\right)\\&\quad+R'_\nu\left(\left(\frac{\pi}{2},\tau\right)^{\perp},\psi+\tau\right)\sin\psi\left(\frac{\pi}{2}-\nu\right)+o\left(\frac{\pi}{2}-\nu\right).\end{aligned} \tag{34}$$

Substituting (33) and (34) into (30) and taking into account the easily establish equalities

$$\int_0^{2\pi}\int_0^{2\pi}H\big((0,\varphi+\tau)\big)\sin\varphi\mathrm{d}\varphi\mathrm{d}\tau=0$$

and

$$\int_0^{2\pi}\int_0^{2\pi}R\left(\left(\frac{\pi}{2},\tau\right)^{\perp},\psi+\tau\right)\big((2\pi-\psi)\cos\psi+\sin\psi\big)\mathrm{d}\psi\mathrm{d}\tau=0 \tag{35}$$

we obtain

$$\begin{aligned}\lim_{\nu\to\frac{\pi}{2}}\frac{\bar{S}(\nu)}{\cos\nu}&=\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}H'_\nu\big((0,\varphi+\tau)\big)\sin^2\varphi\mathrm{d}\varphi\mathrm{d}\tau\\&\quad-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}R'_\nu\left(\left(\frac{\pi}{2},\tau\right)^{\perp},\psi+\tau\right)\sin\psi\big((2\pi-\psi)\cos\psi+\sin\psi\big)\mathrm{d}\psi\mathrm{d}\tau\\&=\int_0^{2\pi}H'_\nu\big((0,\tau)\big)\mathrm{d}\tau-\frac{3}{4}\int_0^{2\pi}\left[R^*\big((\nu,\tau),\mathbf{y}\big)\right]'_{\nu=0}\mathrm{d}\tau.\end{aligned} \tag{36}$$

Theorem 5 *For every* 3*-smooth convex body* $\mathbf{B}\in\mathcal{B}$ *and any direction* $\Omega\in S^2$*, we have*

$$\int_0^{2\pi}\left[R^*\left((\nu,\tau),\boldsymbol{y}\right)\right]'_{\nu=0}\mathrm{d}\tau=0, \tag{37}$$

where $\boldsymbol{y}$ is the direction of the y-axis on $(\nu,\tau)^{\perp}$.

Proof of Theorem 5. *Using spherical geometry, one can prove that* (*see also* (1))

$$\begin{aligned}\left[R^*\left((\nu,\tau),\boldsymbol{y}\right)\right]'_{\nu=0}&=\left[H\left((\nu,\tau)\right)+H''_{\varphi\varphi}\left((\nu,\tau)\right)\right]_{\nu=0}\\&=\left[H\left((\nu,\tau)\right)+H''_{\tau\tau}\frac{1}{\cos^2\nu}-H'_{\nu}\tan\nu\right]_{\nu=0}\\&=\left[H''_{\tau\tau}\right]'_{\nu=0},\end{aligned} \tag{38}$$

where H is the support function of $\mathbf{B}$. Integrating (38), we get

$$\int_0^{2\pi}\left[R^*\left((\nu,\tau),\boldsymbol{y}\right)\right]'_{\nu=0}\mathrm{d}\tau=\int_0^{2\pi}\left[H''_{\tau\tau}\right]'_{\nu=0}\mathrm{d}\tau=0.$$

4. A Representation for Support Functions of Convex Bodies

Let $\mathbf{B}\in\mathcal{B}$ be a convex body and $Q\in\mathbf{R}^3$. By H_Q we denote the support function of $\mathbf{B}$ with respect to Q.

Theorem 6 *Given a* 2-*smooth convex body* $\mathbf{B}\in\mathcal{B}$, *there exists a point* $O^*\in\mathbf{R}^3$ *such that for every* $\Omega\in S^2$ *chosen as the North pole*

$$\int_0^{2\pi}\left[H_{O^*}\left((\nu,\tau)\right)\right]_{\nu=0}\mathrm{d}\tau=0. \tag{39}$$

Proof of Theorem 6. *For a given* $\mathbf{B}$ *and a point* $Q\in\mathbf{R}^3$, *by* K_Q *we denote the following function defined on* S^2

$$K_Q(\Omega)=\int_0^{2\pi}\left[H_Q\left((\nu,\tau)\right)\right]_{\nu=0}\mathrm{d}\tau.$$

Clearly, K_Q is a continuous odd function with maximum $\bar{K}(Q)$:

$$\bar{K}(Q)=\max_{\Omega\in S^2}K_Q(\Omega).$$

It is easy to see that $\bar{K}(Q)\to\infty$ for $|Q|\to\infty$. Since $\bar{K}(Q)$ is continuous, so there is a point O^* for which

$$\bar{K}\left(O^*\right)=\min\bar{K}(Q).$$

Let Ω^* be a direction of maximum now assumed to be unique, *i.e.*

$$\bar{K}\left(O^*\right)=\max_{\Omega\in S^2}K_{O^*}(\Omega)=K_{O^*}\left(\Omega^*\right).$$

If $\bar{K}\left(O^*\right)=0$ the theorem is proved. For the case $\bar{K}\left(O^*\right)=a>0$ let O^{**} be the point for which $\boldsymbol{O}^*\boldsymbol{O}^{**}=\varepsilon\Omega^*$. It is easy to demonstrate that $H_{O^{**}}(\Omega)=H_{O^*}(\Omega)-\varepsilon\left(\Omega,\Omega^*\right)$, hence for a small $\varepsilon>0$ we find that $\bar{K}\left(O^{**}\right)=a-2\pi\varepsilon$, contrary to the definition of O^*. So $\bar{K}\left(O^*\right)=0$. For the case where there are two or more directions of maximum one can apply a similar argument.

Now we take the point O^* of the convex body $\mathbf{B}$ for the origin of $\mathbf{R}^3$. Below H_{O^*}, we will simply denote by H.

By Theorem 6 and Theorem 5, we have the boundary condition (see (36))

$$\left.\frac{\bar{S}(\nu)}{\cos\nu}\right|_{\frac{\pi}{2}}=0. \tag{40}$$

Substituting (29) into (23) we get

$$2\pi H(\Omega)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R\left((0,\tau)^{\perp},\psi\right)\cos\psi\,\mathrm{d}\psi\,\mathrm{d}\tau-\int_0^{\frac{\pi}{2}}\frac{A(\nu)}{\cos\nu}\mathrm{d}\nu$$
$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R\left((0,\tau)^{\perp},\psi\right)\cos\psi\,\mathrm{d}\psi\,\mathrm{d}\tau-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\nu}{\cos\nu} \tag{41}$$
$$\times\int_0^{2\pi}\int_0^{2\pi}\left[\int_0^{\varphi}\left[R\left(\omega^{\perp},\psi\right)'_{\Phi}\sin(\varphi-\psi)+R\left(\omega^{\perp},\psi\right)\cos(\varphi-\psi)\varphi'_{\Phi}\right]\mathrm{d}\psi\right]\sin 2\varphi\,\mathrm{d}\varphi\,\mathrm{d}\tau.$$

Using expressions (19) and integrating by $\mathrm{d}\varphi$ yields

$$2\pi H(\Omega)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R\left((0,\tau)^{\perp},\psi\right)\cos\psi\,\mathrm{d}\psi\,\mathrm{d}\tau$$
$$+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\nu}{\cos\nu}\int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}\left[R\left(\omega^{\perp},\psi\right)'_{\nu}I+R\left(\omega^{\perp},\psi\right)\tan\nu\,II\right]\mathrm{d}\psi, \tag{42}$$

where

$$II=\int_{\psi}^{2\pi}\sin 2\varphi\cos(\varphi-\psi)\sin\varphi\,\mathrm{d}\varphi=\left[\frac{(2\pi-\psi)\cos\psi}{4}+\frac{\sin\psi\left(1+\sin^2\psi\right)}{4}-\sin^3\psi\right],$$

and

$$I=\int_{\psi}^{2\pi}\sin 2\varphi\sin(\varphi-\psi)\cos\varphi\,\mathrm{d}\varphi=\left[\frac{(2\pi-\psi)\cos\psi}{4}+\frac{\sin\psi\left(1+\sin^2\psi\right)}{4}\right].$$

Integrating by parts (42) we get

$$2\pi H(\Omega)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R\left((0,\tau)^{\perp},\psi\right)\cos\psi\,\mathrm{d}\psi\,\mathrm{d}\tau-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{d}\nu\int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}R\left(\omega^{\perp},\psi\right)\frac{\sin\nu\sin^3\psi}{\cos^2\nu}\mathrm{d}\psi$$
$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{d}\tau\int_0^{2\pi}R\left((0,\tau)^{\perp},\psi\right)I\,\mathrm{d}\psi+\lim_{a\to\frac{\pi}{2}}\frac{1}{\pi\cos a}\int_0^{\frac{\pi}{2}}\mathrm{d}\tau\int_0^{2\pi}R\left((a,\tau)^{\perp},\psi\right)I\mathrm{d}\psi. \tag{43}$$

Using (34), Theorem 5 and taking into account that

$$\int_0^{2\pi}I\,\mathrm{d}\psi=0$$

we get

$$2\pi H(\Omega)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R\left((0,\tau)^{\perp},\psi\right)\cos\psi\,\mathrm{d}\psi\,\mathrm{d}\tau-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{d}\nu\int_0^{2\pi}\mathrm{d}\tau\int_0^{2\pi}R\left(\omega^{\perp},\psi\right)\frac{\sin\nu\sin^3\psi}{\cos^2\nu}\mathrm{d}\psi$$
$$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{d}\tau\int_0^{2\pi}R\left((0,\tau)^{\perp},\psi\right)I\,\mathrm{d}\psi. \tag{44}$$

From (44), using (9) we obtain (8). Theorem 2 is proved.

5. Proof of Theorem 3

Necessity: if F is the projection curvature radius function of a convex body $\mathbf{B}\in\mathcal{B}$, then it satisfies (9) (see [16]), the condition (10) (Theorem 5) and F defined by (11) is convex since it is the support function of $\mathbf{B}$ (Theorem 2).

Sufficiency: let F be a positive 2 times differentiable function defined on $\mathcal{F}$ satisfies the conditions (9), (10). We construct the function F on S^2 defined by (11). There exists a convex body $\mathbf{B}$ with support function F since its extension is a convex function. Also Theorem 2 implies that F is the projection curvature radius of $\mathbf{B}$.

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