



On a Nonlinear Volterra-Fredholm Integrodifferential Equation on Time Scales

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Abstract

The main aim in this work is to obtain an integral inequality with a clear estimate on time scales. The obtained inequality is used as a tool to investigate some basic qualitative properties of solutions to certain nonlinear Volterra-Fredholm integrodifferential equations on time scales.

Subject Areas

Mathematical Analysis

Keywords

Integrodifferential Equations, Time Scales, Integral Inequality, Estimate on the Solutions

1. Introduction

The theory of time scales had been begun in 1988 by Stefan Hilger [1], in order to develop a theory that can standardize a continuous and discrete analysis. Recently several authors in this field have investigated various forms of integral and integrodifferential equations under different hypotheses by using different ways, see [4] [5] [6] [7] [8]. In this article we consider the nonlinear integrodifferential equation of the following form

$$y^\Delta(t) = h\left(t, y(t), y^\Delta(t), \int_\alpha^t h_1(t, z, y(z), y^\Delta(z)) \Delta z, \int_\alpha^\beta h_2(t, z, y(z), y^\Delta(z)) \Delta z\right), \quad (1.1)$$

$t \in J_{\mathbb{T}}$ with the initial condition $y(\alpha) = y_0$,

where y is unknown function and $h : J_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h_1, h_2 : J_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and h, h_1, h_2 are given functions, assuming them to be rd-continuous functions, $\alpha < \beta, z \leq t$ and $J_{\mathbb{T}} = J \cap \mathbb{T}, J = [\alpha, \infty)$. We denote a time scale by \mathbb{T} which is nonempty closed subset of \mathbb{R} . \mathbb{R}^n denotes

Euclidean space with a suitable norm defined by $|.|$.

We can investigate the existence and uniqueness results for (1.1) by using the technique present in [6].

2. Preliminaries

The operators $\sigma(t)$ and $\rho(t)$ denote the forward and backward operators respectively which are defined by $\sigma(t)=\inf\{s \in \mathbb{T}: s > t\} \in \mathbb{T}$ and $\rho(t)=\sup\{s \in \mathbb{T}: s < t\} \in \mathbb{T}$, for all $t \in \mathbb{T}$.

For $t \in \mathbb{T}$, If $t < \sup \mathbb{T}$ and $\sigma(t)=t$, then t is said to be right-dense; while If $t > \inf \mathbb{T}$ and $\rho(t)=t$, then t is said to be left-dense. The graininess $\mu: \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t)=\sigma(t)-t$. The set \mathbb{T}^k is denoted by

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{otherwise} \end{cases}$$

Let $z: \mathbb{T} \rightarrow \mathbb{R}$, $t \in \mathbb{T}^k$, then $z^\Delta(t)$ denotes the delta derivative of z at t which is exist with the property that given $\varepsilon > 0$ there is a neighbourhood U of t such that $|z(\sigma(t), \tau) - z(s, \tau) - z^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$ for all $s \in U$. Then $g(t) = \int_\alpha^t z(\tau, \Delta \tau)$ implies $g^\Delta(t) = \int_\alpha^t z^\Delta(\tau, \Delta \tau) + z(\sigma(t), t)$. If a function $g: \mathbb{T} \rightarrow \mathbb{R}$ is continuous at any right-dense point $t \in \mathbb{T}$ and the left-hand limits exists (finite) at any left-dense point $t \in \mathbb{T}$, then g is said to be rd-continuous. C_{rd} denotes the class of all rd-continuous functions. We denote the class of all regressive functions by \mathcal{R} which is defined by

$$\mathcal{R} = \{p \in C_{rd}(\mathbb{T}, \mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0, \forall t \in \mathbb{T}\}$$

For $p \in \mathcal{R}$, we define $e_p(t, s) = \exp\left(\int_s^t \xi_p(\tau)(p(\tau)) \Delta \tau\right)$ for $t, s \in \mathbb{T}$, with

$$\text{the cylinder transformation } \xi_h(\tau) = \begin{cases} \frac{\log(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0 \end{cases}.$$

For more basic information about time scales calculus, see [1] [3].

We need the following result given in [2].

Lemma 2.1. suppose $v, b \in C_{rd}$ and $a \in \mathcal{R}^+$. Then

$$v^\Delta(t) \leq a(t)v(t) + b(t), \text{ for all } t \in \mathbb{T}$$

Implies $v(t) \leq v(\alpha)e_a(t, \alpha) + \int_\alpha^t e_a(t, \sigma(\tau))b(\tau) \Delta \tau$, for all $t \in \mathbb{T}$.

3. Main Results

In the following result we establish an integral inequality on time scales.

Theorem 3.1. Let $v, r, b_1, b_2, p, q, g, d \in C_{rd}(J_{\mathbb{T}}, \mathbb{R}^+)$ and assume that

$$\begin{aligned} v(t) \leq r(t) + b_1(t) \int_\alpha^t [b_2(\tau)p(\tau) + 1]v(\tau) + b_2(\tau) \int_\alpha^\tau p(z)v(z) \Delta z \\ + q(\tau) \int_\alpha^\beta g(z)v(z) \Delta z \Big\} \Delta \tau + d(t) \int_\alpha^\beta g(z)v(z) \Delta z, \quad t \in J_{\mathbb{T}}, \end{aligned} \tag{3.1}$$

If

$$N = \int_{\alpha}^{\beta} g(\gamma) N_2(\gamma) \Delta \gamma < 1, \quad (3.2)$$

Implies

$$\nu(t) \leq N_1(t) + AN_2(t), t \in J_{\mathbb{T}}, \quad (3.3)$$

where

$$\begin{aligned} N_1(t) &= r(t) + b_1(t) \int_{\alpha}^t [b_2^{\sigma}(\tau)p(\tau) + 1] \left[r(\tau) + b(\tau) \int_{\alpha}^{\tau} e_{(b_2^{\sigma} p + p+1)b}(\tau, \sigma(z)) \right. \\ &\quad \times \left. (b_2^{\sigma}(z)p(z) + p(z) + 1)r(z)\Delta z \right] \Delta \tau, \end{aligned} \quad (3.4)$$

$$\begin{aligned} N_2(t) &= d(t) + b_1(t) \int_{\alpha}^t [b_2^{\sigma}(\tau)p(\tau) + 1] \left[d(\tau) + b(\tau) \int_{\alpha}^{\tau} e_{(b_2^{\sigma} p + p+1)b}(\tau, \sigma(z)) \right. \\ &\quad \times \left. (b_2^{\sigma}(z)p(z) + p(z) + 1)d(z) + q(z) \right] \Delta z + q(\tau) \Big\} \Delta \tau, \end{aligned} \quad (3.5)$$

for $t \in J_{\mathbb{T}}$,

$$b_2^{\sigma}(t) = b_2(\sigma(t)) = b_2 \circ \sigma, b(t) = \max_{t \in J_{\mathbb{T}}} \{b_1(t), b_2(t) + b_2^{\Delta}(t)\}, \quad (3.6)$$

$$A = \frac{1}{1-N} \int_{\alpha}^{\beta} g(\gamma) N_1(\gamma) \Delta \gamma, \quad (3.7)$$

Proof. Let

$$\lambda = \int_{\alpha}^{\beta} g(z) \nu(z) \Delta z, \quad (3.8)$$

we shall define functions $B_1(t)$ and $B_2(t)$ by

$$\begin{aligned} B_1(t) &= \int_{\alpha}^t [b_2(t)p(\tau) + 1]\nu(\tau) + b_2(\tau) \int_{\alpha}^{\tau} p(z)\nu(z) \Delta z \\ &\quad + q(\tau) \int_{\alpha}^{\beta} g(z)\nu(z) \Delta z \Big\} \Delta \tau, \end{aligned} \quad (3.9)$$

$$B_2(t) = B_1(t) + \int_{\alpha}^t p(z) [r(z) + b_1(z)B_1(z) + d(z)\lambda] \Delta z, \quad (3.10)$$

then $B_1(\alpha) = 0$, $B_2(\alpha) = 0$, $B_1(t) \leq B_2(t)$ and we have

$$\nu(t) \leq r(t) + b_1(t)B_1(t) + d(t)\lambda, \quad (3.11)$$

from (3.9), we get

$$\begin{aligned} B_1^{\Delta}(t) &= \int_{\alpha}^t b_2^{\Delta}(t)p(\tau)\nu(\tau) \Delta \tau + [b_2^{\sigma}(t)p(t) + 1]\nu(t) \\ &\quad + b_2(t) \int_{\alpha}^t p(z)\nu(z) \Delta z + q(t) \int_{\alpha}^{\beta} g(z)\nu(z) \Delta z \\ &\leq [b_2^{\sigma}(t)p(t) + 1]r(t) + [b_2^{\sigma}(t)p(t) + 1]b_1(t)B_1(t) \\ &\quad + [b_2^{\sigma}(t)p(t) + 1]d(t)\lambda + [b_2(t) + b_2^{\Delta}(t)] \int_{\alpha}^t p(z)\nu(z) \Delta z + q(t)\lambda \\ &\leq [b_2^{\sigma}(t)p(t) + 1]r(t) + b_2^{\sigma}(t)p(t)b_1(t)B_1(t) \\ &\quad + b(t)[B_1(t) + \int_{\alpha}^t p(z)\nu(z) \Delta z] + [b_2^{\sigma}(t)p(t) + 1]d(t)\lambda + q(t)\lambda \\ &\leq [b_2^{\sigma}(t)p(t) + 1][r(t) + b(t)B_2(t) + d(t)\lambda] + q(t)\lambda, \end{aligned} \quad (3.12)$$

integrating the inequality (3.12) and using $B_1(\alpha) = 0$, we have

$$B_1(t) \leq \int_{\alpha}^t \left\{ [b_2^\sigma(\tau)p(\tau) + 1] [r(\tau) + b(\tau)B_2(\tau) + d(\tau)\lambda] + q(\tau)\lambda \right\} \Delta\tau, \quad (3.13)$$

therefore

$$\begin{aligned} B_2^\Delta(t) &= B_1^\Delta(t) + p(t)[r(t) + b_1(t)B_1(t) + d(t)\lambda] \\ &\leq [b_2^\sigma(t)p(t) + 1][r(t) + b(t)B_2(t) + d(t)\lambda] + q(t)\lambda \\ &\quad + p(t)[r(t) + b(t)B_2(t) + d(t)\lambda] \\ &= [b_2^\sigma(t)p(t) + p(t) + 1]b(t)B_2(t) \\ &\quad + [b_2^\sigma(t)p(t) + p(t) + 1][r(t) + d(t)\lambda] + q(t)\lambda, \end{aligned} \quad (3.14)$$

now applying lemma 2.1, we get

$$\begin{aligned} B_2(t) &\leq \int_{\alpha}^t e_{(b_2^\sigma p + p + 1)b}(t, \sigma(z)) \\ &\quad \times ([b_2^\sigma(z)p(z) + p(z) + 1][a(z) + d(z)\lambda] + q(z)\lambda) \Delta z, \end{aligned} \quad (3.15)$$

from (3.11), (3.13) and (3.15), we obtain that

$$v(t) \leq N_1(t) + \lambda N_2(t), \quad (3.16)$$

and from (3.8) and (3.16) we observe that

$$\lambda \leq A, \quad (3.17)$$

using (3.17) in (3.16) we obtain (3.3). \square

We provide the result that includes the estimate on the solutions of (1.1) as follows.

Theorem 3.2. Assume that the following conditions satisfied

$$|h(t, v_1, v_2, v_3, v_4)| \leq L(|v_1| + |v_2| + |v_3| + |v_4|), \quad (3.18)$$

$$|h_1(t, z, u, v)| \leq c_1(t)s_1(z)(|u| + |v|), \quad (3.19)$$

$$|h_2(t, z, u, v)| \leq c_2(t)s_2(z)(|u| + |v|), \quad (3.20)$$

for the functions h, h_1, h_2 in (1.1), where $0 \leq L < 1$ is a constant and

$$c_1, s_1, c_2, s_2 \in C_{rd}(J_{\mathbb{T}}, \mathbb{R}^+)$$

If $y(t)$ is a solution of (1.1) on $J_{\mathbb{T}}$, then

$$|y(t)| + |y^\Delta(t)| \leq M_1(t) + D_1 M_2(t), t \in J_{\mathbb{T}}, \quad (3.21)$$

where

$$\begin{aligned} M_1(t) &= \frac{|y_0|}{1-L} + \frac{L}{1-L} \int_{\alpha}^t \left[c_1^\sigma(\tau)s_1(\tau) + 1 \right] \left[\frac{|y_0|}{1-L} + b(\tau) \int_{\alpha}^{\tau} e_{(c_1^\sigma s_1 + s_1 + 1)b}(\tau, \sigma(z)) \right. \\ &\quad \times \left. (c_1^\sigma(z)s_1(z) + s_1(z) + 1) \frac{|y_0|}{1-L} \Delta z \right] \Delta \tau, t \in J_{\mathbb{T}} \end{aligned} \quad (3.22)$$

$$\begin{aligned} M_2(t) &= \frac{L}{1-L} c_2(t) \\ &\quad + \frac{L}{1-L} \int_{\alpha}^t \left[c_1^\sigma(\tau)s_1(\tau) + 1 \right] \left[\frac{L}{1-L} c_2(\tau) + b(\tau) \int_{\alpha}^{\tau} e_{(c_1^\sigma s_1 + s_1 + 1)b}(\tau, \sigma(z)) \right. \\ &\quad \times \left. (c_1^\sigma(z)s_1(z) + s_1(z) + 1) \frac{L}{1-L} c_2(z) + c_2(z) \right] \Delta z + c_2(\tau) \Big\} \Delta \tau, t \in J_{\mathbb{T}} \end{aligned} \quad (3.23)$$

Assume that

$$b(t) = \max_{t \in J_{\mathbb{T}}} \left\{ \frac{L}{1-L}, c_1(t) + c_1^{\Delta}(t) \right\}, \quad (3.24)$$

$$\lambda = \int_{\alpha}^{\beta} s_2(\gamma) M_2(\gamma) \Delta \gamma < 1, \quad (3.25)$$

$$D_1 = \frac{1}{1-\lambda} \int_{\alpha}^{\beta} s_2(\gamma) M_1(\gamma) \Delta \gamma, \quad (3.26)$$

Proof. Let $a(t) = |y(t)| + |y^{\Delta}(t)|, t \in J_{\mathbb{T}}$, since $y(t)$ is a solution of (1.1), then by using this and the hypotheses, we get

$$\begin{aligned} a(t) &= \left| y_0 + \int_{\alpha}^t h\left(\tau, y(\tau), y^{\Delta}(\tau), \int_{\alpha}^{\tau} h_1(\tau, z, y(z), y^{\Delta}(z)) \Delta z, \right. \right. \\ &\quad \left. \left. \int_{\alpha}^{\beta} h_2(\tau, z, y(z), y^{\Delta}(z)) \Delta z \right) \Delta \tau \right| \\ &\quad + \left| h\left(t, y(t), y^{\Delta}(t), \int_{\alpha}^t h_1(t, z, y(z), y^{\Delta}(z)) \Delta z, \int_{\alpha}^{\beta} h_2(t, z, y(z), y^{\Delta}(z)) \Delta z \right) \right| \\ &\leq |y_0| + \int_{\alpha}^t L \left[a(\tau) + \int_{\alpha}^{\tau} c_1(\tau) s_1(z) a(z) \Delta z + \int_{\alpha}^{\beta} c_2(\tau) s_2(z) a(z) \Delta z \right] \Delta \tau \\ &\quad + L \left[a(t) + \int_{\alpha}^t c_1(t) s_1(z) a(z) \Delta z + \int_{\alpha}^{\beta} c_2(t) s_2(z) a(z) \Delta z \right] \end{aligned}$$

from the above inequality, we have

$$\begin{aligned} a(t) &\leq \frac{|y_0|}{1-L} + \frac{L}{1-L} \left\{ \left[c_1(t) s_1(\tau) + 1 \right] a(\tau) + c_1(\tau) \int_{\alpha}^{\tau} s_1(z) a(z) \Delta z \right. \\ &\quad \left. + c_2(\tau) \int_{\alpha}^{\beta} s_2(z) a(z) \Delta z \right\} \Delta \tau + \frac{L}{1-L} c_2(t) \int_{\alpha}^{\beta} s_2(z) a(z) \Delta z, \end{aligned} \quad (3.27)$$

Now applying theorem 3.1 in (3.27) we obtain (3.21). \square

Remark 3.3. Since $y(t)$ is a solution of (1.1). Then (3.21) yields the bounds on $y(t)$ and $y^{\Delta}(t)$. If the estimate in (3.21) is bounded, implies the solution $y(t)$ and $y^{\Delta}(t)$ are also bounded on $J_{\mathbb{T}}$.

Consider (1.1) with the following corresponding equation

$$\begin{aligned} Y^{\Delta}(t) &= H \left(t, Y(t), Y^{\Delta}(t), \int_{\alpha}^t h_1(t, z, Y(z), Y^{\Delta}(z)) \Delta z, \right. \\ &\quad \left. \int_{\alpha}^{\beta} h_2(t, z, Y(z), Y^{\Delta}(z)) \Delta z \right), t \in J_{\mathbb{T}}, \end{aligned}$$

with the initial condition

$$Y(\alpha) = Y_0 \quad (3.28)$$

where $H \in C_{rd}(J_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, h_1, h_2 as in (1.1).

The next result concerning the closeness of solution of (1.1) and (3.28).

Theorem 3.4. Suppose that the following conditions satisfied

$$\begin{aligned} &|h(t, v_1, v_2, v_3, v_4) - h(t, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)| \\ &\leq L [|v_1 - \bar{v}_1| + |v_2 - \bar{v}_2| + |v_3 - \bar{v}_3| + |v_4 - \bar{v}_4|], \end{aligned} \quad (3.29)$$

$$|h_1(t, z, u, v) - h_1(t, z, \bar{u}, \bar{v})| \leq c_1(t) s_1(z) [|u - \bar{u}| + |v - \bar{v}|], \quad (3.30)$$

$$|h_2(t, z, u, v) - h_2(t, z, \bar{u}, \bar{v})| \leq c_2(t) s_2(z) [|u - \bar{u}| + |v - \bar{v}|], \quad (3.31)$$

where the functions h, h_1, h_2 in (1.1), and $0 \leq L < 1$ is a constant.

Also $c_1, s_1, c_2, s_2 \in C_{rd}(J_{\mathbb{T}}, \mathbb{R}^+)$, and

$$|h(t, v_1, v_2, v_3, v_4) - H(t, v_1, v_2, v_3, v_4)| \leq \varepsilon, \quad (3.32)$$

$$|y_0 - Y_0| \leq \delta, \quad (3.33)$$

where y_0 and H, Y_0 as in (1.1) and (3.28) respectively.

If $y(t)$ and $Y(t)$ be solutions of (1.1) and (3.28) on $J_{\mathbb{T}}$, then

$$|y(t) - Y(t)| + |y^\Delta(t) - Y^\Delta(t)| \leq M_3(t) + D_2 M_2(t), t \in J_{\mathbb{T}}, \quad (3.34)$$

where $M_3(t)$ is described by the right side of (3.22) by substituting

$m(t) = \delta + \varepsilon(1+t-\alpha)$ instead of $|y_0|$, $M_2(t), b(t)$ and λ be as in (3.23), (3.24) and (3.25) respectively and

$$D_2 = \frac{1}{1-\lambda} \int_{\alpha}^{\beta} r_2(\gamma) M_3(\gamma) \Delta \gamma, \quad (3.35)$$

Proof. Let $w(t) = |y(t) - Y(t)| + |y^\Delta(t) - Y^\Delta(t)|, t \in J_{\mathbb{T}}$, we have

$$\begin{aligned} w(t) &\leq |y_0 - Y_0| \\ &+ \left| h\left(\tau, y(\tau), y^\Delta(\tau), \int_{\alpha}^{\tau} h_1(\tau, z, y(z), y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(\tau, z, y(z), y^\Delta(z)) \Delta z \right) \right. \\ &- h\left(\tau, Y(\tau), Y^\Delta(\tau), \int_{\alpha}^{\tau} h_1(\tau, z, Y(z), Y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(\tau, z, Y(z), Y^\Delta(z)) \Delta z \right) \Big| \Delta \tau \\ &+ \left| h\left(\tau, Y(\tau), Y^\Delta(\tau), \int_{\alpha}^{\tau} h_1(\tau, z, Y(z), Y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(\tau, z, Y(z), Y^\Delta(z)) \Delta z \right) \right. \\ &- H\left(\tau, Y(\tau), Y^\Delta(\tau), \int_{\alpha}^{\tau} h_1(\tau, z, Y(z), Y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(\tau, z, Y(z), Y^\Delta(z)) \Delta z \right) \Big| \Delta \tau \\ &+ \left| h\left(t, y(t), y^\Delta(t), \int_{\alpha}^t h_1(t, z, y(z), y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(t, z, y(z), y^\Delta(z)) \Delta z \right) \right. \\ &- h\left(t, Y(t), Y^\Delta(t), \int_{\alpha}^t h_1(t, z, Y(z), Y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(t, z, Y(z), Y^\Delta(z)) \Delta z \right) \Big| \\ &+ \left| h\left(t, Y(t), Y^\Delta(t), \int_{\alpha}^t h_1(t, z, Y(z), Y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(t, z, Y(z), Y^\Delta(z)) \Delta z \right) \right. \\ &- H\left(t, Y(t), Y^\Delta(t), \int_{\alpha}^t h_1(t, z, Y(z), Y^\Delta(z)) \Delta z, \int_{\alpha}^{\beta} h_2(t, z, Y(z), Y^\Delta(z)) \Delta z \right) \Big| \\ &\leq \delta + \int_{\alpha}^t L \left[w(\tau) + \int_{\alpha}^{\tau} c_1(\tau) s_1(z) w(z) \Delta z + \int_{\alpha}^{\beta} c_2(\tau) s_2(z) w(z) \Delta z \right] \Delta \tau \\ &+ \int_{\alpha}^t \varepsilon \Delta \tau + L \left[w(t) + \int_{\alpha}^t c_1(t) s_1(z) w(z) \Delta z + \int_{\alpha}^{\beta} c_2(t) s_2(z) w(z) \Delta z \right] + \varepsilon \\ &= m(t) + L \int_{\alpha}^t \left[w(\tau) + c_1(\tau) \int_{\alpha}^{\tau} s_1(z) w(z) \Delta z + c_2(\tau) \int_{\alpha}^{\beta} s_2(z) w(z) \Delta z \right] \Delta \tau \\ &+ L \left[w(t) + \int_{\alpha}^t c_1(t) s_1(z) w(z) \Delta z + c_2(t) \int_{\alpha}^{\beta} s_2(z) w(z) \Delta z \right] \end{aligned}$$

then we get

$$\begin{aligned} w(t) &\leq \frac{m(t)}{1-L} + \frac{L}{1-L} \int_{\alpha}^t \left\{ [c_1(t) s_1(\tau) + 1] w(\tau) + c_1(\tau) \int_{\alpha}^{\tau} s_1(z) w(z) \Delta z \right. \\ &\quad \left. + c_2(\tau) \int_{\alpha}^{\beta} s_2(z) w(z) \Delta z \right\} \Delta \tau + \frac{L}{1-L} c_2(t) \int_{\alpha}^{\beta} s_2(z) w(z) \Delta z, \end{aligned} \quad (3.36)$$

Now applying theorem 3.1, yields (3.34). \square

The following theorem provide the continuous depends of solutions of (1.1) on given initial values.

Theorem 3.5. Assume that the conditions (3.29), (3.30) and (3.31) are satisfied for the functions h, h_1, h_2 in (1.1). Let $y_1(t)$ and $y_2(t)$ be the solutions of equation

$$\begin{aligned} y^\Delta(t) = & h\left(t, y(t), y^\Delta(t), \int_\alpha^t h_1(t, z, y(z), y^\Delta(z)) \Delta z, \right. \\ & \left. \int_\alpha^\beta h_2(t, z, y(z), y^\Delta(z)) \Delta z\right), t \in J_{\mathbb{T}}, \end{aligned}$$

with the given initial values

$$y_1(\alpha) = c_1 \quad \text{and} \quad y_2(\alpha) = c_2, \quad (3.37)$$

where h, h_1, h_2 as in (1.1), c_1 and c_2 are constants. Then

$$|y_1(t) - y_2(t)| + |y_1^\Delta(t) - y_2^\Delta(t)| \leq M_4(t) + D_3 M_2(t), t \in J_{\mathbb{T}}, \quad (3.38)$$

where $M_4(t)$ is described by the right side of (3.22) by substituting $|c_1 - c_2|$ instead of $|y_0|$, $M_2(t), b(t)$ and λ be as in (3.23), (3.24) and (3.25) respectively and

$$D_3 = \frac{1}{1-\lambda} \int_\alpha^\beta r_2(\gamma) M_4(\gamma) \Delta \gamma, \quad (3.39)$$

Proof. Let $n(t) = |y_1(t) - y_2(t)| + |y_1^\Delta(t) - y_2^\Delta(t)|, t \in J_{\mathbb{T}}$, we get

$$\begin{aligned} n(t) &\leq |c_1 - c_2| \\ &+ \int_\alpha^t \left| h\left(\tau, y_1(\tau), y_1^\Delta(\tau), \int_\alpha^\tau h_1(\tau, z, y_1(z), y_1^\Delta(z)) \Delta z, \int_\alpha^\beta h_2(\tau, z, y_1(z), y_1^\Delta(z)) \Delta z\right) \right. \\ &- h\left(\tau, y_2(\tau), y_2^\Delta(\tau), \int_\alpha^\tau h_1(\tau, z, y_2(z), y_2^\Delta(z)) \Delta z, \int_\alpha^\beta h_2(\tau, z, y_2(z), y_2^\Delta(z)) \Delta z\right) \left. \right| \Delta \tau \\ &+ \left| h\left(t, y_1(t), y_1^\Delta(t), \int_\alpha^t h_1(t, z, y_1(z), y_1^\Delta(z)) \Delta z, \int_\alpha^\beta h_2(t, z, y_1(z), y_1^\Delta(z)) \Delta z\right) \right. \\ &- h\left(t, y_2(t), y_2^\Delta(t), \int_\alpha^t h_1(t, z, y_2(z), y_2^\Delta(z)) \Delta z, \int_\alpha^\beta h_2(t, z, y_2(z), y_2^\Delta(z)) \Delta z\right) \left. \right| \\ &\leq |c_1 - c_2| + L \int_\alpha^t \left[n(\tau) + c_1(\tau) \int_\alpha^\tau s_1(z) n(z) \Delta z + c_2(\tau) \int_\alpha^\beta s_2(z) n(z) \Delta z \right] \Delta \tau \\ &+ L \left[n(t) + \int_\alpha^t c_1(t) s_1(z) n(z) \Delta z + c_2(t) \int_\alpha^\beta s_2(z) n(z) \Delta z \right] \end{aligned}$$

then

$$\begin{aligned} n(t) &\leq \frac{|c_1 - c_2|}{1-L} + \frac{L}{1-L} \int_\alpha^t \left\{ [c_1(t) s_1(\tau) + 1] n(\tau) + c_1(\tau) \int_\alpha^\tau s_1(z) n(z) \Delta z \right. \\ &\quad \left. + c_2(\tau) \int_\alpha^\beta s_2(z) n(z) \Delta z \right\} \Delta \tau + \frac{L}{1-L} c_2(t) \int_\alpha^\beta s_2(z) n(z) \Delta z, \end{aligned} \quad (3.40)$$

Now applying theorem 3.1 in (3.40) we obtain (3.38). \square

Remark 3.6. The inequality (3.38) gives the uniqueness of solutions of (3.37). If we have $c_1 = c_2 = 0$, then we get $M_5(t) = 0$ and $D_3 = 0$, implies the right hand side of (3.37) is equal to zero.

Now consider the initial value problems

$$\begin{aligned} Y^\Delta(t) &= h\left(t, Y(t), Y^\Delta(t), \int_\alpha^t h_1\left(t, z, Y(z), Y^\Delta(z)\right) \Delta z, \right. \\ &\quad \left. \int_\alpha^\beta h_2\left(t, z, Y(z), Y^\Delta(z)\right) \Delta z, \mu\right), t \in J_{\mathbb{T}}, Y(\alpha) = Y_0, \end{aligned} \quad (3.41)$$

$$\begin{aligned} Y^\Delta(t) &= h\left(t, Y(t), Y^\Delta(t), \int_\alpha^t h_1\left(t, z, Y(z), Y^\Delta(z)\right) \Delta z, \right. \\ &\quad \left. \int_\alpha^\beta h_2\left(t, z, Y(z), Y^\Delta(z)\right) \Delta z, \mu_0\right), t \in J_{\mathbb{T}}, Y(\alpha) = Y_0, \end{aligned} \quad (3.42)$$

where $h \in C_{rd}(J_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and μ, μ_0 are parameters.

The dependency of solutions of (3.41) and (3.42) on parameters follows in the next theorem.

Theorem 3.7. Suppose that the conditions (3.30) and (3.31) are satisfied and

$$\begin{aligned} &|h(t, v_1, v_2, v_3, v_4, \mu) - h(t, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \mu)| \\ &\leq L[|v_1 - \bar{v}_1| + |v_2 - \bar{v}_2| + |v_3 - \bar{v}_3| + |v_4 - \bar{v}_4|], \end{aligned} \quad (3.43)$$

$$|h(t, v_1, v_2, v_3, v_4, \mu) - h(t, v_1, v_2, v_3, v_4, \mu_0)| \leq k(t)|\mu - \mu_0|, \quad (3.44)$$

where $0 \leq L < 1$ is a constant and $k \in C_{rd}(J_{\mathbb{T}}, \mathbb{R}^+)$. Let $Y_1(t)$ and $Y_2(t)$ be respectively, the solutions of (3.41) and (3.42) on $J_{\mathbb{T}}$, then

$$|Y_1(t) - Y_2(t)| + |Y_1^\Delta(t) - Y_2^\Delta(t)| \leq M_5(t) + D_4 M_2(t), t \in J_{\mathbb{T}}, \quad (3.45)$$

where $M_5(t)$ is described by the right side of (3.22) by substituting $|\mu - \mu_0| \bar{k}(t)$ instead of $|y_0|$, $M_2(t), b(t)$ and λ be as in (3.23), (3.24) and (3.25) respectively.

Let

$$\bar{k}(t) = k(t) + \int_\alpha^t k(r) \Delta r, \quad (3.46)$$

$$D_4 = \frac{1}{1-\lambda} \int_\alpha^\beta r_2(\gamma) M_5(\gamma) \Delta \gamma, \quad (3.47)$$

Proof. Let $P(t) = |Y_1(t) - Y_2(t)| + |Y_1^\Delta(t) - Y_2^\Delta(t)|, t \in J_{\mathbb{T}}$, we have

$$\begin{aligned} P(t) &\leq \int_\alpha^t \left| h\left(\tau, Y_1(\tau), Y_1^\Delta(\tau), \int_\alpha^\tau h_1\left(\tau, z, Y_1(z), Y_1^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(\tau, z, Y_1(z), Y_1^\Delta(z)\right) \Delta z, \mu \right) \right. \\ &\quad \left. - h\left(\tau, Y_2(\tau), Y_2^\Delta(\tau), \int_\alpha^\tau h_1\left(\tau, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(\tau, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \mu \right) \right| \Delta \tau \\ &\quad + \int_\alpha^t \left| h\left(\tau, Y_2(\tau), Y_2^\Delta(\tau), \int_\alpha^\tau h_1\left(\tau, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(\tau, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \mu \right) \right. \\ &\quad \left. - h\left(\tau, Y_2(\tau), Y_2^\Delta(\tau), \int_\alpha^\tau h_1\left(\tau, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(\tau, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \mu_0 \right) \right| \Delta \tau \\ &\quad + \left| h\left(t, Y_1(t), Y_1^\Delta(t), \int_\alpha^t h_1\left(t, z, Y_1(z), Y_1^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(t, z, Y_1(z), Y_1^\Delta(z)\right) \Delta z, \mu \right) \right. \\ &\quad \left. - h\left(t, Y_2(t), Y_2^\Delta(t), \int_\alpha^t h_1\left(t, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(t, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \mu \right) \right| \\ &\quad + \left| h\left(t, Y_2(t), Y_2^\Delta(t), \int_\alpha^t h_1\left(t, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(t, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \mu \right) \right. \\ &\quad \left. - h\left(t, Y_2(t), Y_2^\Delta(t), \int_\alpha^t h_1\left(t, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \int_\alpha^\beta h_2\left(t, z, Y_2(z), Y_2^\Delta(z)\right) \Delta z, \mu_0 \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\alpha}^t L \left[P(\tau) + \int_{\alpha}^{\tau} c_1(\tau) s_1(z) P(z) \Delta z + \int_{\alpha}^{\beta} c_2(\tau) s_2(z) P(z) \Delta z \right] \Delta \tau \\
&+ \int_{\alpha}^t k(\tau) |\mu - \mu_0| \Delta \tau \\
&+ L \left[P(t) + \int_{\alpha}^t c_1(t) s_1(z) P(z) \Delta z + \int_{\alpha}^{\beta} c_2(t) s_2(z) P(z) \Delta z \right] + k(t) |\mu - \mu_0| \\
&= |\mu - \mu_0| \bar{k}(t) + L \int_{\alpha}^t \left[P(\tau) + c_1(\tau) \int_{\alpha}^{\tau} s_1(z) P(z) \Delta z + c_2(\tau) \int_{\alpha}^{\beta} s_2(z) P(z) \Delta z \right] \Delta \tau \\
&+ L \left[P(t) + \int_{\alpha}^t c_1(t) s_1(z) P(z) \Delta z + \int_{\alpha}^{\beta} c_2(t) s_2(z) P(z) \Delta z \right]
\end{aligned}$$

then we have

$$\begin{aligned}
P(t) &\leq \frac{|\mu - \mu_0| \bar{k}(t)}{1 - L} \\
&+ \frac{L}{1 - L} \int_{\alpha}^t \left\{ [c_1(t) s_1(\tau) + 1] P(\tau) + c_1(\tau) \int_{\alpha}^{\tau} s_1(z) P(z) \Delta z \right. \\
&\quad \left. + c_2(\tau) \int_{\alpha}^{\beta} s_2(z) P(z) \Delta z \right\} \Delta \tau + \frac{L}{1 - L} c_2(t) \int_{\alpha}^{\beta} s_2(z) P(z) \Delta z,
\end{aligned} \tag{3.48}$$

Now applying theorem 3.1, we have (3.45). \square

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Bohner, M. and Georgiev, S.G. (2016) Multivariable Dynamic Calculus on Time Scales. Springer, Berlin. <https://doi.org/10.1007/978-3-319-47620-9>
- [2] Bohner, M. and Peterson, A. (2001) Dynamic Equations on Time Scales: An Introduction with Applications. Springer Science & Business Media. <https://doi.org/10.1007/978-1-4612-0201-1>
- [3] Georgiev, S.G. (2016) Integral Equations on Time Scales, Integral Equations on Time Scales. <https://doi.org/10.2991/978-94-6239-228-1>
- [4] Pachpatte, D. (2009) On a Nonstandard Volterra Type Dynamic Integral Equation on Time Scales. *Electronic Journal of Qualitative Theory of Differential Equations*, **72**, 1-14. <http://www.kurims.kyoto-u.ac.jp/EMIS/journals/EJQTDE/p459.pdf> <https://doi.org/10.14232/ejqtde.2009.1.72>
- [5] Pachpatte, D.B. (2010) Fredholm Type Integrodifferential Equation on Time Scales. *Electronic Journal of Differential Equations (EJDE)*, **2010**, 1-10. <http://www.kurims.kyoto-u.ac.jp/EMIS/journals/EJDE/Volumes/2010/140/pachpatt.e.pdf>
- [6] Pachpatte, D.B. (2013) Properties of Some Dynamic Integral Equation on Time Scales. *Annals of Functional Analysis*, **4**, 12-26. <https://doi.org/10.15352/afa/1399899522> http://emis.ams.org/journals/AFA/AFA-tex_v4_n2_a2.pdf
- [7] Reinfelds, A. and Christian, S. (2019) Volterra Integral Equations on Unbounded Time Scales. *International Journal of Difference Equations*, **14**, 169-177. <https://doi.org/10.37622/IJDE/14.2.2019.169-177>

<http://campus.mst.edu/ijde/contents/v14n2p5.pdf>

- [8] Salih, O.M. and Mahmood, A.H. (2018) Approximate Solutions for Certain Integrodifferential Equations of Volterra_Fredholm Type. *International Journal of Enhanced Research in Science, Technology & Engineering*, **7**, 37-42.