Journal of Applied Mathematics and Physics, 2020, 8, 434-442
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Investigation of the Surface Brightness Model in the Milky Way via Homotopy Perturbation Method

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How to cite this paper: Alatawi, A.A., Aljoufi, M., Alharbi, F.M. and Ebaid, A. (2020) Investigation of the Surface Brightness Model in the Milky Way via Homotopy Perturbation Method. *Journal of Applied Mathematics and Physics*, **8**, 434-442.
https://doi.org/10.4236/jamp.2020.83033

Received: September 17, 2019
Accepted: March 6, 2020
Published: March 9, 2020

Abstract

In this paper, a linear delay model in astronomy, called as Ambartsumian equation, is investigated by two different approaches. The first is the approximate homotopy perturbation method (HPM), while the second is a new closed-form solution for this equation. The results are presented through a table and several plots and have been compared with the relevant literature. It is revealed that the present HPM is of higher accuracy than those approximate techniques used in previously published works, when compared with the obtained analytic solution. The convergence of the new analytic solution has been also discussed.

Keywords

Delay, Homotopy Perturbation Method, Analytic Solution

1. Introduction

The Ambartsumian equation is used in the theory of surface brightness in the Milky Way [1]. It is a linear delay differential equation given by [2]

$$z'(t) = -z(t) + \frac{1}{q} z\left(\frac{t}{q}\right), \quad q > 1, \tag{1}$$

where q is a constant and

$$z(0) = \lambda, \tag{2}$$

where λ is also a constant. Existence and uniqueness were discussed in [3]. Although the Adomian decomposition method (ADM) was effective to deal with various types of equations [4]-[18], the HPM55 [19] [20] is preferred here to analyze (1-2). Details of the HPM were introduced by Ayati and Biazar [20].

Moreover, it will be shown that the present analysis posses more accuracy over the previous method in the literature.

2. Application of the HPM

First, Equation (1) is rewritten as

$$z'(t) = -z(t) + p\left[\frac{1}{q} z\left(\frac{t}{q}\right)\right], \tag{3}$$

where

$$z(t) = \sum_{n=0}^{\infty} p^n z_n(t). \tag{4}$$

On substituting (4) into (3), we have

$$z_0'(t) + z_0(t) + \sum_{n=0}^{\infty} p^{n+1}\left[z_{n+1}'(t) + z_{n+1}(t) - \frac{1}{q} z_n\left(\frac{t}{q}\right)\right] = 0, \tag{5}$$

which yields

$$z_0'(t) + z_0(t) = 0, \quad z_0(0) = \lambda, \tag{6}$$

and

$$z_{n+1}'(t) + z_{n+1}(t) = \frac{1}{q} z_n\left(\frac{t}{q}\right), \quad z_{n+1}(0) = 0, \quad n \geq 0. \tag{7}$$

Hence,

$$z_0(t) = \lambda \mathrm{e}^{-t}. \tag{8}$$

From (7) and (8), the 1st-order system is given as

$$z_1'(t) + z_1(t) = \frac{\lambda}{q} \mathrm{e}^{-t/q}, \quad z_1(0) = 0. \tag{9}$$

Therefore

$$z_1(t) = \frac{\lambda}{q-1}\left(\mathrm{e}^{-t/q} - \mathrm{e}^{-t}\right). \tag{10}$$

The 2nd-order system is given by

$$z_2'(t) + z_2(t) = \frac{1}{q} z_1\left(\frac{t}{q}\right) = \frac{\lambda}{q(q-1)}\left(\mathrm{e}^{-t/q^2} - \mathrm{e}^{-t/q}\right), \quad z_2(0) = 0. \tag{11}$$

By solving the system (11) for $z_2(t)$, we have

$$z_2(t) = \frac{\lambda}{\left(q^2-1\right)(q+1)}\left[q\mathrm{e}^{-t/q^2} - (q+1)\mathrm{e}^{-t/q} + \mathrm{e}^{-t}\right]. \tag{12}$$

Proceeding as above we obtain the 3rd-order system as

$$z_3'(t) + z_3(t) = \frac{1}{q} z_2\left(\frac{t}{q}\right), \quad z_3(0) = 0, \tag{13}$$

with the corresponding solution

$$z_3(t)=\frac{\lambda}{(q^3-1)(q+1)}\left[q^3\mathrm{e}^{-t/q^3}-(q^3+q^2+q)\mathrm{e}^{-t/q^2}+(1+q+q^2)\mathrm{e}^{-t/q}-\mathrm{e}^{-t}\right]. \quad (14)$$

The calculated higher-order solutions are obtained by MATHEMATICA and then implemented to producing the results in Section 5. From Equation (4), the HPM gives the series solution of Equation (1) as $p\to 1$ (see Ayati and Biazar [20]) by $z(t)=\sum_{n=0}^{\infty} z_n(t)$. This infinite series is approximated by replacing infinity with n-term, hence, the approximate solution, denoted by $\psi_n(t)$, is given by [19] [20]:

$$\psi_n(t)=\sum_{i=0}^{n-1} z_i(t). \quad (15)$$

The residual $|RE_n(t)|$ is given as

$$|RE_n(t)|=\left|\psi_n'(t)+\psi_n(t)-\frac{1}{q}\psi_n\left(\frac{t}{q}\right)\right|, \quad n\ge 1, \quad (16)$$

and the approximate solution in Ref. [2] is

$$\chi_m(t)=\lambda\left[1+\sum_{i=1}^{m}\left(\prod_{k=1}^{i}\left(q^{-k}-1\right)\right)\frac{t^i}{i!}\right]. \quad (17)$$

3. Analytic Solution

Equation (1) can be written as

$$z'(t)=-z(t)+\beta\, z(\beta t), \quad \beta=\frac{1}{q}. \quad (18)$$

Assume that

$$z(t)=\sum_{n=0}^{\infty} c_n \mathrm{e}^{-\beta^n t}. \quad (19)$$

Accordingly, we have

$$z'(t)=\sum_{n=0}^{\infty} -\beta^n c_n \mathrm{e}^{-\beta^n t}, \quad (20)$$

and

$$z(\beta t)=\sum_{n=0}^{\infty} c_n \mathrm{e}^{-\beta^{n+1} t}. \quad (21)$$

Inserting Equations (19)-(21) into Equation (18), we get

$$\sum_{n=0}^{\infty} -\beta^n c_n \mathrm{e}^{-\beta^n t}=-\sum_{n=0}^{\infty} c_n \mathrm{e}^{-\beta^n t}+\sum_{n=0}^{\infty}\beta\, c_n \mathrm{e}^{-\beta^{n+1} t}, \quad (22)$$

i.e.,

$$\sum_{n=0}^{\infty}\left[\left(1-\beta^{n+1}\right)c_{n+1}-\beta c_n\right]\mathrm{e}^{-\beta^{n+1} t}=0, \quad (23)$$

and this yields

$$\left(1-\beta^{n+1}\right)c_{n+1}-\beta c_n=0. \quad (24)$$

The last equation implies that

$$c_{n+1}=\left(\frac{\beta}{1-\beta^{n+1}}\right)c_n,\quad n\geq 0. \tag{25}$$

Accordingly,

$$\begin{aligned}
c_1&=\left(\frac{\beta}{1-\beta}\right)c_0,\\
c_2&=c_0\left(\frac{\beta^2}{\prod_{k=1}^{2}\left(1-\beta^k\right)}\right),\\
c_3&=c_0\left(\frac{\beta^3}{\prod_{k=1}^{3}\left(1-\beta^k\right)}\right),\\
&\vdots\\
c_m&=c_0\left(\frac{\beta^m}{\prod_{k=1}^{m}\left(1-\beta^k\right)}\right),\quad m\geq 1.
\end{aligned} \tag{26}$$

Hence

$$z(t)=c_0\left[\mathrm{e}^{-t}+\sum_{n=1}^{\infty}\left(\frac{\beta^n}{\prod_{k=1}^{n}\left(1-\beta^k\right)}\right)\mathrm{e}^{-\beta^n t}\right]. \tag{27}$$

The initial condition (2), gives c_0 by

$$c_0=\lambda\Bigg/\left(1+\sum_{n=1}^{\infty}\frac{\beta^n}{\prod_{k=1}^{n}\left(1-\beta^k\right)}\right). \tag{28}$$

Thus

$$z(t)=\lambda\left(\frac{\mathrm{e}^{-t}+\sum_{n=1}^{\infty}\frac{\beta^n\mathrm{e}^{-\beta^n t}}{\prod_{k=1}^{n}\left(1-\beta^k\right)}}{1+\sum_{n=1}^{\infty}\frac{\beta^n}{\prod_{k=1}^{n}\left(1-\beta^k\right)}}\right), \tag{29}$$

and the convergence of series in (29) is discussed in the next section in detail.

4. Convergence Analysis

Definition 1: Let $\{f_n(x)\}$ be a sequence of real functions, each function of which is defined for all x on a real interval $a\leq x\leq b$. For each particular x such that $a\leq x\leq b$ consider the corresponding sequence of real numbers $\{f_n(x)\}$.

Suppose that the sequence $\{f_n(x)\}$ converges for every x such that $a \le x \le b$, and let $f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in [a,b]$. Then we say that the sequence of real functions $\{f_n(x)\}$ converges pointwise on the interval $a \le x \le b$, and the function $f(x)$ thus defined i called the limit function of the sequence $\{f_n(x)\}$.

Definition 2: Let $\{f_n(x)\}$ be a sequence of real functions, each function of which is defined for all x on a real interval $a \le x \le b$. The sequence $\{f_n(x)\}$ is said to converge uniformly to $f(x)$ on $a \le x \le b$ if, given any $\varepsilon > 0$, there exists $N > 0$ (which depends only upon ε) such that $|f_n(x) - f(x)| < \varepsilon$ $\forall n > N, \forall x \in [a,b]$.

Theorem 1: Let $\{f_n(x)\}$ be a sequence of real functions converges uniformly to $f(x)$ on $a \le x \le b$ and suppose that each function $f_n(x)\ (n = 1,2,3,\cdots)$ is continuous on $a \le x \le b$, then the limit function $f(x)$ is continuous on $a \le x \le b$.

Theorem 2: Weierstrass M-Test

1) Let $\{M_n\}$ be a sequence of positive constants such that the series of constants $\sum_{n=1}^{\infty} M_n$ converge.

2) Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real functions such that $|u_n(x)| \le M_n$ $\forall x \in [a,b]$ for each $n = 1,2,3,\cdots$.

Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $x \in [a,b]$.

Theorem 3: From (29), Let $M_n = \dfrac{\beta^n}{\prod_{k=1}^{n}(1-\beta^k)}$, then $\{M_n\}$ is a sequence of positive constants and the series $\sum_{n=1}^{\infty} M_n$ converges.

Proof: Since $\beta = 1/q$ and $q > 1$, then β is a positive constant and $0 < \beta < 1$. Also, the expression $(1-\beta^k)$ is always positive, where $0 < \beta^k < 1$. Therefore, $\{M_n\}$ is a sequence of positive constants. To prove convergence of the series $\sum_{n=1}^{\infty} M_n$, we have from the ratio test that

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| &= \lim_{n\to\infty}\left|\frac{\beta^{n+1}}{\prod_{k=1}^{n+1}(1-\beta^k)} \times \frac{\prod_{k=1}^{n}(1-\beta^k)}{\beta^n}\right| = \lim_{n\to\infty}\left|\frac{\beta}{1-\beta^{n+1}}\right|, \\ &= \beta \times \lim_{n\to\infty}\left|\frac{1}{1-\beta^{n+1}}\right| = \beta \times 1 < 1 \end{aligned} \tag{30}$$

which proves the convergence of the series $\sum_{n=1}^{\infty} M_n$.

Theorem 4: The solution $z(t)$ given by (29) converges uniformly on the whole domain $0 \le t < \infty$.

Proof: From the previous theorem, it we showed that $\{M_n\}$ is a sequence of positive constants and the series $\sum_{n=1}^{\infty} M_n$ converges. This meets the first requirement of the Weierstrass M-Test in theorem 2 above. Since $\sum_{n=1}^{\infty} M_n$ converges, we suppose that its sum equals M^*, *i.e.*, $\sum_{n=1}^{\infty} M_n = M^*$. In order to satisfy the second requirement of theorem 2, we rewrite the solution (29) in terms of a new variable x, where $x = \mathrm{e}^{-t}$, as:

$$z(x)=\frac{\lambda}{1+M^{*}}\left(x+\sum_{n=1}^{\infty}M_n x^{\beta^n}\right),\quad 0<x\le 1. \tag{31}$$

To prove convergence of $z(x)$ in (31), it is sufficient to prove the convergence of the series $\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}M_n x^{\beta^n}$. At this stage, we have from $u_n=M_n x^{\beta^n}$ that

$$|u_n|=\left|M_n x^{\beta^n}\right|=|M_n|\left|x^{\beta^n}\right|\le M_n, \tag{32}$$

where $\left|x^{\beta^n}\right|\le 1\quad \forall n=1,2,3,\cdots$, and this completes the proof.

5. Validation of Numerical Results

The HPM and another direct approach have been applied in the previous sections to obtain the approximate solutions and the analytic solution, respectively, in terms of exponential functions with negative powers. The convergence of the analytic solutions was discussed in Section 4. The obtained approximate solutions by the HPM are to be analyzed here in view of the analytic solution (29) to stand on their accuracy. This can be achieved via performing comparisons with the results in the literature. The comparisons between the present results and those of Ref. [2] and [21] are presented in **Table 1**. The present HPM is of higher accuracy as observed from **Table 1**. This indicates the advantages of the current approach over those in the literature when analyzing the Ambartsumian equation.

The behavior of $|RE_{11}|$ is displayed in **Figure 1** ($1.1\le q\le 1.4$), **Figure 2** ($1.4\le q\le 2.0$), **Figure 3** ($2.0\le q\le 3.0$), and **Figure 4** ($3.0\le q\le 10$) for $t\in[0,100]$. It can be seen from these figures that the maximum values of $|RE_{11}|$ are 6×10^{-2}, 6×10^{-3}, 2×10^{-9}, and 1.5×10^{-16}, respectively. This proves the efficiency of the HPM over the previous ones in [2] and [21].

6. Conclusion

The HPM was applied to solve the Ambartsumian equation in terms of exponential functions. The obtained solution was valid in the whole domain, while the corresponding solution in the literature [2] was only valid in sub-domains.

Table 1. Comparison of the present results with the corresponding results in the literature.

t	Ref. [2]	HATM [21]	Present			
			HPM (4-term)	HPM (7-term)	HPM 9-term)	Analytic solution (Equation (29))
0.0	1	1	1	1	1	1
0.5	0.8727825992	0.8727825992	0.8728718032	0.8729409264	0.8729409265	0.8729409265
1.0	0.7694328044	0.7694328044	0.7709321110	0.7717847777	0.7717847885	0.7717847885
1.5	0.6788327993	0.6788327993	0.6865930139	0.6899347736	0.6899349261	0.6899349261
2.0	0.5898647673	0.5898647673	0.61449731667	0.6227074556	0.6227083998	0.6227083998

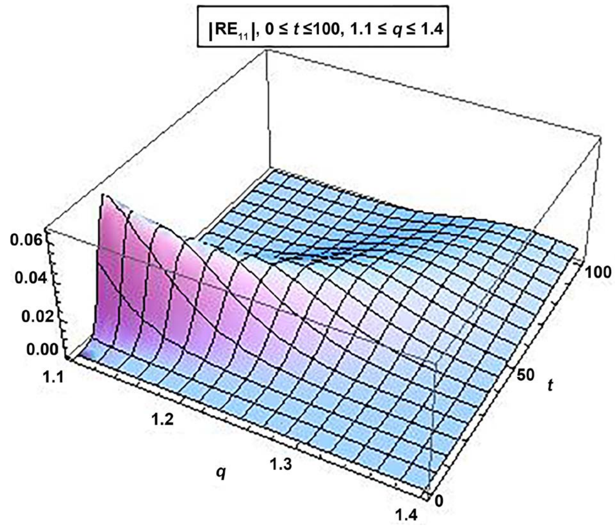

Figure 1. Residual for $1.1 \le q \le 1.4$.

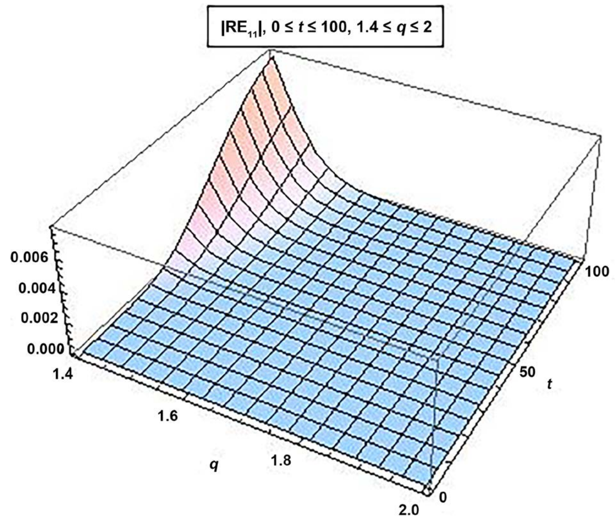

Figure 2. Residual for $1.4 \le q \le 2$.

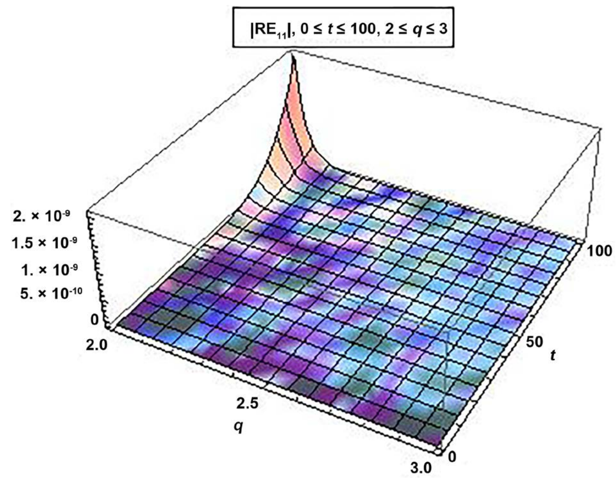

Figure 3. Residual for $2 \le q \le 3$.

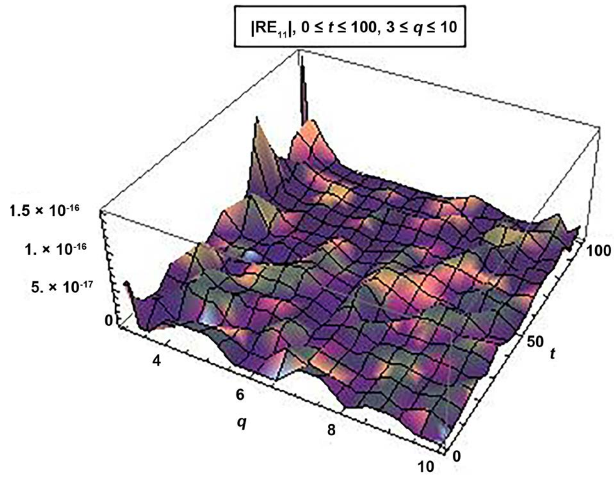

Figure 4. Residual for $3 \le q \le 10$.

Moreover the obtained residual tends to zero as the q increases. In view of references [2] and [21], the present HPM is of higher accuracy when compared with the obtained exact solution.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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