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Review on the Current Stochastic Numerical Methods for Econometric Analysis

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Abstract

The main aim of this paper is to present and emphasize the contribution of stochastic numerical methods as must tools for the modern econometric modelisation. Indeed, the stochastic numerical methods play an important role in mathematical modelling and the econometric analysis because they model uncertainties that govern the real-world data. However these powerful tools are not well-known and understood by many economists and financial econometricians.

Keywords

Stochastic Differential Equations, The Euler-Maruyama Scheme, The Milstein Scheme, The Crank-Nicolson Scheme, Runge-Kutta Method, Itô Integrals, Econometric Analysis

1. Introduction

As mentioned in [1], the theory of stochastic differential equations was originally developed by mathematicians as a tool for explicit construction of the trajectories of diffusion processes for given coefficients of drift and diffusion.

Today, the stochastic differential equation (SDE) models

$$\mathrm{d}X(t)=\Phi\big(X(t),t\big)\mathrm{d}t+\Psi\big(X(t),t\big)\mathrm{d}B(t)$$

or Stochastic partial differential equation (SPDE) models

$$\mathrm{d}X_t=\left[AX_t+F\left(X_t\right)\right]\mathrm{d}t+B\left(X_t\right)\mathrm{d}W_t,\quad X_t\big|_{\partial\mathcal{D}}=0, X_{=}\xi$$

play a prominent role in a range of application areas, including economics, finance, biology, epidemiology, chemistry, microelectronics, and mechanics [2] [3].

By reading [4] [5] and [6] in the field as the reference books of Numerical methods in Economics, one can remark that there exist the high needs to take in content the uncertainties in economic analysis. Therefore, the stochastic numerical methods must be understood by econometricians or economists.

In all, the main challenge of econometricians or economists is how to numerize the stochastic differential equations, that is, how to move from the continuous-time stochastic models to discrete-time stochastic models [7]-[15].

The motivation for these methods came from the need to deal effectively with problems arising in the fields of economics and Finance [2] [16]-[21]. Also, the new direction of the modern econometric theory and applications go to stochastic analysis [22] [23] [24] [25].

The remainder of this paper is organized as follows. Section 2 presents some useful definitions and notations in stochastic analysis. Section 3 gives the stochastic integrals as the tools of evaluation of stochastic differential equations. Section 4 presents some recent stochastic differential equations that can be meaningfully in econometric analysis and their assumptions used for uniqueness and existence of solution. Section 5 presents some powerful numerical methods for stochastic differential equations.

2. Notations and Definitions

In this section we present the notations, definitions and basic facts of stochastic differential equations, stochastic integrals, stochastic numerical methods and convergence which will be used in this paper.

Definition 2.0.1. Let $\left(\Omega,\mathcal{F},P,(\mathcal{F})_{t\in\mathbb{R}_+}\right)$ be a filtered probability space. The σ-algebra on $\mathbb{R}\times\Omega$ generated by all sets of the form A, $A\in\mathcal{F}_0$, and A, $0\le a<b<+\infty$, $A\in\mathcal{F}_a$, is said to be the predictable σ-algebra for the filtration $(\mathcal{F})_{t\in\mathbb{R}_+}$

Definition 2.0.2. A real-valued process $(X_t)_t\in\mathcal{R}_+$ is called predictable with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ or $\mathbb{F}_t$-predictable, if as a mapping from $\mathbb{R}_+\in\Omega\to\mathbb{R}$ predictable σ-algebra generated by this filtration.

Definition 2.0.3 Let $(X_t)_{t\in\mathbb{R}_+}$ be a left-continuous real-valued process adapted to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then X_t is predictable.

Definition 2.0.4. A stochastic process (X_t) is said to be right-continuous with left limits (RCLL) or contu à droite avec limite à gauche (càdlàg) if, almost surely, it has trajectories that are RCLL. That is,

$$X_t = lim x_s$$

Definition 2.0.5. (Wiener process, [26]) Let (Ω,F,P) be a probability space and let $\{F_t,t\ge0\}$ be a filtration defined on it. A process $\{X(t),t\ge0\}$ is called an F_t-Wiener process if it satisfies the following conditions.

1) $X(0)=0$;

2) $X(t)$ is F_t-measurable and $F\left(X(s)-X(t):s\ge t\right)$ is independent of F_t

for all $t \geq 0$;

3) The increments $X(s) - X(t)$ are normally distributed with mean 0 and variance $\sigma^2(s-t) > 0$ for all $s > t \geq 0$;

4) The sample paths of $X(.)$ are in $C[0,\infty)$.

Definition 2.0.6. [27] X is a Markov process if for any t and $s > 0$, the conditional distribution of $X(t+s)$ given F_t is the same as the conditional distribution of $X(t+s)$, given $X(t)$, that is,

$$P\left(X(t+s) \leq Y \mid F_t\right) = P\left(X(t+s) \leq Y \mid X(t)\right)$$

a.s.

Definition 2.0.7. [28] A Brownian motion is a continuous, adapted process $B = \{B_t, \mathcal{F}_t : 0 \leq s < \infty\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the properties that $B_0 = 0$ a.s. and for $0 \leq s < \infty$, the increment $B_t - B_s$ is indepedentof $\mathcal{F}_j$ and is normally distributed with mean zero and variable $t - s$.

The Brownian paths have the following properties [27]. Almost every sample path $B(t), 0 \leq t \leq T$: 1) is a continuous function of t; 2) is not monotone in any interval, no matter how small the interval is; 3) is not differentiable at any point; 4) has infinite variation on any, no matter how small it is; 5) has quadratic variation $[0,t]$ equal to t, for any t.

Definition 2.0.8. (Brownian motion with respect to a filtration, [29]) A vectorial (d-dimensional) Brownian motion on $\mathbf{T}$ with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbf{T}}$ such that 1) $W_0 = 0$; 2) For all $0 \leq s < t$ in $\mathbf{T}$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and follows a centered Gaussian distribution with variance-covariance matrix $(t-s) I_d$.

Some classical properties of Brownian motion are stated in the following proposition.

Proposition 2.0.1. Let $(W_t)_{t \in \mathbf{T}}$ be a Brownian motion with respect to $(\mathcal{F}_t)_{t \in \mathbf{T}}$. 1) symmetry: $(W_t)_{t \in \mathbf{T}}$ is also a Brownian motion. 2) scaling: for all $\lambda > 0$, the process is also a Brownian motion. 3) Invariance by translation: for all $s > 0$, the process $W_{t+s} - W_s$ is a standard Brownian motion independent of $\mathcal{F}_s$.

Definition 2.0.9. [27] A process X is called adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)$, if for all t, $X(t)$ is $\mathcal{F}$-measurable.

Definition 2.0.10. Let X_t be an adapted stochastic process with RCLL trajectories. It is said to be decomposable if it can be written as

$$X_t = X_0 + M_t + Z_t,$$

where $M_0 = Z_0 = 0$, M_t is a locally square-integrable martingale, and Z_t has RCLL-adapted trajectories of bounded variation.

Definition 2.0.11 (Martingale, [30]) Let $\{\mathcal{F}_t\}$ be an indexed set of sub-σ-algebra of $\{\mathcal{F}\}$ such that $\{\mathcal{F}_t\} \supset \{\mathcal{F}_s\}$ if $t > s$. The pair $\{x(t), \mathcal{F}_t\}$ is said to be a $\mathcal{F}_t$-martingale if $E|x(t)| < \infty$ and $x(t)$ is $\mathcal{F}_t$-measurable and

$$E\left[x(t+s) \mid \mathcal{F}_t\right] = x(t)$$

w.p.1. for each t and $s > 0$. If the equality is replaced by $\leq$, we have a Supermartingale, and if it is replaced by $\geq$ we have a Submartingale.

Definition 2.0.12. [31] The quadratic covariation of two processes X and Y is

$$[X,Y]_t = \lim_{\Pi_n \to 0} \sum_{k=1}^{n} \left(X(t_k) - X(t_{k-1})\right)\left(Y(t_k) - Y(t_{k-1})\right). \tag{1}$$

Here $\Pi_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$ is an arbitrary partition of the interval $[0,t]$.

3. Stochastic Integrals

The aim of this section is to provide some backgrounds on the stochastic integrals. These integrals constitute a cornerstone of mathematical modelling and stochastic analysis used in evaluation and resolution of the stochastic diferential equatons [1] [32]-[36].

3.1. The Itô Integral

Itô's theory of stochastic integration was originally motivated as a direct method to construct diffusion processes (as subclass of Markov processes) as solution of stochastic differential equations [35]. As in [27] Itô integral is defined as a sum

$$\int_0^T X(t)\mathrm{d}B(t) = \sum_{i=0}^{n-1} C_i\left(B(t_{i+1}) - B(t_i)\right). \tag{2}$$

Theorem 3.1. (Properties of stochastic integrals, [27]) Let $X(t)$ be a regular adapted such that with probability one $\int_0^T X^2(t)\mathrm{d}t < \infty$. Then Itô integral $\int_0^T X(t)\mathrm{d}B(t)$ is defined and has the following properties.

1) Linearity. If Itô integrals of $X(t)$ and $Y(t)$ are defined and α and β are some constants then

$$\int_0^T \left(\alpha X(t) + \beta Y(t)\right)\mathrm{d}B(t) = \alpha\int_0^T X(t)\mathrm{d}B(t) + \beta\int_0^T Y(t)\mathrm{d}B(t)$$

2) $\int_0^T X(t) I_{(a,b]}(t)\mathrm{d}B(t) = \int_a^b X(t)\mathrm{d}B(t)$. The following two properties hold when the process satisfies an additional assumption

$$\int_0^T E\left(X^2(t)\right)\mathrm{d}t < \infty. \tag{3}$$

3) Zero mean property. If condition 3 holds then $E\left(\int_0^T X(t)\mathrm{d}B(t)\right) = 0$.

4) Isometry property. If condition 3 holds. Then

$$E\left(\int_0^T X(t)\mathrm{d}B(t)\right)^2 = \int_0^T E\left(X^2(t)\right)\mathrm{d}B(t)$$

5) Generalized Itô Isometry [31]. For $f,g \in \mathbb{L}_a^2 d\left(\Omega, \mathbf{L}^2([0,T])\right)$, we have

$$\mathbb{E}\left[\int_0^t f(s)\mathrm{d}W(s)\int_0^t g(s)\mathrm{d}W(s)\right] = \int_0^t \mathbb{E}\left[f(s)g(s)\right]\mathrm{d}t$$

Corollary 3.1.1. If X is a continuous adapted process then the Itô integral $\int_0^T X(t)\mathrm{d}B(t)$ exists. In particular, $\int_0^T f(B(t))\mathrm{d}B(t)$ where f is a continuous function on **R** is well-defined.

A consequence of the isometry property is the expectation of the product of two Itô integrals.

Theorem 3.2. Let $X(t)$ and $Y(t)$ be regular adapted processes, such that $\int_0^T X(t)^2 \mathrm{d}t < \infty$ and $\int_0^T Y(t)^2 < \infty$. Then

$$E\left(\int_0^T X(t)\mathrm{d}B(t)\int_0^T Y(t)\mathrm{d}B(t)\right) = \int_0^T E\left(X(t)Y(t)\right)\mathrm{d}t$$

We denote by $\mathbb{R}^{mn}$ all real-valued $m\times n$ matrices and by

$$W(t) = \left(W_1(t),\cdots,W_n(t)\right)', t \geq 0.$$

Let $[a,b] \in [0,\infty[$ and we put

$$C_W\left([a,b]\right) = \left\{f:[a,b]\times\Omega \to \mathbb{R}^{mn} \mid \forall 1\leq i\leq m, \forall 1\leq j\leq n: f_{ij} \in C_{Wj}\left([a,b]\right)\right\},$$

$$C_{IW}\left([a,b]\right) = \left\{f:[a,b]\times\Omega \to \mathbb{R}^{mn} \mid \forall 1\leq i\leq m, \forall 1\leq j\leq n: f_{ij} \in C_{IWj}\left([a,b]\right)\right\}$$

and $C_I\left([a,b]\right)$ respectively.

Definition 3.2.1. [37] If $f:[a,b]\times\Omega\to\mathbb{R}^{mn}$ belongs to $C_{IW}\left([a,b]\right)$, then the stochastic integral with respect to W is the m-dimensional vector defined by

$$\int_a^b f(t)\mathrm{d}W(t) = \left(\sum_{j=1}^n \int_a^b f_{ij}(t)\mathrm{d}W_j(t)\right)'_{1\leq i\leq m} \tag{4}$$

where each of the integrals on the right-hand side is defined in the sense of Itô.

As in [38] the Itô formula for multidimensional Itô processes is defined in following way. If

$$X_t^i = X_0^i + \int_0^t K_s^i \mathrm{d}s + \sum_{l=1}^k \int_0^t H_s^{il}\mathrm{d}B_s^l,$$

$t\in[0,T]$, $i=1,\cdots,m$, are Itô processes and $F\in C^2\left([0,T]\times\mathbb{R}^m\right)$, then

$$\begin{aligned} &F(T,X_T) - F(0,X_0) \\ &= \sum_{i=1}^m \int_0^T \frac{\partial F}{\partial x_i}(t,X_t)\mathrm{d}X_t^i + \sum_0^T \frac{\partial F}{\partial t}(t,X_t) + \frac{1}{2}\sum_{i,j=1}^m \int_0^T \frac{\partial^2 F}{\partial x_i \partial x_j}(t,X_t)\mathrm{d}\left\langle X^i, X^j\right\rangle_t \end{aligned} \tag{5}$$

$$\begin{aligned} &= \sum_{i=1}^m\sum_{l=1}^k \int_0^T \frac{\partial F}{\partial x_i}(t,X_t)H_t^{il}\mathrm{d}B_t^l + \sum_{i=1}^m \int_0^T \frac{\partial F}{\partial x_i}(t,X_t)K_t^{il}\mathrm{d}t \\ &\quad + \int_0^T \frac{\partial F}{\partial t}(t,X_t)\mathrm{d}t + \frac{1}{2}\sum_{i=1}^m\sum_{l=1}^k \int_0^T \frac{\partial^2 F}{\partial x_i \partial x_j}(t,X_t)H_t^{il}H_t^{jl}\mathrm{d}t. \end{aligned} \tag{6}$$

Morever, if $F\in C^{2,3}\left([0,T]\times\mathbb{R}^m\right)$, then this formula can be written in term of the Stratonovich integral [38]

$$F(T,X_T) - F(0,X_0) = \sum_{i=1}^m \int_0^T \frac{\partial F}{\partial x_i}(t,X_t)\circ\mathrm{d}X_t^i + \int_0^T \frac{\partial F}{\partial t}(t,X_t)\mathrm{d}t \tag{7}$$

$$= \sum_{i=1}^m\sum_{l=1}^k \int_0^T \frac{\partial F}{\partial x_i}(t,X_t)H_t^{il}\circ\mathrm{d}B_t + \sum_{i=1}^m \int_0^T \frac{\partial F}{\partial x_i}(t,X_t)K_t^{il}\mathrm{d}t + \int_0^T \frac{\partial F}{\partial t}(t,X_t)\mathrm{d}t \tag{8}$$

3.2. The Stratonovich Integral

Definition 3.2.2. [35] Let X_t and Y_t be Itô processes. The Stratonovich integral of X_t with respect to Y_t is defined by

$$\int_a^b X_t \circ \mathrm{d}Y_t = \int_a^b X_t \mathrm{d}Y_t + \frac{1}{2}\int_a^b (\mathrm{d}X_t)(\mathrm{d}Y_t), \tag{9}$$

or equivalently in the stochastic differential form

$$X_t \circ \mathrm{d}Y_t = X_t \mathrm{d}Y_t + \frac{1}{2}(\mathrm{d}X_t)(\mathrm{d}Y_t). \tag{10}$$

Theorem 3.3. [35] Let $f(t,x)$ be a continuous function with continuous partial derivatives $\frac{\partial F}{\partial t}$, $\frac{\partial f}{\partial t}$, and $\frac{\partial f}{\partial x}$. Then

$$\int_a^b f(t,B(t)) \circ \mathrm{d}B_t = F(t,B(t))\Big|_a^b - \int_a^b \frac{\partial B}{\partial t}(t,B(t)) \circ \mathrm{d}t. \tag{11}$$

In particular, when the function *f* does not depend on *t*, we have

$$\int_a^b f(B(t)) \circ \mathrm{d}B_t = F(B(t))\Big|_a^b. \tag{12}$$

Theorem 3.4. [35] Let $f(t,x)$ be a continuous function with continuous partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, and $\frac{\partial^2 f}{\partial x^2}$. Then

$$\int_a^b X_t \circ \mathrm{d}Y_t = \lim_{\|\Delta\|\to 0} \sum_{i=1}^n f\left(t_i^*, \frac{1}{2}\left(B(t_{i-1}) + B(t_i)\right)\right)\left(B(t_i) - B(t_{i-1})\right) \tag{13}$$

$$= \lim_{\|\Delta\|\to 0} \sum_{i=1}^n f\left(t_i^*, B\left(\frac{t_{i-1}+t_i}{2}\right)\right)\left(B(t_i) - B(t_{i-1})\right), \tag{14}$$

in probability, where $t_{i-1} \le t_i^* \le t_i$, $\Delta = \{t_0, t_1, \cdots, t_{n-1}, t_n\}$ is a partition of the finite interval $[a,b]$ and $\|\Delta\| = \max_{1\le i\le n}(t_i - t_{i-n})$.

In [39], the multidimensional Stratonovich integrals $S_m(f)$ can be expressed by the following formula using Itô integrals

$$S_m(f) = \sum_{2k\le} \frac{m!}{2^k k!(m-2k)} I_{m-2k}\left(Tr^k f\right). \tag{15}$$

where Tr denoted the iterated traces that are defined formally starting with

$$Trf(s_1, \cdots, s_{m-2}) = \int f(s_1, \cdots, s_{m-2}, s)\mathrm{d}s.$$

Another approach to formula (15) using Hida's theory of white noise. Working on $\mathbb{R}^m$ instead of $\mathbb{R}_+^m$ and assuming that *f* is a test-function, the integral $S_m(f)$ may indead be rewritten as

$$\int f(s_1, \cdots, s_m)\dot{X}_{s_1}(w)\cdots\dot{X}_{s_m}(w)\mathrm{d}s_1\cdots\mathrm{d}s_n = \left\langle f, \dot{X}^{\otimes n}\right\rangle$$

where the derivative of Brownian motion is understood in the distribution sense. In the sense of Hu and Meyer [39], a Stratonovich integral is given in rigorous form as

$$S(f) = \sum_m \frac{1}{m!}\int_{[S)} f_m(s_1, \cdots, s_m)\mathrm{d}X_{s_1}(w)\cdots\mathrm{d}X_{s_m}(w) \tag{16}$$

where *f* is a finite sequence of coefficients $f_m \in L_s^2(\mathbb{R}^m)$ and $n! = n\times(n-1)\times\cdots\times 1$.

3.3. The Skorohod Integral

The Skorohod integral was introduced for the first time by A. Skorohod in 1975 as an extension of the Itô integral to non-adapted processes and is the adjoint of the Malliavin derivative which is fundamentals to the stochastic calculus of variations [40] [41].

Definition 3.4.1. [40] Let $u(t), t \in [0,T]$, be a measurable stochastic process such that for all $t \in [0,T]$ the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $\mathbb{E}[u^2(t)] < \infty$. Let its Wiener-Itô chaos expansion be

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n, t) = \sum_{n=0}^{\infty} I_n(f_n(.,t)). \tag{17}$$

Then we define the Skorohod integral of $u(t)$ by

$$\delta(u) := \int_0^T u(t)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{18}$$

where convergent in $L^2(P)$. Here $\tilde{f}_n, n = 1, 2, \cdots$ are the symmetric functions derived from $f_n(.,t), n = 1, 2, \cdots$. We say that u is Skorohod integrable, and we write $u \in Dom(\delta)$ if the series in (18) converges in $\mathbf{L}^2(P)$.

3.4. The Ogawa Integral

The Itô integral and others are based in a fundamental hypothesis of causal relationship. Shigeyoshi Ogawa [42] defined this following noncausal integral that is so-called Ogawa integral

$$\int_0^T f(t) * \mathrm{d}W(t) = \sum \int_0^t f(s) m_i(s) \mathrm{d}s \int_0^t m_i(s) \mathrm{d}W(s), \ t \in [0,T] \tag{19}$$

where $\{m_i(t)\}$ is the complete orthnormal system on $\mathbb{L}^2([0,T])$. Nualart and Zakai [43] proved that the Ogawa integral is equivalent to the Stratonovich integral of the Ogawa integral exists with the Stratonovich integral defined [31] as

$$\int_0^t f(t) \circ \mathrm{d}W(t) = \lim_{|\Pi_n| \to 0} \sum_{i=1}^{n} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} f(s) \mathrm{d}s \left(W(t_{i+1}) - W(t_i)\right). \tag{20}$$

Here $\Pi_n = \{0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$ is an arbitrary partition of the interval $[0,t]$. The Ogawa integral coincides with the Stratonovich integral defined at the midpoints [31]

$$\int_0^t f(t) \circ \mathrm{d}W(t) = \lim_{|\Pi_n| \to 0} \sum_{i=1}^{n} f\left(\frac{t_i + t_{i+1}}{2}\right) \left(W(t_{i+1}) - W(t_i)\right). \tag{21}$$

Here $\Pi_n = \{0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t\}$ is an arbitrary partition of the interval $[0,t]$.

4. Stochastic Differential Equations

This section presents four types of stochastic differential equations that can be useful in econometric modelling such as the stochastic ordinary differential equation, stochastic partial differential equation, Stochastic Differential Equation with Jumps, and Stochastic Delay Differential Equations [15] [26] [44]-[51].

4.1. Stochastic Ordinary Differential Equations

Let $X(t)$ be a diffusion in *n* dimensions described by the multi-dimensional stochastic differential equation

$$\mathrm{d}X(t) = \Phi(X(t),t)\mathrm{d}t + \Psi(X(t),t)\mathrm{d}B(t), \tag{22}$$

where Ψ is $n \times d$ matrix valued function, *B* is d-dimensional Brownian motion and and *X* and Φ are vector *n*-dimensional vector valued functions. The vector $\Phi(X,t)$ and the matrix $\Psi(X,t)$ are the coefficients of the stochastic differential equation.

Theorem 4.1. (Unique and Existence of Solution). If the coefficients are locally Lipschitz in X with a constant independent of t, that is, for every N, there is a constant K depending only on T and N such that for all $|x|,|y| \le N$ and all $0 \le t \le T$,

$$|\Phi(x,t) - \Phi(y,t)| + |\Psi(x,t) - \Psi(y,t)| \le K|x-y|, \tag{23}$$

for any given $X(0)$ the strong solution to stochastic differentional equation 26 is unique. If in addition to condition 23 the linear growth condition holds

$$|\Phi(x,t)| + |\Psi(x,t)| \le K_\tau(1+|x|), \tag{24}$$

$X(0)$ is independent of *B*, and $E|X(0)|^2 < \infty$, then the strong solution exists and is unique on $[0,T]$, moreover,

$$E\left(\sup|X(t)|^2\right) < C\left(1 + E|X(0)|^2\right),$$

where constant *C* depends only on *K* and *T*.

The following theorem gives the solution of stochastic differential equations as Markov processes.

Theorem 4.2. [1] (The solution of SDEs as Markov processes) If Equation (26) satisfies the conditions of the existence and uniqueness theorem 4.1, the solution X_t of the equation for arbitrary initial values is a Markov process on the interval $[t_0,T]$ whose initial probability distribution at the instant to is the distribution of C and whose transition probabilities are given by

$$P(s,xt,B) = P(X_t \in B \mid X_s = x) = P(X_t(s,x) \in B) \tag{25}$$

where $X_t(s,x)$ is the solution of equation.

Theorem 4.3. [1] (The solution of SDEs as Diffusion processes). The condition of the existence and uniqueness Theorem 4.1 are satisfied for the SDE

$$\mathrm{d}X(t) = \Phi(X(t),t)\mathrm{d}t + \Psi(X(t),t)\mathrm{d}B(t), X_{t_0} = C, t_0 \le t \le T, \tag{26}$$

where $X_t \in R^d$, $\Phi(t,x) \in R^d$, $B \in R^m$ and $\Psi(t,x)$ is a $d \times m$ matrix. If in addition, the functions Φ and Ψ are continuous with respect to t, the solution X_t is a d-dimensional diffusion process on $[t_0,T]$ with drift vector and diffusion matrix $\Pi(t,x) = \Psi(t,x)\Psi(t,x)'$. In particular, the solution of an autonomous SDE is always a homogeneous diffusion process on $[t_0,\infty)$.

4.2. Stochastic Partial Differential Equations

Consider the Itô Stochastic Partial Differential Equation of the form as mentioned in [52]

$$\mathrm{d}X_t = \left[AX_t + F\left(X_t\right)\right]\mathrm{d}t + B\left(X_t\right)\mathrm{d}W_t,\ \ X_t\big|_{\partial\mathcal{D}} = 0, X_{=}\xi \tag{27}$$

for $t \geq 0$, where W_t, is an infinite dimensional Wiener process of the

$$W_t\left(x,w\right) = \sum_{j=1}^{\infty} C_j W_t^{\,j}\left(w\right)\phi_j\left(x\right), t \geq 0, x \in \mathcal{D} \tag{28}$$

with independent scalar Wiener processes $W_t^{\,j}$, $j \in \mathbb{N}$. Here the family ϕ_j, $j \in \mathbb{N}$, is an orthonormal basis in, e.g., $\mathbf{L}^2\left(\mathcal{D},\mathbb{R}\right)$.

Assumptions: For uniqueness and existence of solution of this SPDE the following assumptions hold. **A1**) Linear operator A. Let $\mathcal{L}$ be a finite or countable set. In addition, let $\left(\lambda_i\right)_{i\in\mathcal{L}}$ be a family of real numbers with $inf_{i\in\mathcal{L}} > -\infty$ and let $\left(\vartheta_i\right)_{i\in\mathcal{L}}$ be an orthonormal basis of *H*. The linear operator $A:\mathcal{D}\left(A\right)\to H$ is given by $A\upsilon = \sum_{i=\mathcal{L}} -\lambda\left\langle\vartheta_i,\upsilon\right\rangle\vartheta_i$ for all $\upsilon\in\mathcal{D}\left(A\right)$ with $\mathcal{D}\left(A\right) = \left\{\sum_{i=\mathcal{L}}\left|\lambda\right|^2\left|\left\langle\vartheta_i,\upsilon\right\rangle\right|^2\right\} \in H$.

A2) Drift term F. Let $\alpha,\delta\in\mathbb{R}$ be real numbers with $\delta-\alpha<1$ and let $F: H_\delta \to H_\alpha$ be a globally Lipschitz continuous mapping.

A3) Diffusion term B.Let $\alpha,\delta\in\mathbb{R}$ be real numbers with $\delta-\beta<\frac{1}{2}$ and let $F: H_\delta \to HS\left(\upsilon_0, H_\beta\right)$ be a globally Lipschitz continuous mapping.

A4) Initial value ξ: Let $\gamma\in\left[\delta,\min\left(\alpha+1,\beta+1/2\right)\right]$ and $p\in\left[2,\infty\right)$ be real numbers and let $\xi:\Omega\to H_\gamma$ be an $\mathcal{F}_0/\mathcal{B}\left(H_\gamma\right)$-measurable mapping with $\mathbf{E}\left[\left\|\xi\right\|_{H_\gamma}^p\right]<\infty$.

The literature contains many existence and uniqueness theorems for mild solutions of SPDEs. Theorem below provides an existence, uniqueness, and regularity result for solutions of SPDEs with globally Lipschitz continuous coefficients in the Equation (27).

Theorem 4.4. [52] Assume that the Assumptions **A1**)-**A4**) are fulfilled. Then there exists a unique predictable stochastic process $X:\left[0,T\right]\times\Omega\to H_\gamma$ satisfying $\sup_{t\in[0,T]}\mathbf{E}\left[\left\|\xi\right\|_{H_\gamma}^p\right]<\infty$ and

$$X_t = \mathrm{e}^{At}\xi + \int_0^t \mathrm{e}^{A(t-s)}F\left(X_s\right)\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}B\left(X_s\right)\mathrm{d}W_s \tag{29}$$

$\mathbb{P}-a.s.$ for all $t\in\left[0,T\right]$. In addition,

$$X\in\bigcap_{r\in(-\infty,\gamma]} C^{\min(\gamma-r,1/2)}\left(\left[0,T\right],\mathbf{L}^p\left(\Omega,H_r\right)\right).$$

Here we assume that the Assumptions that $X:\left[0,T\right]\times\Omega\to H$ is a predictable stochastic process, which satisfies 27. Let $t_0\in\left[0,T\right)$. Then the solution process X also satisfies

$$X_t = \mathrm{e}^{A(t-t_0)}X_{t_0} + \int_{t_0}^t \mathrm{e}^{A(t-s)}F\left(X_s\right)\mathrm{d}s + \int_{t_0}^t \mathrm{e}^{A(t-s)}B\left(X_s\right)\mathrm{d}W_s, \mathbb{P}-a.s. \tag{30}$$

for every $t \in [t_0 \in T]$.

Proposition 2 Let assumption **A1**)-**A4**) be satisfied and let $\gamma \in (0,1)$ be given by Assumption A3. Then there is an up-to-modification unique predictable stochastic process $X : [0,T] \times \Omega \to D\left((\kappa - A)^{\gamma}\right)$ with

$$\sup_{0 \le t \le T} \mathbb{E}\left\|(\kappa - A)^{\gamma} A_t\right\|_H^p < \infty$$

for $p \in [0,\infty)$ and with

$$P\left[X_t = \mathrm{e}^{A(t-t_0)} X_0 + \int_0^t \mathrm{e}^{A(t-s)} F(X_s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)} B(X_s)\,\mathrm{d}W_s\right] = 1, \tag{31}$$

for all $t \in [0,T)$. Moreover, X is the unique mild solution of the SPDE 27 in the sense of Equation (31).

4.3. Stochastic Differential Equation with Jumps

In real world, some phenomena or economic policy decisions are governed under uncertainty with jumps. Therefore, stochastic differential equation with jumps modeling can be considered as a useful econometric approach [53]. Consider a one-dimensional SDE, $d = 1$, in the form

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t + \int_{\varepsilon} c(t, X_{t-}, v)\, p_{\varphi}(\mathrm{d}v, \mathrm{d}t) \tag{32}$$

for $t \in [0,T]$, with $X_0 \in \mathbb{R}$, and $W = \{W_t, t \in [0,T]\}$ an $\mathcal{F}_t$-*adapted* one-dimensional Wiener process. We assume an an $\mathcal{F}_t$-*adapted* Poisson measure $p_{\varphi}(\mathrm{d}v, \mathrm{d}t)$ with mark space $\varepsilon \subseteq \mathbb{R} \setminus \{0\}$ and with intensity measure $\varphi(\mathrm{d}v)\mathrm{d}t = \lambda F(\mathrm{d}v)\mathrm{d}t$, where $F(.)$ is a given probability distribution function for the realizations of the marks.

Consider a one-dimensional SDE with Jumps (32) in integral form, is of the form

$$X_t = X_0 + \int_0^t a(s, X_s)\,\mathrm{d}s + \int_0^t b(s, X_s)\,\mathrm{d}W_s + \sum_{i=1}^{p_{\varphi}(t)} c\left(\tau_i, X_{\tau_i}\right) \tag{33}$$

4.4. Stochastic Delay Differential Equations

Consider the following Stochastic Delay Differential Equations with constant delay in Stratonovich form [54]

$$\mathrm{d}X(t) = f\left(X(t), X(t-\tau)\right)\mathrm{d}t + \sum_{l=1}^{r} g_l\left(X(t), X(t-\tau)\right) \circ \mathrm{d}W_l(t), \tag{34}$$

$$X(t) = \phi(t) \tag{35}$$

where $\tau > 0$ is a constant $(W(t), \mathcal{F}_t) = \left(\{W_l(t), 1 \le l \le r\}, \mathcal{F}\right)$ is a system of one dimensional independent standard Wiener process, the function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g_l : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\phi(t) : [-\tau, 0] \to \mathbb{R}^d$ are continuous with $\mathbb{E}\|\phi\|_{L^\infty}^2 < \infty$. and ϕ is $\mathcal{F}$-measurable. For mean-square stability of (35), we assume that $f, g_l, \partial_x g_l g_q$ and $\partial_{x_\tau} g_l g_q$ (∂_x and ∂_{x_τ} denote the derivatives with respect to the first and second variables respectively), $l, q = 1, 2, \cdots, r$, in (35) satisfy the Lipschitz and linear growth conditions.

5. Numerical Methods for Stochastic Differential Equations

In this section we review shortly some numerical methods used in the stochastic analysis that can be useful for economists and other social scientists. These main books that can helpfully to econometricians and economists are [26] [52] [55]-[61].

5.1. Numerical Methods for Stochastic Ordinary Differential Equations

The Euler-Maruyama Scheme. The Euler-Maruyama method is a method for the approximate numerical solution of a stochastic differential equation. It is a simple generalization of the Euler method for ordinary differential equations to stochastic differential equations. It is named after a Swiss mathematician, physicist, geograph, astronomer, engineer, and logician Leonhard Euler (1707-1783) and a Japanese mathematician Gisiro Maruyama (1916-1986). Consider a scalar Itô stochastic ordinary differential equation [52]

$$\mathrm{d}X_t = f\left(t, X_t\right)\mathrm{d}t + g\left(t, X_t\right)\mathrm{d}W_t \tag{36}$$

with a standard scalar Wiener process $W_t, t \geq 0$. This Equation (36) is in fact a symbolic representation for the stochastic integral equation

$$X_t = X_{t_0} + \int_{t_0}^{t} f\left(t, X_t\right)\mathrm{d}t + \int_{t_0}^{t} g\left(t, X_t\right)\mathrm{d}W_t \tag{37}$$

The simplest numerical scheme for the stochastic ordinary differential Eqaution (36) is the *Euler-Maruyama Scheme* given by

$$Y_{n+1} = Y_n + f\left(t_n, Y_n\right)\int_{t_n}^{t_{n+1}}\mathrm{d}s + g\left(t_n, Y_n\right)\int_{t_n}^{t_{n+1}}\mathrm{d}W_s \tag{38}$$

where one usually writes

$$\Delta_n = \int_{t_n}^{t_{n+1}}\mathrm{d}s,\ \ \Delta W_n = \int_{t_n}^{t_{n+1}}\mathrm{d}W_s,$$

for $n = 0, 1, \cdots, M_T - 1$ and where $t_0 < t_1 < \cdots < t_M = T$ with $M_T \in \mathbb{N}$ is an arbitrary partition of $\left[t_0, T\right]$. The Euler-Maruyama approximation of an m-dimensional stochastic differential equation $X^h = \left(X_1^h, X_2^h, \cdots, X_m^h\right)$ is defined by [38]

$$X_{t_{p+1}}^h = X_{t_p}^h + \mu\left(t_p, X_{t_p}^h\right)h + \sigma\left(t_p, X_{t_p}^h\right)\Delta B_p \tag{39}$$

$$X_0^h = x, \Delta B_p := B_{t_{p+1}} - B_{t_p}, t_p = ph.$$

As a strong approximation, it is of order 1/2, while as a weak approximation it s of order 1. In other words, $\sup_{t\leq T}\mathbf{E}\left|X_t^h - X_t\right| = O\left(h^{1/2}\right)$ and $\mathbf{E}f\left(X_t^h\right) - \mathbf{E}f\left(X_t\right) = O\left(h\right)$, $h \to 0$, for all $f \in \mathbf{C}_\mu^4\left(\mathbb{R}^m\right)$.

The Milstein Scheme The Mistein method is a technique for the approximate numerical solution of a stochastic differential equation. It is named after Russian mathematician Grigori N. Milstein (who first published the method in 1974. The another useful numerical scheme for the SODE (36) is the *Milstein Scheme* given in [52] by

$$
\begin{aligned}
Y_{n+1} &= Y_n + f\left(t_n, Y_n\right)\int_{t_n}^{t_{n+1}} \mathrm{d}s + g\left(t_n, Y_n\right)\int_{t_n}^{t_{n+1}} \mathrm{d}W_s \\
&\quad + g\left(t_n, Y_n\right)\frac{\partial g}{\partial x}\left(t_n, Y_n\right)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} \mathrm{d}W_u \mathrm{d}W_s
\end{aligned} \tag{40}
$$

The Milstein approximation of an m-dimensional stochastic differential equation $X^h = \left(X_1^h, X_2^h, \cdots, X_m^h\right)$ is defined by [38]

$$
X_{it_{p+1}}^h = X_{it_p}^h + \mu_i\left(t_p, X_{it_p}^h\right)h + \sum \sigma_{ij}\left(t_p, X_{it_p}^h\right)\Delta B_p^j + \sum_{j,l,q} \frac{\partial \sigma_{iq}}{\partial x_j}\sigma_{jl}\Delta C_p^{lq},
$$

$$
\Delta C_p^{lq} := \int_{t_p}^{t_{p+1}}\left(B_s^l - B_{t_p}\right)\mathrm{d}B_s^q, X_{it_p}^h = x^i, t_p = ph.
$$

The Runge-Kutta Scheme. The Runge-Kutta methods are a family of implicit and explicit iterative methods, which include the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations. These methods were developed around 1900 by the German mathematicians Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944). Consider an m-dimensional Stratonovich differential equation of the form [62] [63]

$$
\mathrm{d}X = f\left(t, X\right)\mathrm{d}t + g\left(t, X\right)\circ \mathrm{d}W\left(t\right),\ X\left(t_0\right) = X_0, \tag{41}
$$

where f is an m-vector-valued function, g is an $m \times p$ matrix-valued function and $W\left(t\right)$ is a p-dimensional process having independent scalar Wiener process components and the solution $X\left(t\right)$ is an m-vector process. A general class of stochastic Runge-Kutta method in which [62] [63]

$$
X_i = x_n + h\sum_{j=1}^{s} a_{ij} f\left(X_j\right) + \sum_{j=1}^{s} Z_{ij} g\left(X_j\right), i = 1, \cdots, s \tag{42}
$$

$$
x_{n+1} = x_n + h\sum_{j=1}^{s} \alpha_i f\left(X_j\right) + \sum_{j=1}^{s} Z_j g\left(X_j\right), \tag{43}
$$

where Z and z are respectively, an $s \times s$ matrix and $s \times 1$ vector whose elements are themselves arbitrary random variables. By letting

$$
Z_{ij} = b_{ij}^{(1)} J_1 + b_{ij} J_{10}/h, i, j = 1, \cdots, s,
$$

$$
Z_j = \gamma_j^{(1)} J_1 + \gamma_j^{(2)}/h,\ j = 1, \cdots, s.
$$

Here J_{10} represents the Stratonovich multiple integral of order two given by

$$
J_{10} = \int_{t_0}^{t}\int_{t_0}^{s} \circ \mathrm{d}W\left(s_t\right)\mathrm{d}s.
$$

This method is defined to be of order p if the local truncation error is $O\left(h^p\right)$.

5.2. Numerical Methods for Stochastic Differential Equations with Jumps

The Euler scheme for SDE with jumps (32), is given by the algorithm, Platen [53] [64] [65]

$$
Y_{n+1} = Y_n + a\Delta_n + b\Delta W_n + \int_{t_n}^{t_{n+1}}\int_{\varepsilon} c\left(v\right) p_{\varphi}\left(\mathrm{d}v, \mathrm{d}z\right)
$$

$$Y_{n+1} = Y_n + a\Delta_n + b\Delta W_n + \sum_{i=p_\varphi(t)+1}^{p_\varphi(t_{n+1})} c(\xi_i) \tag{44}$$

for $n \in \{0,1,\cdots,N-1\}$ with initial value $Y_0 = X_0$. Here $\Delta_n = t_{n+1} - t_n$ is the length of the time interval $[t_n, t_{n+1}]$ and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ is the n^{th} Gaussian $N(0,\Delta n)$ distributed increment of the Wiener process W, $n \in \{0,1,\cdots,N-1\}$, $p_\varphi(t) = p_\varphi(\varepsilon,[0,t])$ represents the total number of jumps of Poisson random measure up to time t, which is Poisson distributed with mean λt.

In the multidimensional case with mark-indepedent jump size we obtain the k^{th} component of the Euler scheme

$$Y_{n+1}^k = Y_n^k + a^k\Delta_n + \sum_{i=p_\varphi(t)+1}^{p_\varphi(t_{n+1})} b^{k,j}\Delta W_n + c^k\Delta p_n. \tag{45}$$

5.3. Numerical Methods for Stochastic Partial Differential Equations

This material is from [66]

$$\begin{aligned} X\left(t_{j+1}^n\right) &= S\left(t_{j+1}^n - t_j^n\right)X\left(t_j^n\right) + \int_{t_j^n}^{t_{j+1}^n} S\left(t_{j+1}^n - s\right)B\Big(S\left(s - t_j^n\right)X\left(t_j^n\right) \\ &\quad + \int_{t_j^n}^{s} BX(r)\mathrm{d}r + \int_{t_j^n}^{s} S(s-r)G(X(r))\mathrm{d}M(r)\Big)\mathrm{d}s \\ &\quad + \int_{t_j^n}^{t_{j+1}^n} S\left(t_{j+1}^n - s\right)B\Big(S\left(s - t_j^n\right)X\left(t_j^n\right) \\ &\quad + \int_{t_j^n}^{s} BX(r)\mathrm{d}r + \int_{t_j^n}^{s} S(s-r)G(X(r))\mathrm{d}M(r)\Big)\mathrm{d}M(s) \end{aligned} \tag{46}$$

For SPDE with multiplicative noise, (27), there are two stochastic numerical methods that are used in the literature the linear-mplicit Euler and the linear-implicit Crank-Nicolson schemes [52].

The Euler-Maruyama scheme

$$\begin{aligned} Y_{k+1}^{N,M,L} &= \left(I - hA_N\right)^{-1}\left(Y_{k+1}^{N,M,L} + hF_N\left(Y_{k+1}^{N,M,L}\right)\right) \\ &\quad + \left(I - hA_N\right)^{-1} B_{N,L}\left(Y_{k+1}^{N,M,L}\right)\Delta W_k, \mathbb{P} - a.s. \end{aligned} \tag{47}$$

The Crank-Nicolson scheme. The Crank-Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. It is implicit in time and can be written as an implicit Runge-Kutta method, and it is numerically stable. The method was developed by a British mathematical physicist John Crank (1867-1944) and a British mathematician Phyllis Nicolson (1916-1968) in the mid 20th century [67].

$$\begin{aligned} Y_{k+1}^{N,M,L} &= \left(I - \frac{h}{2}A_N\right)^{-1}\left(\left(I + \frac{h}{2}A_N\right)Y_{k+1}^{N,M,L} + hF_N\left(Y_{k+1}^{N,M,L}\right)\right) \\ &\quad + \left(I - \frac{h}{2}A_N\right)^{-1} B_{N,L}\left(Y_k^{N,M,L}\right)\Delta W_k, \mathbb{P} - a.s. \end{aligned} \tag{48}$$

for $k \in \{0,1,\cdots,M-1\}$ and $N, M, L \in \mathbb{N}$. Here it is necessary to assume that $\lambda \ge 0$ for all $i \in \mathcal{L}$ in Assumption ( ) in order to ensure that $(I - hA)$ is invertible for every $h \ge 0$.

Convergence of SPDE with multiplicative noise. The convergence of the exponential Euler scheme will proved under the following assumptions.

Assumption 5.0.1 (A5). (Linear operator A). There exist sequences of real eigenvalues $0<\lambda_1\le\lambda_2\le\cdots$ and orthonormal eigenfunctions $(e_n)_{n\ge1}$ of $-A$ such that the linear operator $A:D(A)\in H\to H$ is given by

$$Av=\sum_{n=1}^{\infty}-\lambda_n\langle e_n,v\rangle,$$

for all $v\in D(A)$ with $D(A)=\left\{v\in H:\sum_{n=1}^{\infty}|\lambda_n|^2\langle e_n,v\rangle^2<\infty\right\}$.

(A6) (nonlinearity F). The nonlinearity $F:H\to H$ is two times continuously Fréchet differentiable and its derivatives satisfy

$$\|F'(x)-F'(y)\|\le L|x-y|_H,$$

$$\left|(-A)^{-(-r)}F'(x)(-A)^r v\right|_H\le L|v|_H,$$

for all $x,y\in H$, $v\in D\left((-A)^r\right)$, and $r=0,1/2,1$, and

$$\left|A^{-1}F''(x)(v,w)\right|\le L\left|(-A)^{-1/2}v\right|_H\left|(-A)^{-1/2}w\right|_H,$$

for all $x,y\in H$, where $L>0$ is a positive constant.

(A7) (Cylindrical Q-Wiener process W_t). There exist a sequence $(q_n)_{n\ge1}$ of positive real numbers and a real number $\gamma\in(0,1)$ such that

$$\sum_{n=1}^{\infty}\lambda_n^{2\gamma-1}q_n<\infty$$

and pairwise independent scalar $\mathcal{F}_t$-adapted Wiener process $(W_t)_{t\ge0}$ for $n\ge1$. The cylindrical Q-Wiener process W_t is given formally by

$$W_t=\sum_{n=1}^{\infty}\sqrt{q_n}W_t^n e_n. \tag{49}$$

(A8) (Initial value). The random variable $x_0:\Omega\to D\left((-A)^\gamma\right)$ satisfies $E\left|(-A)^\gamma x_0\right|_H^4<\infty$, where $\gamma>0$ is given in A7.

5.4. Numerical Methods for Stochastic Delay Differential Equations

There are many numerical schemes for solving stochastic delay differential equations. As given in [54], we give three schemes to solve (35). The first scheme is the **Predictor-correction scheme** given by

$$\begin{aligned}X_{n+1}&=X_n+\frac{h}{2}\left[f(X_n,X_{n-m})+f\left(\bar{X}_{n+1},X_{n-m+1}\right)\right]\\&\quad+\frac{1}{2}\sum_{l=1}^{r}\left[g_l(X_n,X_{n-m})+g_l\left(\bar{X}_{n+1},X_{n-m+1}\right)\right]\Delta W_{l,n},\end{aligned} \tag{50}$$

$$\bar{X}_{n+1}=X_n+hf(X_n,X_{n-m})+\sum_{l=1}^{r}g_l(X_n,X_{n-m})\Delta W_{l,n} \tag{51}$$

The second is the **Midpoint scheme** given by

$$X_{n+1} = X_n + hf\left(\frac{X_n + X_{n+1}}{2}, \frac{X_{n-m} + X_{n-m+1}}{2}\right) + \sum_{l=1}^{r}\left[g_l\left(\frac{X_n + X_{n+1}}{2}, \frac{X_{n-m} + X_{n-m+1}}{2}\right)\right]\Delta\bar{W}_{l,n} \tag{52}$$

where we have $\Delta W_{l,n}$ with $\Delta\bar{W}_{l,n}$.

The last scheme is the **Milstein-like scheme** given by

$$\begin{aligned} X_{n+1} &= X_n + hf\left(X_n, X_{n-m}\right) + \sum_{l=1}^{r} g_l\left(X_n, X_{n-m}\right) I_0 \\ &+ \sum_{l=1}^{r}\sum_{l=1}^{r} \partial_x g_l\left(X_n, X_{n-m}\right) g_q\left(X_n, X_{n-m}\right) I_{q,l,t_n,t_{n+1},0} \\ &+ \sum_{l=1}^{r}\sum_{l=1}^{r} \partial_{x_\tau} g_l\left(X_n, X_{n-m}\right) g_q\left(X_{n-m}, X_{n-2m}\right) \chi_{t_n \ge \tau} I_{q,l,t_n,t_{n+1}} \end{aligned} \tag{53}$$

where $I_0 = \int_{t_n}^{t_{n+1}} \mathrm{d}\tilde{W}_l(t)$, $I_{q,l,t_n,t_{n+1},0} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \mathrm{d}\tilde{W}_q(s)\mathrm{d}\tilde{W}_l(t)$,

$I_{q,l,t_n,t_{n+1},\tau} = \int_{t_n}^{t_{n+1}}\int_{t_n-\tau}^{t-\tau} \mathrm{d}\tilde{W}_q(s)\mathrm{d}\tilde{W}_l(t)$, $t_{n+1} \ge 0$,

$\int_{t_n}^{t_{n+1}} \mathrm{d}\tilde{W}_l(t) = \Delta W_{l,n} = W_l(t_{n+1}) - W_l(t_n)$.

6. Convergence and Implementation of Numerical Methods

6.1. Convergences of Numerical Methods

Definition 6.0.1. (Strong Convergence) We say that a numerical scheme for solving the SDE (36) converges strongly on $[0,T]$ to the solution X of the SDE if for the final time T have

$$\lim_{\delta\to 0}\mathbb{E}\left(\left|X(T) - Y^\delta(T)\right|\right) = 0. \tag{54}$$

A strongly convergent scheme is said o have convergence rate γ if for some constants C and $\delta_0 > 0$ we have

$$\lim_{\delta\to 0}\mathbb{E}\left(\left|X(T) - Y^\delta(T)\right|\right) \le C\delta^\gamma, \forall\delta \in [0,\delta_0]. \tag{55}$$

Theorem 6.1. (Strong Convergence: Euler-Maruyama scheme) Under assumptions of Lipschitz and linear growth of coefficients and additionally

$$|a(t,x) - a(s,x)| + |\sigma(t,x) - \sigma(s,x)| \le K(1+|x|)|t-s|^{1/2} \tag{56}$$

for some suitable constant K, the Euler-Maruyama scheme converges strongly with a convergence rate of $\gamma = 1/2$.

Theorem 6.2. (Strong convergence of Milstein scheme) In addition to the assumption of Theorem , let $\sigma(t,x)$ and $\frac{\partial\sigma(t,x)}{\partial x}$ satisfy the conditions on the coefficients of the Theorem . If further we have $a \in C^{1,1}$, $\sigma \in C^{1,2}$, then the Milstein converges strongly with a convergence rate of $\gamma = 1$.

The convergence of error for SPDE is given by the following theorem.

Theorem 6.3. (Convergence Theorem, [52]) Suppose that assumptions (A1)-(A8) are satisfied. Then there is a constant $C_T > 0$ such that

$$\sup_{k=0,\cdots,M}\left(\mathbb{E}\left|X_{t_k}-Y_k^{(N,M)}\right|_H^2\right)^{\frac{1}{2}}\le C_T\left(\lambda_N^{-\gamma}+\frac{\log(M)}{M}\right), \tag{57}$$

holds for all $N,M\in\mathbb{N}$, where X_t is the solution of SPDE (27), $Y_k^{(N,M)}$ is the numerical solution given by (46), $t_k=T\frac{k}{M}$ for $k=0,1,\cdots,M$, and $\gamma>0$ is the constant given in Assumption (**A8**).

6.2. Implementation of Numerical Methods

As with the Euler-Maruyama method, the Milstein method is very easy to implement which is a reason that it is also quite popular among practitioners in finance.

Algorithm 6.3.1. (The Euler-Maruyama Scheme). Let $\Delta t:=T/N$ for a given N. Then approximate the SDE via

1) Set $Y_N(0)=X(0)=x_0$

2) For $j=0$ to $N-1$ do

a) Simulate a standard normally distributed random number Z_j

b) Set $\Delta W(j\Delta t)=\sqrt{\Delta t}Z_j$ and

$Y_N((j+1)\Delta t)=Y_N(j\Delta t)+a(j\Delta t,Y_N(j\Delta t))\Delta t+\sigma(j\Delta t,Y_N(j\Delta t))\Delta W(j\Delta t)$.

Algorithm 6.3.2. (The Milstein Scheme) Let $\Delta t:=T/N$ for a given N. Then approximate the SDE via

1) Set $Y_N(0)=X(0)=x_0$

2) For $j=0$ to $N-1$ do

a) Simulate a standard normally distributed random number Z_j

b) Set $\Delta W(j\Delta t)=\sqrt{\Delta t}Z_j$ and

$$\begin{aligned}Y_N((j+1)\Delta t)&=Y_N(j\Delta t)+a(j\Delta t,Y_N(j\Delta t))\Delta t+\sigma(j\Delta t,Y_N(j\Delta t))\Delta W(j\Delta t)\\&\quad+\frac{1}{2}\sigma(j\Delta t,Y_N(j\Delta t))\sigma'(j\Delta t,Y_N(j\Delta t))\left(\Delta W(j\Delta t)^2-\Delta t\right).\end{aligned}$$

7. Conclusion

This paper surveys the recent development of numerical methods used in stochastic analysis that can be useful in econometric analysis. As well-known, the discretization of the stochastic continuous-time models through the numerical methods is one of main cornerstones and problems of the modern econometric analysis. Modelling and analyzing economical dynamical systems under uncertainties through the stochastic differential equations are considered as the challenges for economists. In this paper we give these numerical methods such as Euler-Maruyama scheme, Runge-Kutta scheme, Milstein scheme and Crank-Nicolson scheme that are used in literature. Since the Black-Scholes-Merton works awarded by Nobel Prize Committee in 1997 in Economics field, the stochastic differential equations are used in economics and finance as one of the best ways to model uncertainties.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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