Open Access Library Journal
2019, Volume 6, e5973
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Fixed Point Theorem for Meir-Keeler Type Function in b_2-Metric Spaces

Zhuoyi Tian, Jinxing Cui, Linan Zhong*

Department of Mathematics, Yanbian University, Yanji, China
Email: *zhonglinan2000@126.com

How to cite this paper: Tian, Z.Y., Cui, J.X. and Zhong, L.N. (2019) Fixed Point Theorem for Meir-Keeler Type Function in b_2-Metric Spaces. *Open Access Library Journal*, **6**: e5973.
https://doi.org/10.4236/oalib.1105973

Received: December 3, 2019
Accepted: December 21, 2019
Published: December 24, 2019

Abstract

In this paper, we prove fixed point theorems of a generalization which is related to the concept of Meir-Keeler function in a complete b_2-metric space. And we know it extends and generalizes some known results in metric space to b_2-metric space.

Subject Areas

Function Theory, Functional Analysis, Mathematical Analysis

Keywords

Fixed Point, b_2-Metric Space, Meir-Keeler Function

1. Introduction

Many mathematicians have studied fixed point theory over the last several decades since Banach contraction principle [1] was introduced in 1992. The notion of Meir-Keeler function [2] was introduced in 1969. Then the concept of weaker Meir-Keeler function [3] was introduced by Chi-Ming Chen in 2012. And in this paper, we establish fixed point for Meir-Keeler function and weaker Meir-Keeler function in a complete new type of generalized matric space, which is called by b_2-metric space, and this space was generalized from both 2-metric space [4] [5] [6] and b-metric space [7] [8].

2. Preliminaries

Throughout this paper N will denote the set of all positive integers and R will denote the set of all real numbers.

Before stating our main results, some necessary definitions might be introduced as follows.

*Corresponding author.

Definition 2.1 [2] Let X be a nonempty subsets, $m \in N$ and $f : X \to X$ an operator. Then $X = \bigcap_{i=1}^{m} A_i$ is called a cyclic representation of X with respect to f if

1) $A_i, i = 1, 2, \cdots, m$ are empty subsets of X,

2) $f(A_i) \subset A_2, f(A_2) \subset A_3, \cdots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Definition 2.2 [2] A function $\phi : (0 \to \infty] \to (0 \to \infty]$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for each $t \in (0 \to \infty]$ with $\eta \le t \le \eta + \delta$, we have $\phi(t) < \eta$.

Definition 2.3 [3] We call $\phi : (0 \to \infty] \to (0 \to \infty]$ a weak Meir-Keeler function if for each $\eta > 0$ such that for each $t \in (0 \to \infty]$ with $\eta \le t \le \eta + \delta$, there exists $n_0 \in N$ such that $\phi^{n_0}(t) < \eta$.

Definition 2.4 [4] [5] [6] Let X be an nonempty set and let $d : X \times X \times X \to R$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \ne 0$.

2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$,

3) The symmetry:

$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,x,y)$ for all $x, y, z \in X$.

4)The rectangle inequality:

$d(x,y,z) \le d(x,y,a) + d(y,z,a) + d(z,x,a)$ for all $x, y, z, a \in X$.

Then d is called a 2 metric on X and (X, d) is called a 2 metric space.

Definition 2.5 [7] [8] Let X be a nonempty set and $s \ge 1$ be a given real number. A

function $d : X \times X \to R^+$ is a b metric on X if for all $x, y, z \in X$, the following conditions hold:

1) $d(x, y) = 0$ if and only if $x = y$.

2) $d(x, y) = d(y, x)$.

3) $d(x, y) \le s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b metric space.

Definition 2.6 [9] Let X be a nonempty set, $s \ge 1$ be a real number and let $d : X \times X \times X \to R$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \ne 0$.

2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$,

3) The symmetry:

$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,x,y)$ for all $x, y, z \in X$.

4) The rectangle inequality:

$d(x,y,z) \le s[d(x,y,a) + d(y,z,a) + d(z,x,a)]$, for all $x, y, z, a \in X$.

Then d is called a b_2 metric on X and (X, d) is called a b_2 metric space with parameter s. Obviously, for $s = 1$, b_2 metric reduces to 2-metric.

Definition 2.7 [9] Let $\{x_n\}$ be a sequence in a b_2 metric space (X, d).

1) A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if all $a \in X$ $\lim_{n\to\infty} d(x_n, x, a) = 0$.

2) $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$. for all $a \in X$.

3) (X, d) is said to be complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

Definition 2.8 [9] Let (X, d) and (X', d') be two b_2-metric spaces and let $f: X \to X'$ be a mapping. Then f is said to be b_2-continuous,at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $d'(fz, fx, a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.

Definition 2.9 [9] Let (X, d) and (X', d') be two b_2-metric spaces. Then a mapping $f: X \to X'$ is b_2-continuous at a point $x \in X'$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.

3. Main Results

In this section, we give and prove a generalization of the Meir-Keeler fixed point theorem [2].

Theorem 3.1. Let (X, d) be a complete b_2-metric space and let f be a mapping on X, for each $\varepsilon > 0$, there exists $\delta \in (s\varepsilon, (2s-1)\varepsilon)$ such that

(a) $\frac{1}{2s} d(x, fx, a) < d(x, y, a)$ and $d(x, y, a) < \varepsilon + \delta$ imply $d(fx, fy, a) \le \varepsilon$

(b) $\frac{1}{2s} d(x, fx, a) < d(x, y, a)$ implies $d(fx, fy, a) < d(x, y, a)$ for all $x, y \in X$. Then there exists a unique fixed point z of f. Moreover $\lim_{n\to\infty} f^n x = z$ for all $x \in X$.

Proof If $fx \ne x$, then we can easily get that $d(x, fx, a) < 2sd(x, fx, a)$. So, by hypothesis, $d(fx, f^2x, a) < d(x, fx, a)$ holds for all $x \in X$ with $fx \ne x$. We also get

$$d(fx, f^2x, a) \le d(x, fx, a) \text{ for all } x \in X \tag{3.1}$$

Fix point x_0 in X and define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^n x_0$ for $n \in N$. From the above (3.1) we get $d(x_n, x_{n+1}, a) \le d(x_{n-1}, x_n, a)$, so we know that $\{d(x_n, x_{n+1}, a)\}$ is a decreasing sequence, and the sequence $\{d(x_n, x_{n+1}, a)\}$ converges to some $\beta \ge 0$. We assume that $\beta > 0$, then we know that $d(x_n, x_{n+1}, a) > \beta$ for every $n \in N$, then there exists δ such that (a) is true with $\varepsilon = \beta$, for the definition of β, there exists $i \in N$ such that $d(x_i, x_{i+1}, a) < \beta + \delta$, so we have $d(x_{i+1}, x_{i+2}, a) \le \beta$, which is a contraction. Therefore $\beta = 0$, and that is:

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0.$$

Now we show that $d(x_i, x_j, x_k) = 0$.

From part 2 of Definition 2.6, the equation $d(x_m, x_m, x_{m-1}) = 0$ is obtained.

Since $\{d(x_n, x_{n+1}, a)\}$ is decreasing, if $d(x_{n-1}, x_n, a) = 0$, then $d(x_n, x_{n+1}, a) = 0$, then it is easy to get

$$d(x_n, x_{n+1}, x_m) = 0, \text{ for all } n+1 \ge m. \tag{3.2}$$

For $0 \le n+1 < m$, we get $m-1 \ge n+1$ and that is $m-2 \ge n$, from (3.2)

$$d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0, \tag{3.3}$$

From (3.2) and triangular inequality,

$$\begin{aligned} & d(x_n, x_{n+1}, x_m) \\ & \le sd(x_n, x_{n+1}, x_{m-1}) + sd(x_{n+1}, x_m, x_{m-1}) + d(x_m, x_n, x_{m-1}) \\ & = sd(x_n, x_{n+1}, x_{m-1}). \end{aligned}$$

And since $d(x_n, x_{n+1}, x_{n+1}) = 0$, and from the inequality above,

$$d(x_{n+1}, x_n, x_m) \le s^{m-n-1} d(x_{n+1}, x_{n+1}, x_n) = 0, \text{ for all } 0 \le n+1 \le m. \tag{3.4}$$

Now for all $i, j, k \in N$, the condition of $j > i$ is considered here, from the above equation

$$d(x_{j-1}, x_j, x_i) = d(x_k, x_{j-1}, x_j) = 0 \tag{3.5}$$

From (3.5) and triangular inequality, therefore

$$\begin{aligned} d(x_i, x_k, x_j) & \le s\left[d(x_i, x_j, x_{j-1}) + d(x_j, x_{k-1}, x_k) + d(x_i, x_{j-1}, x_k)\right] \\ & \le \cdots \\ & \le s^{j-1} d(x_i, x_k, x_i) \\ & = 0 \end{aligned}$$

In conclusion, the result below is true

$$d(x_j, x_k, x_i) = 0, \text{ for all } i, j, k \in N. \tag{3.6}$$

Now we fix $\varepsilon > \frac{\delta}{2s-1}$, then there exists δ such that (a) is true. Let $N_1 \in N$ such that

$$d(x_n, x_{n+1}, a) < \frac{\delta - (s-1)\varepsilon}{s}, \text{ for all } n \in N \text{ with } n \ge N_1. \tag{3.7}$$

Now we will show that

$$d(x_k, x_{k+m}, a) < \varepsilon + \delta \text{ for } m \in N \tag{3.8}$$

By induction, when $m = 1$, it is true for (3.8). We assume that (3.8) holds for some $m \in N$.

In one case $d(x_k, x_{k+m}, a) \le \varepsilon$, we have

$$d(x_k, x_{k+m+1}, a) \le s\left[d(x_k, x_{k+m}, a) + d(x_{k+m}, x_{k+m+1}, a) + d(x_k, x_{k+m+1}, x_{k+m})\right]$$

From (3.6) and (3.7) we have

$$\begin{aligned} d(x_k, x_{k+m+1}, a) & \le s\left[d(x_k, x_{k+m}, a) + d(x_{k+m}, x_{k+m+1}, a)\right] \\ & \le s\varepsilon + s\frac{\delta - (s-1)\varepsilon}{s} \\ & \le \varepsilon + \delta \end{aligned} \tag{3.9}$$

In other case, where $\varepsilon < d(x_k, x_{k+m}, a) < \varepsilon + \delta$, since

$$d(x_k, x_{k+1}, a) < \frac{\delta - (s-1)\varepsilon}{s} < \varepsilon < d(x_k, x_{k+m}, a) < 2sd(x_k, x_{k+m}, a)$$

We get $d(x_{k+1}, x_{k+m+1}, a) \le \varepsilon$ and then we have

$$d(x_k, x_{k+m+1}, a) \le s\left[d(x_k, x_{k+m}, a) + d(x_{k+m}, x_{k+m+1}, a)\right] \le \varepsilon + \delta \quad (3.10)$$

So for (3.9) and (3.10), (3.8) is true for every $m \in N$. Therefore we have $\limsup_{n\to\infty} d(x_n, x_m, a) = 0$, for all $n < m$. This shows that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point $z \in X$ such that sequence $\{x_n\}$ converges to it. From the following two respectively cases, we will show that this point is a fixed point for *f*.

Case one: There exists $u \in N$ such that $x_u = x_{u+1}$.

Case two: $x_n \ne x_{n+1}$, for all $n \in N$.

In the first case, we know that $x_n = x_u$ for $n \ge u, u \in N$. Since $\{x_n\} \to z$ as $n \to \infty$, then we get $x_n = z$ for $n \ge u, n \in N$. This prove that $fz = z$.

In the second case, we know that $x_n \ne x_{n+1} = fx_n$, for all $n \in N$, so we get sequence $\{d(x_n, x_{n+1}, a)\}$ is strictly decreasing. If we assume that

$d(x_n, x_{n+1}, a) \ge 2sd(x_n, z, a)$ and $d(x_{n+1}, x_{n+2}, a) \ge 2sd(x_{n+1}, z, a)$

for some $n \in N$. For the first inequality of the above assumption, we choose $a = x_{n+1}$, then we have

$$d(x_n, x_{n+1}, z) = 0 \quad (3.11)$$

Then we have

$$\begin{aligned} d(x_n, x_{n+1}, a) &\le s\left[d(x_n, z, a) + d(x_{n+1}, z, a) + d(x_n, x_{n+1}, a)\right] \\ &\le s\left[d(x_n, z, a) + d(x_{n+1}, z, a)\right] \\ &\le \frac{d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a)}{2} \\ &< d(x_n, x_{n+1}, a). \end{aligned}$$

This is a contraction. So we get either

$d(x_n, x_{n+1}, a) < 2sd(x_n, z, a)$ or $d(x_{n+1}, x_{n+2}, a) < 2sd(x_{n+1}, z, a)$ for all $n \in N$. Since $x_n \to z$ as $n \to \infty$, the above inequality prove that there exists a sub sequence of sequence $\{x_n\}$, which converges to *fz*. This shows that z is a fixed point of *f*. Next we prove that z is the unique fixed point of *f*. Suppose that z and y are two different fixed point of *f*, from the assumption of this theorem, we get

$d(z, fz, a) = 0 < 2d(y, z, a)$ from the above inequality we have

$$d(z, y, a) = d(fx, fy, a) < d(y, z, a)$$

This is a contraction. Hence z is a unique fixed point of *f*. □

In this section, we prove a fixed point theory for the cyclic weaker Meir-Keeler function in b_2-metric space. Now we give some comments as follows:

$\Omega = \omega : (0 \to \infty] \to (0 \to \infty]$ is a set, where ω is a weaker Meir-Keeler function and satisfying the following conditions:

(ω_1) $\omega(t)>0$ for $t>0$, and $\omega(0)=0$;

(ω_2) For all $t\in(0\to\infty]$, $\{\omega^n(t)\}_{n\in N}$ is decreasing;

(ω_3) For $t_n\in(0\to\infty]$, if $\lim_{n\to\infty}t_n=\gamma$, then $\lim_{n\to\infty}\omega(t_n)<\gamma$.

$\Theta=\theta:[0\to\infty)\to[0\to\infty)$, where θ is a non-increasing and continuous function with $\theta(t)>0$ for all $t>0$ and $\theta(0)=0$.

We now introduce the following definition of cyclic weaker (ω,θ)-contraction mapping in b_2-metric space:

Definition 3.2 Let (X,d) be a b_2-metric space, $A_1,A_2,\cdots,A_m$ are all non-empty subsets of $X=\bigcup_{i=1}^m A_i$. A mapping $f:X\to X$ is said to be cyclic weaker (ω,θ)-contraction in b_2-metric space if satisfying the following condition:

1) $X=\bigcup_{i=1}^m A_i$ with respect of *f*, it is a cyclic representation of *X*.

2) $i=1,2,\cdots,m$, for any $x\in A_i, y\in A_{i+1}$, such that

$d(fx,fy,a)\le\omega(d(x,y,a))-\theta(d(x,y,a))$, where $A_{m+1}=A_1$, $\omega\in\Omega$ and $\theta\in\Theta$.

Theorem 3.3 Let (X,d) be a b_2-metric space, $A_1,A_2,\cdots,A_m$ are all non-empty subsets of $X=\bigcup_{i=1}^m A_i$. Let $f:X\to X$ be cyclic weaker (ω,θ)-contraction in b_2-metric space, then *f* has a unique fixed point in $\bigcap_{i=1}^M A_i$.

Proof Let x_0 be an arbitrary point in *X* and we define a sequence $\{x_n\}$ by $x_{n+1}=fx_n=f^{n+1}x_0$, for all $n\in N$, if there exists some $n_0\in N$ such that $fx_{n_0-1}=fx_{n_0}$ then $fx_{n_0}=fx_{n_0}$. Thus x_{n_0} is a fixed point of *f*. Suppose that $fx_{n-1}\ne fx_n$ for all $n\in N$, we know that there exists $i_n\in1,2,\cdots,m$ such that $x_{n-1}\in A_{i_n}$ and $x_n\in A_{i_{n+1}}$ for any $n>0$. Since $f:X\to X$ be cyclic weaker (ω,θ)-contraction, we get

$$\begin{aligned}d(x_n,x_{n+1},a)=d(fx_{n-1},fx_n,a)&\le\omega(d(x_{n-1},x_n,a))-\theta(d(x_{n-1},x_n,a))\\&\le\omega(d(x_{n-1},x_n,a))\\&\le\omega(\omega(d(x_{n-1},x_{n-2},a)))=\omega^2(d(x_{n-1},x_{n-2},a))\\&\le\cdots\\&\le\omega^n(d(x_0,x_1,a)).\end{aligned}$$

Since sequence $\{\omega^n(d(x_0,x_1,a))\}$ is decreasing for all $n\in N$, and this sequence must converge to some $\rho\ge0$. We get $\rho=0$ by the following assumption.

First we assume that $\rho>0$, since ω is defined as a weaker Meir-Keeler function, there exists δ such that $\rho\le d(x_0,x_1,a)<\delta+\rho$ for $x_0,x_1\in X$, there exists $n_0\in N$ such that $\omega^n(d(x_0,x_1,a))<\rho$, from $\lim_{n\to\infty}\omega^n d(x_0,x_1,a)=\rho$, we know that there exists $p_0\in N$ such that $\rho<\omega^p(d(x_0,x_1,a))<\rho+\delta$, for all $p\ge p_0$. Thus we get a conclusion $\omega^{p_0+n_0}(d(x_0,x_1,a))<\rho$, which is a contraction. Thus $\lim_{n\to\infty}\omega^n d(x_0,x_1,a)=0$, and that is, $\lim_{n\to\infty}d(x_n,x_{n+1},a)=0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence.

Suppose to the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sub sequences $\{n_i\}$ and $\{m_i\}$ such that $i < m_i < n_i$ and

$$d\left(x_{m_i}, x_{n_i}, a\right) \geq \varepsilon \quad \text{and} \quad d\left(x_{m_i}, x_{n_i\text{-}1}, a\right) < \frac{\varepsilon}{s} < \varepsilon \tag{3.12}$$

From the part 4 of Definition 3.6 and (3.6), we get

$$\begin{aligned} d\left(x_{m_i}, x_{n_i}, a\right) &\leq s\left[d\left(x_{m_i}, x_{m_i+1}, a\right) + d\left(x_{m_i+1}, x_{n_i}, a\right) + d\left(x_{m_i}, x_{n_i}, x_{m_i+1}\right)\right] \\ &\leq s\left[d\left(x_{m_i}, x_{m_i+1}, a\right) + d\left(x_{m_i+1}, x_{n_i}, a\right)\right] \end{aligned}$$

Taking $i \to \infty$, from (3.6) and (3.12) we have

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} d\left(x_{m_i+1}, x_{n_i}, a\right) \tag{3.13}$$

Now by using the condition that *f* is a cyclic weaker (ω, θ)-contraction, we get

$$\begin{aligned} d\left(x_{m_i+1}, x_{n_i}, a\right) &= d\left(x_{m_i}, x_{n_i-1}, a\right) \\ &\leq \omega\left(d\left(x_{m_i}, x_{n_i-1}, a\right)\right) - \theta\left(d\left(x_{m_i}, x_{n_i-1}, a\right)\right) \\ &\leq \omega\left(d\left(x_{m_i}, x_{n_i-1}, a\right)\right) \end{aligned}$$

Letting $i \to \infty$ and using the condition of ω, we get

$$\lim_{n\to\infty} d\left(x_{m_i+1}, x_{n_i}, a\right) < \frac{\varepsilon}{s} \tag{3.14}$$

From (3.13) and (3.14) $\frac{\varepsilon}{s} \leq \lim_{n\to\infty} d\left(x_{m_i+1}, x_{n_i}, a\right) < \frac{\varepsilon}{s}$, which is a contraction. Therefore $\{x_n\}$ is a Cauchy sequence in *X*.

Since *X* is a complete set, there exists a point $z \in \bigcup_{i=1}^m A_i$ such that $n \to \infty$, $\{x_n\} \to z$. For $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of *X* respect to *f*, thus in each A_i for $i \in \{1, 2, \cdots, m\}$, the sequence $\{x_n\}$ has infinite term. A sub sequence $\{x_{n_k}\}$ of $\{x_n\}$, we take this sub sequence and it also all converge to *z*, for all $i = 1, 2, \cdots, m$. Since

$$\begin{aligned} d\left(x_{n_k+1}, fz, a\right) &= d\left(fx_{n_k}, fz, a\right) \\ &\leq \omega\left(d\left(x_{n_k}, z, a\right)\right) - \theta\left(d\left(x_{n_k}, z, a\right)\right) \\ &\leq \omega\left(d\left(x_{n_k}, z, a\right)\right), \end{aligned}$$

From the above inequality, letting $k \to \infty$, we get $d(z, fz, a) = 0$, so $z = fz$.

Now we prove the fixed point is unique for *f*. Suppose there exists another fixed point *y*, since *f* gets the cyclic character, we have $z, y \in \bigcup_{i=1}^m A_i$. Since *f* is a cyclic weaker (ω, θ)-contraction, we get

$$\begin{aligned} d(z, y, a) &= d(z, fy, a) \\ &= \lim_{n\to\infty} d\left(x_{n_k+1}, fy, a\right) = \lim_{n\to\infty} d\left(fx_{n_k}, fy, a\right) \\ &\leq \lim_{n\to\infty}\left[\omega\left(d\left(x_{n_k}, y, a\right)\right) - \theta\left(d\left(x_{n_k}, y, a\right)\right)\right] \\ &\leq d(z, a, y) - \theta\left(d(y, z, a)\right), \end{aligned}$$

then we get

$\theta(d(y,z,a))=0$, that is $y=z$, we get the result of the uniqueness of point *z*. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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