Journal of Applied Mathematics and Physics, 2019, 7, 2283-2302
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Dynamics of Structurally Inhomogeneous Lamellar and Shell Mechanical Systems. Part 1

Mirsaidov Mirziyod Mirsaidovich[1], **Safarov Ismoil Ibrokhimovich**[2], **Teshaev Mukhsin Khudoyberdievich**[3]

[1]Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, Tashkent, Uzbekistan
[2]Tashkent Chemical and Technological Institute, Tashkent, Uzbekistan
[3]Bukhara Engineering and Technological Institute, Bukhara, Uzbekistan
Email: muhsin.teshayev@rambler.ru

How to cite this paper: Mirsaidovich, M.M., Ibrokhimovich, S.I. and Khudoyberdievich, T.M. (2019) Dynamics of Structurally Inhomogeneous Lamellar and Shell Mechanical Systems. Part 1. *Journal of Applied Mathematics and Physics*, **7**, 2283-2302.
https://doi.org/10.4236/jamp.2019.710155

Received: August 12, 2019
Accepted: October 11, 2019
Published: October 14, 2019

Abstract

A vibrational statement, method and algorithm to assess the damping capacity of structurally inhomogeneous viscoelastic mechanical systems, consisting of a package of rectangular plates and shells with point relations and concentrated masses at different rheological properties of deformable elements, are proposed in the paper. To describe rheological properties of the material, the linear hereditary Boltzmann-Volter theory was used. To assess the damping capacity of the system, the problem in question in each case was reduced to solving the proper problems of algebraic equations with complex parameters solved by the Muller method. The accuracy of the methods was demonstrated by comparing the calculated results with known published data and a numerical experiment. Complex natural frequency of the system was used to assess the damping capacity of inhomogeneous viscoelastic systems. Various eigenvalue problems have been solved for structurally inhomogeneous mechanical systems consisting of a package of plate and shell systems with concentrated masses and shock absorbers. A number of new mechanical effects have been discovered, related to the manifestation of the damping capacity of mechanical systems under consideration.

Keywords

Inhomogeneous System, Viscoelasticity, Complex Natural Frequency, Global Damping Coefficient, Package of Plates, Concentrated Mass, Shock Absorber

1. Introduction

Currently, many technical structures are widely used shell and plate structures. Thin-walled pipes (shells), plates and panels in real conditions, as a rule, interact with other structures and bodies, rely on point rigid and (or) elastic, articulated and (or) pinched supports, have point-attached masses. Boundary conditions often differ from homogeneous. If under static loads these features cause a local redistribution of stress and strain fields, then in dynamic problems point connections significantly affect all the dynamic characteristics of a plate or shell—the spectrum of natural frequencies, the forms of natural vibrations, resonant frequencies and amplitudes, etc., for example:

Leissa [1] [2] developed exact solutions for problems with free vibrational vibrations of rectangular plates.

Boay [3] analyzed the natural frequencies of plates with and without concentrated masses using the Rayleigh energy method.

Avalos *et al.* [4] dealt with solving vibration problems using a simple mounted concentrated mass using the well-known normal mode.

Xiang *et al.* [5] used the Ritz method in combination with the variation method to solve the problem of vibration of rectangular plates, which rely on elastic edge supports.

Laura and Grossi [6] [7] calculated the fundamental frequency coefficient for a rectangular plate with edges elastically held against both displacements and rotations using polynomial coordinates and the functions of the Rayleigh-Ritz method.

Wu and Luo [8] solved the problem of eigenfrequencies and corresponding mode shapes of a uniform rectangular plane of a plate carrying any number of point masses using analytical and combined numerical-analytical methods.

Nicholson and Bergman [9] used the Green's function in describing natural modes for damping oscillatory systems. Gorman [10] solved the problem of free vibrations of deformable systems during shear of plates based on homogeneous elastic substrates using the modified Galerkin method. This work is devoted to the application of the differential quadrature method to the vibration of plates resting on elastic bases and bearing any number of spring-loaded masses.

The design of submarine tunnels is inevitably characterized by the presence of attached masses due to structural, strength or operational necessity [11] [12] [13]. Such a system "Shell-mass" in operating conditions is subjected to intense dynamic, in particular, periodic loads, which requires increased accuracy of calculations [14].

Rod systems have a high degree of vibration isolation. E. Skudrzyk [15] considered vibration isolation of a structure consisting of masses and rods connected in series. Lamellar mechanical systems with point supports and lumped masses are also dedicated to the work [16]-[24].

To generalize the methods of the above works for a finite number of mechanical systems, consisting of dissipative plates and shells is an impossible task.

The task is complicated if the plate or shell has hereditary properties, which has recently become increasingly relevant, especially in vibration protection systems [25] [26]. Free vibrations of a dissipative system, which are of a damped nature, were studied in [27] [28] [29] [30] [31]. The amplitudes of the vibration modes decrease over time, therefore, such a process, strictly speaking, is not periodic. But the frequencies of the corresponding forms remain constant, and in this sense, the dissipative system can be investigated as a system with its own vibrations.

Of great practical interest are the dynamic problems of dissipative mechanical systems, the elements of which have different dissipative properties. In the future, such mechanical systems are called structurally inhomogeneous viscoelastic systems, the elements of which have different rheological properties.

An analysis of the work on the natural vibrations of elastic plate mechanical systems shows that practically no methods have been developed for the dynamic calculation of plate mechanical systems whose elements have different rheological properties (structurally inhomogeneous mechanical system).

Structurally heterogeneous mechanical systems with a finite number of degrees of freedom are addressed by I. I. Safarov and *et al.* [32].

In the works of I. I. Safarov and *et al.* [33] [34] [35] ehe linear problem of the natural vibrations of structurally inhomogeneous viscoelastic systems is considered. The structural heterogeneity of the system is determined by the presence of viscoelastic elements with different dissipative properties in it (otherwise it is a structurally homogeneous viscoelastic system).

In the scientific literature there are a large number of publications on the study of the dynamic characteristics of shells bearing attached masses [36] [37]. However, to date, theoretical results are not always consistent with numerical and experimental data [38]. For example, in [39] it was found that the decrease in the smaller of the split eigenfrequencies is stronger than that predicted by the traditional theory [40], but also depends on the geometric parameters of the shell [41]. This circumstance requires clarification of the traditional mathematical model, the development of an effective methodology and algorithm for studying the vibrations of plate and shell dissipative mechanical systems consisting of a package of rectangular plates having point bonds and concentrated masses with various rheological properties of deformable elements [42] [43] [44] [45].

For a package of plates (or shells) with point bonds and concentrated masses, the traditional approach is possible when the directions and coordinates of the active and passive loads must be known.

This is a difficult task. Therefore, the traditional approach is applied for a single plate with concentrated masses. For two or more plates with concentrated masses and point bonds, a variational approach is applied, which allows one to obtain a system of integro-differential equations satisfying boundary conditions and allows one to analyze the dissipative properties of the mechanical system as a whole.

To date, the question of assessing the level of oscillations and dissipative properties of plates and plate systems of new structural elements of various apparatuses is still insufficiently studied and requires extensive research. Therefore, this problem is relevant, requiring its gradual solution.

2. Vibrational Statement of the Problem of Natural Vibrations for Structurally Inhomogeneous Plate and Shell Mechanical Systems with Point Relations and Concentrated Masses

Consider a mechanical system consisting of N isotropic viscoelastic bodies (a package of rectangular plates or cylindrical shells) occupying volume V_n and restricted by surfaces $\Omega_n\left(n=1,2,\cdots,N\right)$. It is assumed that one linear size of each body is much smaller than the other two (the class of plates and shells). For each *n*, homogeneous boundary conditions Ω_n^{bo} are set on parts of the surface of the *n*-th body; kinematic and dynamic relations are imposed on the remaining free surface $\Omega_n^{fr}=\Omega_n/\Omega_n^{bo}$ at a finite number of points: point-like rigid, elastic and (or) viscoelastic hinge-type supports (rigid supports can be the fixed ones), rigid elastic and (or) viscoelastic shock absorbers connecting the bodies (at $N>1$), concentrated masses $M_{qn}\left(q=1,2,\cdots,Q\right)$. The arrangement of relations and masses on surfaces Ω_n^{fr} is arbitrary.

A special case of such a structurally inhomogeneous viscoelastic system is a system with elastic and viscoelastic elements.

Then, the system under consideration is transformed into a non-conservative one with complex Eigen frequencies, *i.e.* $\omega=\omega_R+i\omega_I$. The real part of the complex natural frequency ω_R means the natural frequency of the system, the fictitious part ω_I determines the damping rate of vibrations and has the meaning of the coefficient of damping. In the general case, the dissipative (damping) properties of the elements of such systems are different.

Further, in the problems under consideration, it is required to determine the natural vibrations frequencies of the inhomogeneous viscoelastic system, and to estimate its damping capacity.

In mathematical statement, a viscoelastic problem is as follows: let all points of the *n*-th body obey the harmonic law of vibrations, *i.e.*

$$U_{nj}\left(\bar{x}^n,t\right)=U_{nj}^0\left(\bar{x}^n\right)\mathrm{e}^{-i\omega t},n=1,\cdots,N;\ j=1,\cdots,J\,, \tag{1}$$

where $U_{nj}^0\left(\bar{x}^n\right)$ is the *j*-th component of the displacement vector of the *n*-th body, J is the number of components of the displacements vector, $\bar{x}^n=\left(\bar{x}_1^n,\bar{x}_2^n,\bar{x}_3^n\right)$ is the radius vector of the point of the *n*-th body, $\omega=\omega_R+i\omega_I$ is the sought-for complex frequency of the system, ω_R is the natural frequency and ω_I is the coefficient of damping ($\omega_I<0$). Since each component of the displacements vector already has an index *n*, the latter is not used in what follows to designate the components of the radius vector. For rectangular plates $J=1$

$$U_{n1}^0\left(x_1,x_2\right)=W_n^0\left(x,y\right),$$

and for the shells of revolution $J=3$

$$U_{n1}^0\left(x_1,x_2\right)=U_n^0\left(x,y\right),\ U_{n2}^0\left(x_1,x_2\right)=V_n^0\left(x,y\right),\ U_{n3}^0\left(x_1,x_2\right)=W_n^0\left(x,y\right),$$

where *x*, *y* are the coordinates.

Based on the principle of possible displacements, we equate the sum of the virtual work of all active forces, including inertial ones, on possible displacements $\delta U_{nj}^{0}(\overline{x},t)$, to zero

$$\delta A_\sigma + \delta A_a + \delta A_m = 0, \tag{2}$$

where $\delta A_\sigma, \delta A_a, \delta A_m$ is a virtual work of the internal forces of the bodies of springs, and the forces of inertia, with account for concentrated masses. This work can be represented by the following relations:

$$\delta A_\sigma = -\sum_{n=1}^{N} \sigma_{mk}^{n} \delta\varepsilon_{mk}^{n} \mathrm{dV},$$

$$\delta A_a = -\sum_{n=1}^{N-1}\sum_{l=1}^{L_n} \sigma_l^n \delta\varepsilon_l^n - \sum_{n=1}^{N}\sum_{l'=1}^{L'_n} \sigma_{l'}^n \delta\varepsilon_{l'}^n \tag{3}$$

$$\delta A_m = -\sum_{n=1}^{N} \rho_n \int_{V_n} \left(\sum_{j=1}^{J} \ddot{U}_{nj}(\overline{x},t)\delta U_{nj} \right) \mathrm{d}V - \sum_{n=1}^{N}\sum_{q=1}^{Q_n} M_{qn} \sum_{j=1}^{J} \ddot{U}_{nj}(\overline{x}_n^q, t)\delta U_{nj}$$

where $\overline{x}_n^q = (\overline{x}_{n1}^q, \overline{x}_{n2}^q, \overline{x}_{n3}^q)$ are the coordinates, L_n is the number of deformable elements (springs or shock absorbers) between the n-th and (n + 1)-th bodies, Q_n is the number of concentrated masses on the n-th body, L'_n is the number of elastic (viscoelastic) supports on the n-th body, $\sigma_{mk}^n, \varepsilon_{mk}^n, \sigma_l^n, \varepsilon_l^n, \sigma_{l'}^n, \varepsilon_{l'}^n$ are the components of stress and strain tensors, respectively, of the n-th body, l-th spring (deformable element or shock absorber), and l'-th elastic (viscoelastic) support.

To describe viscoelastic properties of a body material with the linear hereditary Boltzmann-Volter theory, physical relations for the n-th viscoelastic element of the system are defined as [14]

$$\sigma_{mk}^n(t) = \tilde{\lambda}_n \Theta^n(t)\delta_{mk} + 2\tilde{\mu}_n \varepsilon_{mk}^n(t) \tag{4}$$

where $\tilde{\lambda}_n, \tilde{\mu}_n$ are the Volter integral operators, replaced below by one operator. The Poisson's ratio ν_n in the proposed statement of the problem is assumed constant. This means that for a structurally homogeneous viscoelastic system, the modes of natural vibrations are equal to the eigenvectors of corresponding elastic problem [21] [22]. Expressing $\tilde{\lambda}_n, \tilde{\mu}_n$ by known formulas through $\tilde{E}_n, \tilde{\nu}_n$, and considering that $\tilde{\nu}_n = \nu_n = const$, instead of (4) we get

$$\sigma_{mk}^n(t) = \frac{\tilde{E}_n}{1+\nu_n}\left[\frac{\nu_n}{1-2\nu_n}\Theta^n(t)\delta_{mk} + \varepsilon_{mk}^n(t) \right] \tag{5}$$

where $\tilde{E}_n$ is the Volter operator of the form [42] [43]:

$$\tilde{E}_n\varphi(t) = E_{0n}\left[\varphi(t) - \int_0^t R_{En}(t-\tau)\varphi(\tau)\mathrm{d}\tau \right] \tag{6}$$

here E_{0n} is the instantaneous modulus of elasticity, and R_{En} is the kernel of relaxation.

Given (1), the time function in Equality (6) is $\varphi(t) = \exp(-i\omega t)$ at a slowly varying amplitude. Assuming the smallness of the integral term $\int_0^\infty R(\tau)\mathrm{d}\tau$, using the freeze-etch method [38], the relation (6) is substituted by the approximate one:

$$\tilde{E}_n\varphi(t) \cong E_{0j}\left[1-\Gamma_n^c(\omega_R)-i\Gamma_n^s(\omega_R)\right]\varphi(t) \equiv \bar{E}_n\varphi(t), \tag{7}$$

where

$$\Gamma_n^c(\omega_R)=\int_0^\infty R_n(\tau)\cos(\omega_R\tau)\mathrm{d}\tau,\ \ \Gamma_n^s(\omega_R)=\int_0^\infty R_n(\tau)\sin(\omega_R\tau)\mathrm{d}\tau .$$

This allows eliminating the integral terms and, ultimately, time from the vibrational equation. In symbolic form, it can be represented as

$$\delta G\left(U_{nj}^0(\bar{x}),\omega^2\right)=0. \tag{8}$$

Write out a specific representation of the functional *G*, for example, for a package of rectangular plates with point relations:

$$\begin{aligned}
&G\left[W_n^0(x,y),\omega^2\right]\\
&=-\frac{1}{2}\sum_{n=1}^{N}\bar{D}_n\int_0^{a_n}\int_0^{b_n}\left[\left(\frac{\partial^2W_n^0}{\partial x^2}+\frac{\partial^2W_n^0}{\partial y^2}\right)^2-2(1-\nu_n)\left(\frac{\partial^2W_n^0}{\partial x^2}\frac{\partial^2W_n^0}{\partial y^2}-\left(\frac{\partial^2W_n^0}{\partial x\partial y}\right)^2\right)\right]\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{1}{2}\sum_{n=1}^{N-1}\sum_{l=1}^{L_n}\bar{D}_n\left[W_n^0\left(x_n^l,y_n^l\right)-W_{n+1}^0\left(x_n^l,y_n^l\right)\right]^2-\frac{1}{2}\sum_{n=1}^{N}\sum_{l'=1}^{L'_n}C_{l'n}\left(W_n^0\right)^2\left(x_n^{l'},y_n^{l'}\right)\\
&\quad+\frac{\omega^2}{2}\sum_{n=1}^{N}\rho_nh_n\int_0^{a_n}\int_0^{b_n}\left(W_n^0\right)^2\mathrm{d}x\mathrm{d}y+\frac{\omega^2}{2}\sum_{n=1}^{N}\sum_{q=1}^{Q_n}M_{qn}\left(W_n^0\right)^2\left(x_n^q,y_n^q\right),
\end{aligned}$$

where h_n, a_n, b_n are the thickness and linear dimensions of the *n*-th plate, x_n^q, y_n^q are the coordinates of the *n*-th concentrated mass, x_n^l, y_n^l are the coordinates of the *l*-th spring (shock absorber), $x_n^{l'}, y_n^{l'}$ are the coordinates of the l'-th elastic (viscoelastic) support.

If the *n*-th plate, *l*-th spring, and l'-th support are viscoelastic, then $\bar{D}_n, \bar{C}_{ln}, \bar{C}_{l'n}$ are represented by the following formulas:

$$\bar{D}_n=D_nf_n(\omega_R), \bar{C}_{ln}=C_{ln}f_{ln}(\omega_R), \bar{C}_{l'n}=C_{l'n}f_{l'n}(\omega_R),$$

where $f(\omega_R)=1-\Gamma_c(\omega_R)-i\Gamma_s(\omega_R)$ is a complex function whose numerical coefficients depend on the parameters of the relaxation kernel of corresponding viscoelastic elements, $D_n=E_nh_n^3\big/\left(12\left(1-\nu_n^2\right)\right)$, $C_{ln}, C_{l'n}$ is the generalized instantaneous stiffness of the *n*-th plate, *l*-th shock absorber and l'-th support, respectively. In the elastic case, $\bar{D}_n=D_n, \bar{C}_{ln}=C_{ln}, \bar{C}_{l'n}=C_{l'n}$ where $D_n, C_{ln}, C_{l'n}$ is the generalized stiffness of the *n*-th plate, *l*-th shock absorber and l'-th support, respectively. A similar functional can be written for the system of the shells of revolution. The components of the displacements vector $U_{nj}^0(x)$ are the sought for functions of vibrational Equation (8) and must satisfy the boundary conditions on the surface Ω_n^{bo},

$$L_nU_{nj}^0(\bar{x})=0,\ \bar{x}\in\Omega_n^{bo}. \tag{9}$$

It remains to impose on the system rigid point relations that do not perform work under vibrations. The conditions of rigid hinge support of the *n*-th body in point supports S_n are written as

$$U_{nj}^{0}\left(\overline{x}_{n}^{s}\right)=0\ \left(s=1,\cdots,S_{n};\,j=1,\cdots,J\right),\tag{10}$$

where $\overline{x}_n^s$ are the coordinates of the s-the support of the n-the body.

If the part of supports is fixed, then the following conditions are added.

$$\frac{\partial U_{nj}^{0}\left(\overline{x}_{n}^{s}\right)}{\partial \alpha_{n}^{s}}=0,\ \ \left(s=1,\cdots,S_{n}^{\alpha};\,j=1,\cdots,J\right)\tag{11}$$

where α_n^s is the direction of the unit vector along which the rigid fixing of the body is done at the point x_n^s.

In the program implementing the algorithm, condition (11) is considered only for the shells of revolution. The presence of rigid posts between the n-th and (n+1)-the bodies at $N \geq 2$ is taken into account by the relationships

$$U_{nj}^{0}\left(\overline{x}_{n}^{r}\right)-U_{n+1,j}^{0}\left(\overline{x}_{n}^{r}\right)=0\ \left(r=1,\cdots,R_{n};\,j=1,\cdots,J\right),\tag{12}$$

where $\overline{x}^r$ is the coordinate of the r-the post, R_n is the number of posts between the n-the and (n + 1)-the bodies. At $N=1$ conditions (12) are absent.

Thus, restrictions of the type (10)-(12) are imposed on the displacement vector. The imposition on the system of point relations is considered using the Lagrange multipliers method. Then the vibrational Equation (8) is rewritten as

$$\begin{aligned}\delta\Bigg\{&G\left(U_{nj}^{0}\left(\overline{x}\right),\omega^{2}\right)+\sum_{n=1}^{N}\sum_{s=1}^{S_n}\sum_{j=1}^{J}\lambda_{nj}^{s}U_{nj}^{0}\left(\overline{x}_{n}^{s}\right)+\sum_{n=1}^{N}\sum_{s=1}^{S_n^{\alpha}}\sum_{j=1}^{J}k_{nj}^{s}\frac{\partial U_{nj}^{0}\left(\overline{x}_{n}^{s}\right)}{\partial\alpha_{n}^{s}}\\&+\sum_{n=1}^{N-1}\sum_{r=1}^{R_n}\sum_{j=1}^{J}\mu_{nj}^{r}\left[U_{nj}^{0}\left(\overline{x}_{n}^{r}\right)-U_{n+1,j}^{0}\left(\overline{x}_{n}^{r}\right)\right]\Bigg\}=0,\end{aligned}\tag{13}$$

where $\lambda_{nj}^s, k_{nj}^s, \mu_{nj}^s$ are the Lagrange multipliers. It is necessary to find the spectrum of complex natural frequencies, $\omega^k=\omega_R^k+i\omega_I^k$ where ω_R^k are the frequencies, and ω_I^k are the coefficients of damping of natural vibrations.

3. Algorithm for the Implementation of the Vibrational Method to Solve Viscoelastic Problem of Natural Vibrations

Approximate solution of vibrational Equation (13) is sought using an expansion in approximating forms composed of fundamental functions that satisfy the equation and the given geometric boundary conditions on Ω_n^{fr} surfaces of each body. It is assumed that functions $\Phi_{nj}^k\left(\overline{x}\right)$ for such bodies are known (for rectangular plates and circular cylindrical shells this is a fundamental sequence of beam functions). Then the approximating forms can be constructed as a finite expansion in known functions [35] [36]:

$$U_{nj}^{0}\left(\overline{x}\right)=\sum_{k=1}^{K}\gamma_{nj}^{k}\Phi_{nj}^{k}\left(\overline{x}\right),\tag{14}$$

where γ_{nj}^k is the sought for complex coefficient.

Preliminary $\Phi_{nj}^k\left(\overline{x}\right)$ can be normalized. The sum (14) automatically satisfies the boundary conditions on Ω_n^{fr} by virtue of terms choice. Varying Equation (13) on the generalized coordinates $\lambda_{nj}^s, k_{nj}^s, \mu_{nj}^s, \gamma_{nj}^s$, a homogeneous system of linear equations is obtained. Dimension of this systems is $J\cdot N'\times J\cdot N'$, where;

$N' = \sum_{n=1}^{N}\left(S_n + S_n^{\alpha} + R_n\right) + N \cdot K$; J is the number of components of the displacements vector $U_{nj}^{0}\left(\overline{x}\right)$. Without specific calculations, we write this system in a matrix form:

$$\left(A + \sum_{n=1}^{N_n} f_n\left(\omega_R\right) A_n^n + \sum_{n=1}^{N-1}\sum_{l=1}^{L_n} f_{ln}\left(\omega_R\right) A_{ln}^n + \sum_{n=1}^{N}\sum_{l'=1}^{L_n} f_{l'n}\left(\omega_R\right) A_{l'n}^n - \omega^2 B\right)\overline{\xi} = 0, \quad (15)$$

where $\overline{\xi}$ is a column vector of generalized coordinates; N_n -the number of viscoelastic bodies of the system; B is a symmetric, degenerate matrix of the generalized masses of the system; $A_n^n, A_{ln}^n, A_{l'n}^n$ -square matrices of dimension $J \cdot N' \times J \cdot N'$ consisting of zeros, except for sub matrices of instantaneous stiffness of the n-the viscoelastic body, l-the shock absorber and l'-the viscoelastic support, respectively; A is a symmetric matrix (its sub matrix is A^0 of dimension $J \cdot K \times J \cdot K$); it presents a generalized total stiffness of elastic elements of the system, and the sub matrices $A_H = A_b^{\mathrm{T}}$ take into account kinematic conditions imposed on the system of rigid point relations.

Structurally, the matrices A and B are similar to those described in [17] for elastic mechanical systems. The generalized stiffness and masses of the n-th elastic element are calculated using the method given in [17], if the elements of the system are rectangular plates. If all viscoelastic elements of the system have the same rheological properties, then $f_1\left(\omega_R\right) = f_2\left(\omega_R\right) = \cdots$ so, the second, third and fourth terms in (15) are replaced by one matrix of total instantaneous stiffness of all viscoelastic elements (in case of structurally homogeneous viscoelastic system). The degeneracy of the matrix B, as in the elastic problem [17], is due to the introduction of additional point relations (rigid supports and posts) into the system. By the method described in [17], we eliminate linearly dependent coordinates and bring system (15) to the standard generalized eigenvalue problem. The converted matrices have the dimension $N'' \times N''$, where

$N'' = J \cdot N' - 2J\sum_{n=1}^{N}\left(S_n + S_n^{\alpha} + R_n\right)$. Equating the system determinant to zero, we obtain the frequency equation, which, unlike the case of elastic problem [17], is a complex one, *i.e.* we obtain a complex problem on eigenvalues. The most effective way to solve such equations is the Muller method, used here. Without disclosing the frequency determinant, we calculate its value at each step for a fixed value of $\omega = \omega_R + i\omega_I$. As mentioned above, the fictitious part ω_I of the complex value $\omega = \omega_R + i\omega_I$ determines the damping rate of vibrations.

In engineering, an estimated logarithmic decrement of vibrations is assessed using ω_I, and determined [23] as follows

$$\delta = -\frac{2\pi\omega_I}{\omega_R}$$

4. Evaluation of the Practical Convergence of the Algorithm and the Reliability of Numerical Results on the Problems of Natural Vibrations of Dissipative Mechanical Systems

In this section we will not consider the question of strict convergence of the method from a mathematical point of view, since it is not important for the following reasons. The energy approach used in the statement of the problem is essentially the Ritz method, the convergence of which is proved, for example, in [25] [28]. Accounting with the help of Lagrange multipliers of point bonds superimposed on plates is also a well-known method for finding a conditional extremum. Superimposed point relationships affect the rate of convergence, but not the convergence itself. To illustrate the convergence of the method, we compare the data available in the literature (theoretical and experimental) with the results of calculations by the described method that we obtained.

In [25] [28], a relation was obtained that allows (in a first approximation) to calculate the first (main) frequency of a square elastic plate supported at the edges, with an attached mass M in the center:

$$\omega_1 = \frac{2\pi^2}{a^2}\sqrt{\frac{D}{\rho h + 4M/a^2}}. \tag{16}$$

Studies have shown that the first approximation (for supported on the edges of the plate) is quite close to the exact one. So, an increase in the number of members of fundamental functions to $K = 16$ "clarifies" the main frequency by only 1% - 2%; if the mass of the attached load is commensurate with the weight of the plate, the relative error is slightly larger (up to 6%) for $K = 16$.

At the edges of the free square plate is supported by four symmetrical supports located diagonally. For such problems, to determine the fundamental natural frequency, a theoretical and experimental formula is given [25]

$$\omega_1 = \frac{\bar{\omega}}{a^2}\sqrt{\frac{D}{\rho h}}, \tag{17}$$

where is the frequency coefficient determined experimentally depending on the location of the supports on the diagonals. Consider the two extreme cases—bearing on one support in the center and bearing in the corners (see **Table 1**). Data analysis indicates satisfactory convergence of the method and the dependence of the convergence rate on the number of supports is observed. The more point dependencies, the more the convergence rate worsens, *i.e.* the speed of convergence depends on the number of supports. The more point bonds are superimposed on the plate, at a fixed row length (14), the greater the error. In [28] for symmetric natural vibrations of a duralumin square plate (a = 13 cm, h = 0.193 cm) experimental results of the first frequency.

A square plate with a free circuit, spaced from the corners by a distance of r = 0.5 cm, is supported at four symmetric points:

$$\omega_1' = 246; \omega_2' = 1290.3; \omega_3' = 2815; \omega_4' = 3840.$$

When $K = 25$ in the formula (17), the calculated frequencies are as follows [44] [45]:

$$\omega_1 = 250.7; \omega_2 = 1380.3; \omega_3 = 2839.8; \omega_4 = 4077.3.$$

We reduce the initial problem to the following: a square plate is pivotally supported along the contour, one of its sides has discrete uniformly located

Table 1. The frequencies of the two extreme cases—bearing on the corners and bearing on one support in the center in comparison with the formula (16).

K		6	9	16	25
Leaning in the corners	ϖ	10.9082			
	ϖ_K	13.8061	-	11.6074	11.5811
	$\delta_1\%$	20.1833	-	6.9229	5.7265
Leaning in the center	ϖ	6.7806			
	ϖ_K	-	7.2712	7.2397	6.8214
	$\delta_1\%$	-	7.212	6.7030	0.6573

pinch points in the direction perpendicular to this side. We varied the number of fundamental functions of the sum (14) and the number of pinches that must satisfy (for a pivotally supported plate) the boundary conditions. The relative error was calculated by the formula [28]:

$$\delta_{ij} = -\frac{\omega_{ij} - \omega_{ij}^T}{\omega_{ij}} 100\% ,$$

where ω_{ij}^T -is the natural frequency determined exactly; ω_{ij} -natural frequency, determined approximately; i and j are the number of half-waves of the form along the axes 0X and 0Y, respectively.

Thus, the developed algorithm and the results obtained on the basis of the developed programs are reliable and the solution converges. The physical interpretation of the graph is as follows: the left branch of the graph is the energy of the plate, sufficient for the implementation of close to real (true) forms of vibration; curve minimum-saturation of the mechanical system with bonds; the right branch shows that the total energy of the plate decreases due to an excess of bonds, and, at higher frequencies, the shape of the vibrations is more and more distorted.

5. Solution and Analysis of the Problem of Natural Vibrations of Structurally Inhomogeneous Viscoelastic Systems

The examples given below are mostly theoretical in nature, but they allow us to draw practical conclusions. The problems are solved using the algorithm described in paragraph 3 of this paper.

Problem 1. An inhomogeneous system is considered, which consists of rectangular plates with point relations. The kernel of relaxation for deformable viscoelastic elements (shock absorbers) is chosen in the form of the Rzhanitsyn-Koltunov kernel [26]

$$R(t) = A\mathrm{e}^{-\beta t} t^{\alpha-1}$$

where A, α, β are the kernel parameters [24]. The viscosity of the shock absorber is taken such that its creep strain during a quasistatic process is a small fraction (~12%) of the total strain.

In the general case, the methods to determine the kernel parameters (16) for various materials are given in [26], and in [33], using these methods, specific parameters for some materials are given.

In this case, in specific calculations, the kernel parameters are as follows:

$$A = 0.01, \alpha = 0.1, \beta = 0.05.$$

In contrast to the elastic problem, the dependence of the two lowest frequencies and corresponding damping coefficients on the magnitude of the instantaneous stiffness of the shock absorber was studied here. The stiffness varied from 10^{-4} to 10^{-1}. On the right, this range is limited to C_2, since at $C = C_2$, the second mode changes.

Figure 1 shows the dependence of the first two frequencies ω_R^1, ω_R^2 and corresponding damping coefficients on the magnitude of instantaneous stiffness of the shock absorber C. From the analysis of graphs, it follows that the dissipative properties of this system are determined not only by the rheology of its elements, but greatly depend on the interaction of natural modes of vibration. This effect is expressed in the fact that under certain conditions and to some value of the rigidity of the shock absorber, the mode energetically more capacious (in this case, the second one) dissipates less energy than the mode less energy-intensive. Then, beginning from some value of instantaneous stiffness of the shock absorber C^* (in this case, $C^* = 5.4 \times 10^{-3}$), the process of energy dissipation by natural modes is normalized and proceeds according to the energy hierarchy of modes.

Real illustration of this effect is the presence of the point of graphs intersection of the damping coefficients ω_I^1 and ω_I^2 at $C = C^*$.

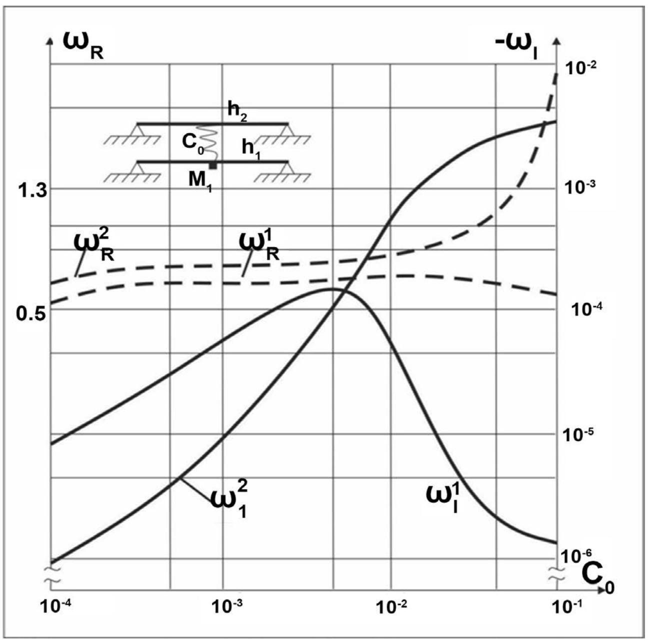

Figure 1. Dependence of frequencies ω_R and damping coefficients ω_I on the stiffness of shock absorber C_0 for the considered mechanical systems.

And one more feature: at this point, the difference in damping rates of two modes of structure vibrations changes the sign if the spring is viscoelastic (that is, the system is structurally inhomogeneous). As in the elastic problem, the Eigen frequencies ω_R^1, ω_R^2 that are less than ω_1, ω_2 at this point converge. Analysis of the problems of this type showed that the effect of the interaction of Eigen modes is observed only in structurally inhomogeneous systems (in this case with elastic and viscoelastic elements) and at a noticeable convergence of the real parts of the Eigen frequencies ω_R^1, ω_R^2; the fictitious parts of the frequencies (the damping coefficients) change dramatically (increases or decreases). The absence of at least one of these conditions eliminates the manifestation of the effect.

Physical explanation of the observed effect should be sought in the nature of dynamic redistribution of energy of the system between two modes of the properties described above.

Now consider what differences exist between structurally homogeneous and inhomogeneous viscoelastic systems.

Structurally homogeneous viscoelastic system (all elements are viscoelastic with the same rheological properties) is characterized by the fact that, in formula (15), firstly, there is no matrix A (sub matrices A_H and A_b can be transferred to the next matrix) and, secondly, all functions $f_n(\omega_R)$ are identical. Then the system of Equations (15) in the matrix form can be rewritten as:

$$\left[f(\omega_R)A^n - \omega^2 B\right]\bar{\xi} = 0, \tag{18}$$

where A^n is a numerical matrix of total instantaneous stiffness of all viscoelastic elements of the system.

After elimination of linearly dependent components from system (18), the transformed matrices of generalized instantaneous stiffness $\tilde{A}^n$ and $\tilde{B}$ can be written in a canonical form, *i.e.* by special transformation of the generalized coordinates they are reduced to a matrix of diagonal form. This means that mechanical system is a set of independent partial systems with one degree of freedom. In other words, the Eigen modes of such a system are independent and can be considered and calculated separately.

Another situation is formed for a structurally inhomogeneous viscoelastic system, for which the matrix of generalized stiffness of elastic elements A is added to (18):

$$\left[\bar{A}(\omega_R) - \omega^2 B\right]\bar{\xi} = 0, \tag{19}$$

where $\bar{A}(\omega_R) = A + f(\omega_R)$.

In the general case, for two matrices $A(\omega_R)$ and B (one of which is functional), after eliminating linearly dependent components, it is impossible to simultaneously select a non-degenerate coordinate transformation, leading to a canonical form. This means that natural modes of such a mechanical system cannot be considered separately from each other, *i.e.* they are interdependent. Consequently, under free oscillations, energy exchange occurs between the modes. This is seen when the modes have close natural frequencies. Then at the point of approaching of the graphs of natural frequencies ω_R^1, ω_R^2 (at the point $C = C^*$), the fictitious parts of the frequency ω_I^1 and ω_I^2 intersect.

At the intersection point of the graphs (**Figure 1**) of damping coefficients ω_I^1 and ω_I^2 both modes dissipate energy in the same way, although they differ from each other (up to a phase). Up to the point $C = C^*$ there is a "transfer" of energy from the second mode to the first one, therefore the latter dissipates energy more intensively. Past the intersection point, the difference between the first Eigen frequencies increases, the interaction of corresponding modes decreases, and their dissipative properties take on an ordinary character.

From the above, the following practical conclusion can be made: the damping capacity of inhomogeneous viscoelastic systems basically determines the minimum damping coefficient (in this case, the vibrations of this particular mode are damped out last); the system's global (determining) damping coefficient is ω_I^1 up to the intersection point and then ω_I^2. The optimal, in the sense of damping, vibration mode of the structure is at $C = C^*$, when this global damping coefficient has a maximal value.

Problem 2. A similar effect was found for a dissipative inhomogeneous viscoelastic structure consisting of two circular coaxially located cylindrical elastic shells interconnected by a massless viscoelastic element. Parameters of its relaxation kernel are [35]: $A = 0.078, \alpha = 0.1, \beta = 0.05$. The structure materials considered here are hypothetical: the instantaneous stiffness of the shock absorber changes from 10^{-2} to 10^2. Mechanical characteristics of the shells are the same and equal to $E = 1, \rho = 1, \nu = 0.35$. Geometric parameters of the first (inner) shell are $L = 10, R_1 = 1, h_1 = 0.1$; of the second (outer) shell: $L = 10, R_1 = 2, h_1 = 0.2$. On the second shell there is a concentrated mass $M = 0.5$ attached at a point $X_M = 5, Y_M = 0$.

The shock absorber connecting the shells has the coordinates of the application $X_a = 5, \varphi_a = \frac{Y_a^1}{R_1} = \frac{Y_a^2}{R_2} = \frac{\pi}{2}$. Shells are hinge supported (Navier conditions) [25].

Dependence of the frequencies ω_R and damping coefficients ω_I on the rigidity of the shock absorber C_0 for the considered mechanical systems is shown in **Figure 2**. From **Figure 2** it is clear that the described effect in this problem is less pronounced both damping coefficients ω_I^1, ω_I^2 have a non-monotonic dependence, are close to each other.

This is explained by the fact that the deformable element less effectively damps vibrations of shells, *i.e.* torsional and longitudinal modes of vibrations remain undammed.

Unlike the previous problem, the global damping coefficient is, in the order of succession, not ω_I^2 and ω_I^1, but ω_I^1 and ω_I^2, respectively. The stiffness corresponding to the point of intersection of the graphs of damping coefficients is an optimal instantaneous stiffness of the shock absorber [27].

Problem 3. Consider a mechanical system consisting of two parallel, identical (in geometry and mechanical properties) elastic plates ($E = 2 \times 10^{11}\ \mathrm{N/m^2}$,

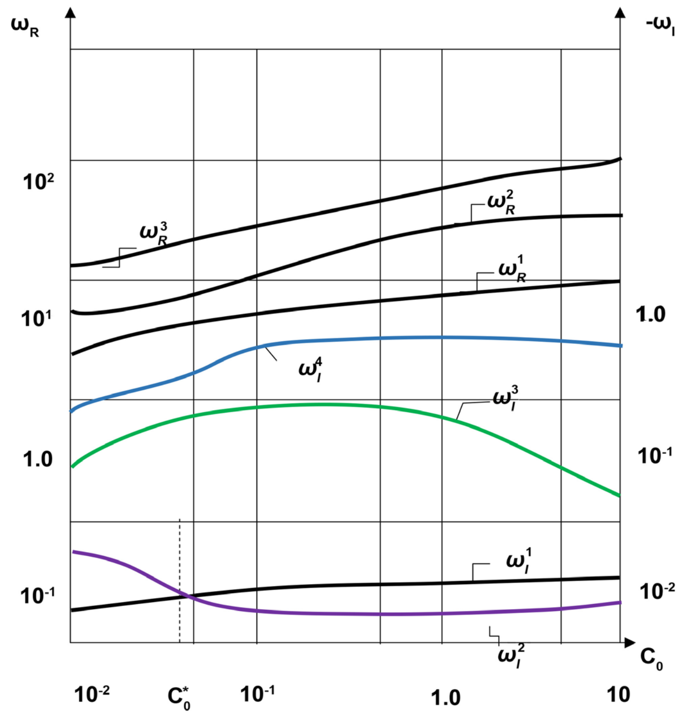

Figure 2. Dependence of frequencies ω_R and damping coefficients ω_I on the stiffness of shock absorber C_0 for the considered mechanical systems.

$\rho = 7.8 \times 10^3\ \text{kg/m}^2$, $\nu = 0.35$, $h = 0.001\ \text{m}$), connected by one massless viscoelastic deformable element. The parameters of its relaxation kernel are $A = 0.01, \alpha = 0.1, \beta = 0.05$ and instantaneous stiffness is $C = 10$.

The plates are of square form ($a = b = 0.2\ \text{m}$), supported along the contour, each having one added mass ($M_1 = M_2 = 0.05\ \text{kg}$) [28] [29].

The aim of the study is to determine the nature of dependence of damping capacity of structure on the location of shock absorber and the masses on the area of the plates.

In this problem, the following option is investigated: a shock absorber is located in the center of the plate, the added masses are on the central axis of the structure, M_1 is attached to the first (lower) plate at 0.04 m from the shock absorber; on the second (upper) plate a shock absorber is located in the center and positions of mass M_2 on the second plate are varied along the central axis of the structure.

Figure 3 shows the graphs of dependence of the first two Eigen frequencies and their damping coefficients on the location of mass M_2. Calculations showed a strong dependence of the damping coefficient of the first global mode ω_I^1 on the location of the coordinates of mass M_2. The damping coefficient of the second mode ω_I^2 and natural frequencies ω_R^1, ω_R^2 remained practically unchanged.

The presence of two points on the *x*-axis, where the first global mode of the structure does not damp, is explained by the fact that the wave modes of the

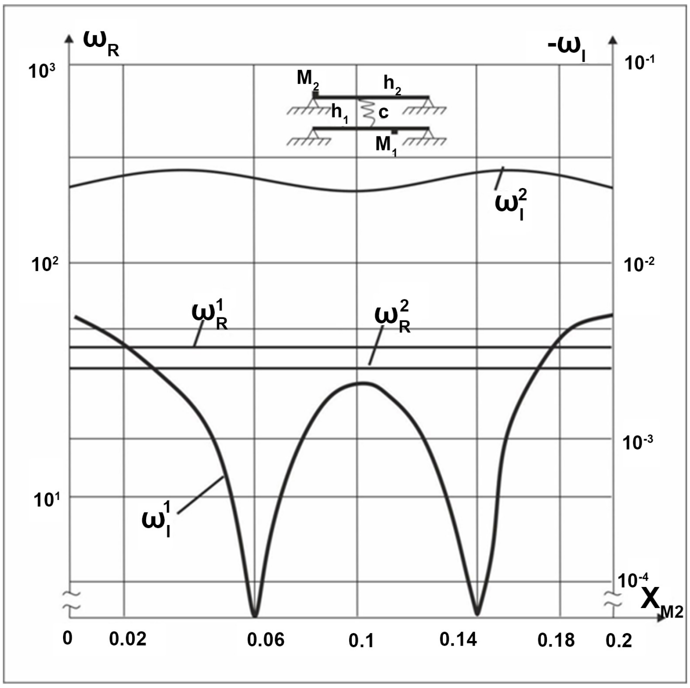

Figure 3. Dependence of frequencies ω_R and damping coefficients ω_I on the location of mass M_2.

plate are indistinguishable in this case. When mass M_2 is located at these points, the plates become identical in terms of inertial and stiffness characteristics [30].

If to move the mass M_2 on the upper plate in other directions, then the points could be found at which the damping coefficient ω_I^1 is zero. Thus, for mass M_2, there are an infinite number of positions (at fixed positions of the shock absorber and mass M_1) in which the first global mode does not damp. These points on the second plate form a closed curve (close to a circle).

Analysis of the results (**Figure 3**) shows that to damp the first global mode, it is necessary to break the symmetry of the plates, for example, by appropriate location of the added masses. The symmetry of the right and left branches of the graph provides the possibility of obtaining the maximum (and identical) efficiency of the shock absorber with the loads location both on one side (right branch) and on both sides of it (left branch).

The effect described shows that the magnitude of dissipative energy of the system depends on damping (rheological) properties of the material and on structure geometry as a whole. This effect does not manifest itself if the viscoelastic system is dissipative homogeneous [31] [32]

Problem 4. Consider the structure with the same initial data as in Problem 3. The masses are located at distances of 0.04 m on both sides of the massless deformable element. The Eigen frequencies of two modes of vibrations are almost unchanged. Consider the dependence of frequency and damping characteristics of the system on simultaneous displacement of a massless deformable element and mass.

As in the previous example, there is a point (when the shock absorber is in the center) where $\omega_I^1 = 0$. At this point, the second damping coefficient is a maximal one, *i.e.* ω_I^2. In the presence of concentrated masses the dependence of damping properties of the system is so strong that, in the absence of them, the damping coefficient ω_I^1 (global one) is zero for any location of the shock absorber (the corresponding graph is shown in **Figure 3**).

Knowledge of the law of variation determining damping coefficient ω_I^1 made it possible to make various recommendations of an optimization nature. The optimization of the damping properties of structure did not affect the spectrum of the lowest frequencies. The instantaneous stiffness of the shock absorber, the magnitude of the added mass and viscosity did not vary, since these parameters did not (qualitatively) affect the results obtained. Only the quantitative characteristics of the system varied, which is not fundamental.

The effect revealed in this problem does not manifest itself if to consider a dissipatively homogeneous viscoelastic system.

After analyzing problems 3 and 4, we can conclude that for given damping coefficients of the system, it is possible to determine the necessary rheological properties of the shock absorber, the size and location of the added masses, its instantaneous stiffness and the location of masses and shock absorber in the plane of the plates [27].

Problem 5. Consider a viscoelastic shell, the rheological properties of which are described by a kernel of the form given in (17); the parameters are: $A = 0.01, \alpha = 0.1, \beta = 0.05$. Geometric and mechanical characteristics are: $E = 1, \rho = 1, \nu = 0.35, R = 1, L = 7, h = 0.01$. Coordinates of rigid support are: $x = 2, y = 3\pi/2$. Shell thickness h varied from 0.01 to 0.2. Results of calculations for the two lower frequencies ω_R^1 and ω_R^2 of corresponding damping coefficients ω_I^1, ω_I^2 are shown in **Table 2**.

For comparison, the first two Eigen frequencies of elastic shell ω_1, ω_2 are given. A twentyfold increase in the shell thickness increases the first and second Eigen frequencies and the damping coefficient by almost 4 times.

6. Conclusion and Recommendations

6.1. Conclusions

1) The mathematical formulation and methods for solving the problem of natural vibrations of structurally inhomogeneous dissipatively homogeneous and heterogeneous mechanical systems consisting of a package of plates (or shells) with point supports and attached masses are formulated.

2) The convergence of solution is numerically proved depending on the terms taken in the sought-for solution (*i.e.*, in solution expansion in fundamental functions) and on various parameters of mechanical systems.

3) The problems of natural vibrations of dissipative homogeneous and inhomogeneous lamellar systems with concentrated supports and added masses are solved.

Table 2. Calculation results for two lower frequencies.

h	ω_1	ω_R^1	$-\omega_I^1$	ω_2	ω_R^2	$-\omega_I^2$
0.01	0.0927	0.0872	0.4×10^{-4}	0.142	0.135	0.94×10^{-4}
0,1	0.274	0.26	0.34×10^{-3}	0.281	0.27	0.44×10^{-3}
0.2	0.339	0.322	0.53×10^{-3}	0.502	0.48	0.11×10^{-2}

4) To describe dissipative properties of the system as a whole, the concept of "global damping coefficient" is introduced. In the case of a structurally homogeneous mechanical system, the global damping coefficient is determined by the fictitious part of the first (in modulus) complex Eigen frequency. In the case of a structurally inhomogeneous mechanical system, the fictitious parts of the first and second Eigen frequencies play the role of a global damping coefficient, depending on the stiffness magnitude of a shock absorber.

5) It is established that optimal damping of vibrations of inhomogeneous systems occurs when the values of close frequencies of vibration modes converge, and the fictitious parts of these frequencies are equal to each other (*i.e.*, at C=C*). In this case, both modes of vibrations provide the same energy dissipation.

6) It is revealed that for given damping coefficients of the system, it is possible to determine the necessary rheological properties of a shock absorber, the size and location of the added masses in the plane of the plates, to ensure effective control of system vibrations with maximum damping capacity.

6.2. Recommendations

1) To reduce the amplitude of the interference of radio electronic equipment in resonance mode, we recommend that we use the developed methodology and algorithm of structurally inhomogeneous dissipative mechanical systems, which effectively reduces the amplitudes of mixing and other force factors to 60%.

2) Significant intensification of dissipative processes in structurally heterogeneous dynamical systems occurs when the corresponding natural frequencies approach each other.

3) The effect of the interaction of various forms of motion of solid bodies has a fundamental perspective for the synthesis of structurally heterogeneous machine-building structures that are optimal in dissipative properties and material consumption.

4) Analytically, to obtain several eigenfrequency modes of structurally heterogeneous dynamic mechanical systems, depending on the physical and geometric parameters, it remains open.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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