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On Holomorphic Curvature of Complex Finsler Square Metric

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Abstract

The notion of the holomorphic curvature for a Complex Finsler space (M,F) is defined with respect to the Chern complex linear connection on the pull-back tangent bundle. This paper is about the fundamental metric tensor, inverse tensor and as a special approach of the pull-back bundle is devoted to obtaining the holomorphic curvature of Complex Finsler Square metrics. Further, it proved that it is not a weakly Kähler.

Keywords

Complex Square Metric, Holomorphic Flag Curvature, Riemannian Curvature

1. Introduction

The notion of holomorphic curvature of a complex Finsler space is defined with respect to the Chern complex linear connection in briefly Chern (c.l.c) as a connection in the holomorphic pull back tangent bundle $\pi^*(T'M)$ (here π represented as projection). In [1], Nicolta Aldea has obtained the characterization of the holomorphic bisectional curvature and gave the generalization of the holomorphic curvature of the complex Finsler spaces which are called holomorphic flag curvature. After that in (2006) he devoted to obtaining the characterization of holomorphic flag curvature.

In complex Finsler geometry, it is systematically used the concept of holomorphic curvature in direction η. But, the holomorphic curvature is not an analogue of the flag curvature from real Finsler geometry.

This problem sets up the subject of the present paper. Our goal is to determine the conditions in which complex Finsler spaces with square metric of holomorphic curvature. As per our claim, we shall use the holomorphic curvature of complex Finsler spaces, with respect to Chern (c.l.c) on $\pi^*(T'M)$ (definition (2.4) and (2.5)). We shall see that the fundamental metric tensor $g_{i\bar{j}}$ and its inverse are obtained (see in Section-3). Moreover, we determine the holomorphic curvature of complex square metric (theorem (4.3)) and some special properties of holomorphic curvature are obtained (proposition (4.4)).

2. Preliminaries

This section, includes the basic notions of Complex Finsler spaces.

An $\mathbb{R}$-Complex Finsler metric on M is continuous function $F : TM \to \mathbb{R}$ satisfying:

1) $L = F^2$ is a smooth on $T'\tilde{M}/0$;
2) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;
3) $F(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = |\lambda| F(z,\eta,\bar{z},\bar{\eta})$, for all $\lambda \in \mathbb{R}$.

Let M be a complex manifold, $dim_c M = n$ and $T'M$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by (z^k, η^k). The complexified tangent bundle of $T'M$ is decomposed in $T_c(T'M) = T'(T''M) \oplus T''(T'M)$, where operator $\oplus$ becomes direct sum.

Considering the restriction of the projection to $T'\tilde{M} = T'M/0$, for pulling back of the holomorphic tangent bundle $T'M$ then it obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \to \widetilde{T'M}$, called the pull-back tangent bundle over the slit $T'\tilde{M}$. We denote by $\left\{ \frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \bar{z}^k}^* \right\}$, the local frame and by $\{dz^{*k}, d\bar{z}^{*k}\}$ the local frame and its dual.

Let $V(T'M) = ker\pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\left\{ \frac{\partial}{\partial \eta^k} \right\}$. A complex nonlinear connection, briefly (c.n.c), determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, that is

$$T'(T'M) = H(T'M) \oplus V(T'M).$$

The adapted frames is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where $N_k^j(z,\eta)$ are the coefficients of the (c.n.c). Further we shall use the abbreviations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\bar{i}} = \frac{\delta}{\delta \bar{z}^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \bar{\eta}^j}$, and their conjugates [2] [3] [4].

On $T'M$, let $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space $(M, L = F^2)$.

The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism of $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\bar{j}}$ defines an Hermitian metric structure $G(z,\eta) = g_{j\bar{k}} d\bar{z}^{*j} \otimes d\bar{z}^{*k}$ on $\pi^*(T'M)$, with respect to the natural complex structure. Further, the Hermitian metric structure G on $\pi^*(T'M)$ induces a Hermitian inner product $h(\xi,\gamma) := ReG(\chi,\bar{\gamma})$ and the angle

$$\cos(\chi\gamma)=\frac{ReG(\chi,\overline{\gamma})}{\|\chi\|\|\overline{\gamma}\|},$$

for any χ, γ the sections on $\pi^*(T'M)$, where $\|\chi\|^2=\|\overline{\chi}\|^2=G(\chi,\overline{\chi})$ (for details see in [5]).

On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic. Therefore, the structures on $\pi^*(T_CM)$ can be pulled-back to $H(T'M)\oplus\overline{H(T'M)}$. By this isomorphism the natural co-basis dz^{*j} is identified with dz^j. In view of this constructions the pull-back tangent bundle $\pi^*(T'M)$ admits a unique complex linear connection ∇, called the Chern (c.l.c), which is metric with respect to G and of $(0,1)$-type.

$$\omega_j^i(z,\eta)=L_{jk}^i(z,\eta)dz^k+C_{jk}^i(z,\eta)\delta\eta^k; \tag{2.1}$$

The Chern (c.l.c) on $\pi^*(T'M)$ determines the Chern-Finsler (c.n.c) on ($T'M$), with the coefficients $N_k^i=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}\eta^j$, and its local coefficients of torsion and curvature are

$$T_{jk}^i:=L_{jk}^i-L_{kj}^i; \tag{2.2}$$

$$R_{j\overline{h}k}^i:=-\delta_{\overline{h}}L_{jk}^i-\delta_{\overline{h}}\left(N_k^l\right)C_{jl}^i;\ \sigma_{j\overline{h}k}^i:=-\delta_{\overline{h}}C^i jk=\sigma_{k\overline{h}j}^i;$$

$$L_{jk}^i=g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k};\ C_{jk}^i=g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial\eta^k};$$

$$P_{j\overline{h}k}^i:=-\dot{\partial}_{\overline{h}}L_{jk}^i-\dot{\partial}_{\overline{h}}\left(N_k^l\right)C_{jl}^i,;\ S_{j\overline{h}k}^i:=-\dot{\partial}_{\overline{h}}C_{jk}^i=S_{k\overline{h}j}^i.$$

The Riemann type tensor

$$R(W,\overline{z},X,\overline{y}):=G\left(R\left(X,\overline{Y}\right)W,\overline{Z}\right),$$

has properties:

$$R\left(W,\overline{Z},X,\overline{Y}\right)=W^i\overline{Z}^jX^k\overline{Y}^hR_{\overline{i}jk\overline{h}};\ R_{\overline{j}i\overline{h}k}:=R_{i\overline{h}k}^lg_{l\overline{j}}; \tag{2.3}$$

$$R_{\overline{i}jk\overline{h}}=-R_{\overline{i}j\overline{h}k}=\overline{R_{j\overline{i}h\overline{k}}}=R_{\overline{j}i\overline{h}k};$$

$$R_{\overline{j}j\overline{h}k}^i=R_{kh\overline{j}}^i\quad\text{then } R_{\overline{i}jk\overline{h}}=R_{\overline{k}jk\overline{h}}=R_{kh\overline{i}\overline{j}}.$$

According to [2] the complex Finsler space (M,F) is strongly Kähler if and only if $T_{jk}^i=0$, Kähler if and only if $T_{jk}^i\eta^j=0$ and weakly Kähler if and only if $g_{i\overline{l}}T_{jk}^i\eta^j\overline{\eta}^l=0$. Note that for a complex Finsler metric which comes from a Hermitian metric on *M*, so-called purely Hermitian metric. That is $g_{i\overline{j}}=g_{i\overline{j}}(z)$, the three nuances of Kähler spaces consider, in [6].

The holomorphic curvature of F in direction η, with respect to the Chern (c.l.c) is,

$$\kappa_F(z,\eta):=\frac{2R(\eta,\overline{\eta},\eta,\overline{\eta})}{G^2(\eta,\overline{\eta})}=\frac{2\overline{\eta}^j\eta^kR_{\overline{j}k}}{L^2(z,\eta)}, \tag{2.4}$$

where η is viewed as local section of $\pi^*(T'M)$, that is $\eta:=\eta^i\frac{\partial}{\partial z^i}$. Further on, we shall simply call it holomorphic curvature. It depends both on the position $z \in M$ and the direction η.

Definition 2.1. [7] *The complex Finsler space* (M,F) *is called generalized Einstein if* $R_{\bar{j}k}$ *is proportional to* $t_{k\bar{j}}$, *that is if there exists a real valued function* $K(z,\eta)$, *such that*

$$R_{\bar{j}k} = K(z,\eta)t_{k\bar{j}}, \tag{2.5}$$

where $R_{\bar{j}k} := R_{i\bar{j}k\bar{h}}\eta^i\bar{\eta}^h = -g_{l\bar{j}}\delta_{\bar{h}}\left(N_k^l\right)\bar{\eta}^h$, $t_{k\bar{j}} := L(z,\eta)g_{k\bar{j}} + \eta_k\bar{\eta}_j$, $\eta_k := \dfrac{\partial L}{\partial \eta^k}$, $\bar{\eta}_j := \dfrac{\partial L}{\partial \bar{\eta}^j}$.

By finding the Chern (c.l.c) on $\pi_*(T'M)$ determines the Chern-Finsler on $T'M$, with the coefficient $N_k^i = g^{\bar{m}i}\dfrac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j$ determines, we need to find the fundamental metric tensor followed by the invariants are given below:

Now, from definition of Complex Finsler metric follows that L is $(2,0)$-homogeneous with respect to the real scalar λ and is proved that the following identities are fulfilled in [8].

$$\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i} = 2L;\quad g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}, \tag{2.6}$$

$$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ij}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0;\quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0, \tag{2.7}$$

$$2Lg_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{2.8}$$

where,

$$g_{ij} = \frac{\partial^2 L}{\eta^i\eta^j};\quad g_{i\bar{j}} = \frac{\partial^2 L}{\eta^i\bar{\eta}^j};\quad g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial\bar{\eta}^i\partial\bar{\eta}^j}.$$

Here, to find the inverse of fundamental metric tensor $g_{i\bar{j}}$ we use the following proposition.

Proposition 2.1. *Suppose*:

- (Q_{ij}) is a non-singular $n \times n$ complex matrix with inverse Q^{ji};
- C_i and $C_{\bar{i}} = \bar{C}_i, i = 1,\cdots,n$ are complex numbers;
- $C^i := Q^{ji}C_j$ and its conjugates; $C^2 := C^iC_i = \bar{C}^iC_{\bar{i}}$; $H_{ij} := Q_{ij} \pm C_iC_j$.

Then,

1) $det(H_{ij}) = (1 \pm C^2)det(Q_{ij})$ (Here, det indicates determinant),

2) whenever $(1 \pm C^2) \neq 0$, the matrix (H_{ij}) is invertible and in this case its inverse is $H^{ij} = Q^{ji} \pm \dfrac{1}{1 \pm C^2}C^iC^j$.

3. Notation of Complex Square Metrics

The $\mathbb{R}$-complex Finsler space produce the tensor fields g_{ij} and $g_{i\bar{j}}$. The tensor field must $g_{i\bar{j}}$ be invertible in Hermitian geometry. These problems are about to Hermitian $\mathbb{R}$-complex Finsler spaces, if $det(g_{i\bar{j}} \neq 0)$ and non-Hermitian

$\mathbb{R}$-complex Finsler spaces, if $det\left(g_{ij} \neq 0\right)$. In this section, we determine the fundamental tensor of complex Square metric and inverse also.

Consider $\mathbb{R}$-complex Finsler space with Square metric,

$$L(\alpha,\beta)=\left(\frac{\left(\alpha+|\beta|\right)^2}{\alpha}\right)^2 \tag{3.1}$$

then it follows that $F=\left(\frac{\left(\alpha+|\beta|\right)^2}{\alpha}\right)$.

Now, we find the following quantities of *F*.

From the equalities (2.6) and (2.7) with metric (3.1), we have

$$\alpha L_\alpha+\beta L_\beta=2L,\ \alpha L_{\alpha\alpha}+\beta L_{\alpha\beta}=L_\alpha, \tag{3.2}$$

$$\alpha L_{\alpha\beta}+\beta L_{\beta\beta}=L_\beta,\ \alpha^2 L_{\alpha\alpha}+2\alpha\beta L_{\alpha\beta}+\beta^2 L_{\beta\beta}=2L,$$

where

$$L_\alpha=\frac{\partial L}{\partial\alpha},\ L_\beta=\frac{\partial L}{\partial\beta},\ L_{\alpha\beta}=\frac{\partial^2 L}{\partial\alpha\partial\beta},\ L_{\alpha\alpha}=\frac{\partial^2 L}{\partial\alpha^2},\ L_{\beta\beta}=\frac{\partial^2 L}{\partial\beta^2}. \tag{3.3}$$

$$L_\alpha=\frac{4\left(\alpha+|\beta|\right)^3}{\alpha^2}-\frac{2\left(\alpha+|\beta|\right)^4}{\alpha^3}, \tag{3.4}$$

$$L_\beta=\frac{2\alpha^4}{\left(\alpha-\beta\right)^3}, \tag{3.5}$$

$$L_{\alpha\alpha}=2\left(1+\frac{|\beta|}{\alpha}\right)^2\left(\frac{3|\beta|^2}{\alpha^2}-\frac{2|\beta|}{\alpha}+4\right), \tag{3.6}$$

$$L_{\beta\beta}=12\left(1+\frac{|\beta|}{\alpha}\right)^2, \tag{3.7}$$

$$L_{\alpha\beta}=4\left(1+\frac{|\beta|}{\alpha}\right)^2\left(1-\frac{2|\beta|}{\alpha}\right), \tag{3.8}$$

$$\alpha L_\alpha+\beta L_\beta=\frac{2F}{\alpha}\left[\alpha-|\beta|+2\alpha|\beta|\right], \tag{3.9}$$

$$\alpha L_{\alpha\alpha}+\beta L_{\alpha\beta}=\frac{2F}{\alpha^2}\left[4\alpha^3+8|\beta|^3-\alpha|\beta|\left(2\alpha+3|\beta|\right)+4\alpha|\beta|\right]. \tag{3.10}$$

We propose to determine the metric tensors of an $\mathbb{R}$-complex Finsler space using the following equalities:

$$g_{ij}=\frac{\partial^2 L\left(z,\eta,\bar{z},\lambda\bar{\eta}\right)}{\partial\eta^i\partial\eta^j},\ g_{i\bar{j}}=\frac{\partial^2 L\left(z,\eta,\bar{z},\lambda\bar{\eta}\right)}{\partial\eta^i\partial\bar{\eta}^j}.$$

Each of these being of interest in the following:

Consider,

$$\frac{\partial\alpha}{\partial\eta^i}=\frac{1}{2\alpha}\left(a_{ij}\eta^j+a_{i\bar{j}}\bar{\eta}^j\right)=\frac{1}{2\alpha l_i},\ \frac{\partial\beta}{\partial\eta^i}=\frac{1}{2}b_i.$$

$$\frac{\partial \alpha}{\partial \bar{\eta}^i} = \frac{1}{2\alpha}\left(a_{\bar{i}\bar{j}}\bar{\eta}^j + a_{\bar{i}j}\eta^j\right) = \frac{\partial \beta}{\partial \bar{\eta}^i} = \frac{1}{2}b_{\bar{i}},$$

where,

$$l_i = \left(a_{ij}\eta^j + a_{i\bar{j}}\eta^{\bar{j}}\right),\ l_{\bar{j}} = a_{\bar{i}\bar{j}}\bar{\eta}^i + a_{i\bar{j}}\eta^i.$$

Then, we can find,

$$l_i\eta^i + l_{\bar{j}}\bar{\eta}^j = 2\alpha^2.$$

We denote:

$$\eta^i = \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i}F^2 = 2F\frac{\partial}{\partial \eta^i}\left(\frac{\alpha^2}{\alpha-\beta}\right),$$

$$\eta_i = \rho_0 l_i + \rho_1 b_i,$$

where

$$\rho_0 = \frac{1}{2}\alpha^{-1}L_\alpha, \tag{3.11}$$

and

$$\rho_1 = \frac{1}{2}L_\beta. \tag{3.12}$$

Differentiating ρ_0 and ρ_1 with respect to η^j and $\bar{\eta}^j$ respectively, which yields:

$$\frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2}l_j + \rho_{-1}b_j,$$

and

$$\frac{\partial \rho_0}{\partial \bar{\eta}^j} = \rho_{-2}l_{\bar{j}} + \rho_{-1}b_{\bar{j}}.$$

Similarly,

$$\frac{\partial \rho_1}{\partial \eta^i} = \eta_{-1}l_i + \mu_0 b_i,\ \frac{\partial \rho_1}{\partial \bar{\eta}^i} = \rho_{-1}l_{\bar{i}} + \mu_0 b_{\bar{i}},$$

where,

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha - L_\alpha}}{4\alpha^3},\ \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha},\ \mu_0 = \frac{L_{\beta\beta}}{4}. \tag{3.13}$$

By direct computation using (3.11), (3.12), (3.13), we obtain the invariants of $\mathbb{R}$-complex Finsler space with Square metric: ρ_0, ρ_1, ρ_{-1}, ρ_{-2} are given below:

$$\rho_0 = \frac{1}{2\alpha}\left\{\frac{4\left(\alpha+|\beta|\right)^3}{\alpha^2} - \frac{2\left(\alpha+|\beta|\right)^4}{\alpha^3}\right\}, \tag{3.14}$$

$$\rho_1 = \frac{2\left(\alpha+|\beta|\right)^3}{\alpha^2}, \tag{3.15}$$

$$\rho_{-2} = 2\alpha\left(1+\frac{|\beta|}{\alpha}\right)^2\left(\frac{3|\beta|^2}{\alpha^2}-\frac{2|\beta|}{\alpha}+4\right)-\frac{4(\alpha+|\beta|)^3}{\alpha^2}+\frac{2(\alpha+|\beta|)^4}{\alpha^3}, \tag{3.16}$$

$$\rho_{-1} = \frac{\left(1+\frac{|\beta|}{\alpha}\right)^2\left(1-\frac{2|\beta|}{\alpha}\right)}{\alpha}, \tag{3.17}$$

$$\mu_0 = 3\left(1+\frac{|\beta|}{\alpha}\right)^2. \tag{3.18}$$

Fundamental Metric Tensor of $\mathbb{R}$-Complex Finsler Space with Square Metric

The fundamental metric tensors of $\mathbb{R}$-complex Finsler space with (α,β) metric are given by [9]:

$$g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \rho_{-2} l_i l_{\bar{j}} + \mu_0 b_i b_{\bar{j}} + \rho_1\left(b_{\bar{j}} l_i + b_i l_{\bar{j}}\right) \tag{3.19}$$

By using the Equations (3.14) to (3.18) in (3.19) we have

$$\begin{aligned} g_{ij} &= \frac{F}{\alpha}\left\{2\left(1+\frac{|\beta|}{\alpha}\right)-F\right\}a_{ij} \\ &+\alpha F\left\{8\alpha^3-4\alpha^2+2\alpha|\beta|(3|\beta|+\alpha)-4\alpha^3|\beta|-9\alpha|\beta|+\alpha F\right\}l_i l_j \\ &+\frac{3F^2-\alpha^2-2\alpha|\beta|}{\alpha(\alpha-F)}b_i b_j+\frac{\alpha-2|\beta|}{2F\left(\alpha^2+|\beta|\right)^2\left(2-\alpha^2+|\beta|\right)}\eta_i\eta_j. \end{aligned} \tag{3.20}$$

$$\begin{aligned} g_{i\bar{j}} &= \frac{F}{\alpha}\left\{2\left(1+\frac{|\beta|}{\alpha}\right)-F\right\}a_{i\bar{j}} \\ &+\alpha F\left\{8\alpha^3-4\alpha^2+2\alpha|\beta|(3|\beta|+\alpha)-4\alpha^3|\beta|-9\alpha|\beta|+\alpha F\right\}l_i l_{\bar{j}} \\ &+\frac{3F^2-\alpha^2-2\alpha|\beta|}{\alpha(\alpha-F)}b_i b_{\bar{j}}+\frac{\alpha-2|\beta|}{2F\left(\alpha^2+|\beta|\right)^2\left(2-\alpha^2+|\beta|\right)}\eta_i\eta_{\bar{j}} \end{aligned} \tag{3.21}$$

Or, equivalently,

$$g_{ij} = Aa_{ij} + Bl_i l_j + Cb_i b_j + D\eta_i\eta_j, \tag{3.22}$$

$$g_{i\bar{j}} = Aa_{i\bar{j}} + Bl_i l_{\bar{j}} + Cb_i b_{\bar{j}} + D\eta_i\eta_{\bar{j}}, \tag{3.23}$$

where,

$$A = \frac{F}{\alpha}\left\{2\left(1+\frac{|\beta|}{\alpha}\right)-F\right\}, \tag{3.24}$$

$$B = \alpha F\left\{8\alpha^3-4\alpha^2+2\alpha|\beta|(3|\beta|+\alpha)-4\alpha^3|\beta|-9\alpha|\beta|+2\alpha F\right\}, \tag{3.25}$$

$$C = \frac{3F^2-\alpha^2-2\alpha|\beta|}{\alpha(\alpha-F)}, \tag{3.26}$$

$$D = \frac{\alpha-2|\beta|}{2F\left(\alpha^2+|\beta|^2\right)\left(2-\alpha^2+|\beta|\right)}. \tag{3.27}$$

Next to determine the determinant and inverse of the tensor field $g_{i\bar{j}}$ through the theorem below by using Proposition (2.1). The solution of the non-Hermitian metric $Q_{i\bar{j}}$ as follows.

Theorem 3.2. *For a non-Hermitian* $\mathbb{R}$*-Complex Finsler space with Square metric* $F=\frac{(\alpha+|\beta|)^2}{\alpha}$, *then they have the following*:

1) The contravariant tensor g^{ij} of the fundamental tensor g_{ij} is:

$$g^{\bar{j}i}=\frac{\alpha^4}{F\left\{2\left(\alpha+|\beta|\right)-\alpha^3-\alpha|\beta|\left(|\beta|-2\alpha\right)\right\}}\left[Aa^{\bar{j}i}\left(\frac{\alpha FB}{\alpha^3+\alpha FB\mu}+\frac{c\left(\alpha FB\epsilon\right)^2}{\delta\left(\alpha^3+\alpha FB\mu\right)^2}\right)\eta^i\eta^{\bar{j}}+\frac{C}{\delta}b^ib^{\bar{j}}+\frac{FBC\epsilon}{\delta(1+B\mu)}\left(b^i\eta^{\bar{j}}+\eta^ib^{\bar{j}}\right)+\frac{M^2\eta^i\eta^{\bar{j}}+MN\left(\eta^ib^{\bar{j}}+b^i\eta^{\bar{j}}\right)+N^2b^ib^{\bar{j}}}{L}\right], \tag{3.28}$$

where

$$X=\left[1+\left(\frac{B}{1+B\mu}+\frac{CB^2\epsilon^2}{\delta(1+B\mu)^2}\right)\right]\mu+\frac{BC\epsilon}{\delta(1+B\mu)^2},\quad \text{and } Y=\frac{C}{\delta}+\frac{BC\epsilon\mu}{\delta(1+B\mu)}. \tag{3.29}$$

2) $$det\left(a_{i\bar{j}}+pl_il_{\bar{j}}+qb_ib_{\bar{j}}+r\eta_i\eta_{\bar{j}}\right)=\left[1+\left(X\mu-Y\epsilon\right)\sqrt{D}\right]\left[1+\omega+\frac{B\epsilon^2}{1+B\mu}\right](1+B\mu)Adet\left(q_{i\bar{j}}\right)$$

where, $D=\frac{\alpha-2|\beta|}{2F\left(\alpha^2+|\beta|^2\right)\left(2-\alpha^2+|\beta|\right)}$.

Proof. We prove this theorem by following three steps:

Step 1: We write $g_{i\bar{j}}$ from (3.21) in the form.

$$g_{i\bar{j}}=\left[Aa_{i\bar{j}}+Bl_il_{\bar{j}}+Cb_ib_{\bar{j}}+D\eta_i\eta_{\bar{j}}\right]. \tag{3.30}$$

We take $Q_{i\bar{j}}=a_{i\bar{j}}$ and $C_i=\sqrt{B}l_i$. By applying the proposition 2.1 we obtain $Q^{i\bar{j}}=a^{ji}$, $C^2=C_iC^i=\sqrt{B}l_i\times Q^{\bar{j}i}\times C_j=\sqrt{B}l_i\times a^{\bar{j}i}\times\sqrt{B}l_j=B\times l_ia^{i\bar{j}}l_{\bar{j}}=B\gamma$, and $1+C^2=(1+B\gamma)$.

So, the matrix $H_{i\bar{j}}=Aa_{i\bar{j}}-Bl_il_{\bar{j}}$, is invertible with

$$H^{i\bar{j}}=Aa^{\bar{j}i}+\frac{1}{1+AB\mu}\eta^i\eta^{\bar{j}},$$

$$det\left(Aa_{i\bar{j}}+Bl_il_{\bar{j}}\right)=(1+B\mu)=Adet\left(a_{i\bar{j}}\right).$$

Step 2: Now, we consider

$$Q_{i\bar{j}}=Aa_{i\bar{j}}+Bl_il_{\bar{j}},\text{ and } C_i=\sqrt{C}b_i,$$

By applying the proposition 2.1 we have

$$Q^{\bar{j}i} = Aa^{\bar{j}i} + \frac{B\eta^i \eta^{\bar{j}}}{1+AB\mu},$$

$$C^2 = C_i C^i = Q^{\bar{j}i} \times C_{\bar{j}} = \sqrt{C} b_i \left[Aa^{i\bar{j}} + \frac{B\eta^i \eta^{\bar{j}}}{1+AB\mu} \sqrt{C} b^{\bar{j}} \right],$$

$$c^2 = C\left[A\omega + \frac{B\epsilon^2}{1+AB\mu} \right].$$

Therefore,

$$1+C^2 = 1+C\left[A\omega + \frac{B\epsilon^2}{1+AB\mu} \right] \neq 0,$$

where, $\epsilon = b_j \eta^j$, $\omega = b_j b^j$.

It results that the inverse of $H_{ij} = Aa_{i\bar{j}} + Bl_i l_{\bar{j}} + Cb_i b_{\bar{j}}$ exists and it is

$$H^{\bar{j}i} = Q^{\bar{j}i} + \frac{1}{1+C^2} C^i C^{\bar{j}},$$

$$H^{\bar{j}i} = Aa^{\bar{j}i} + \frac{B\eta^i \eta^{\bar{j}}}{1+AB\mu} + \frac{B\left[b^i + \frac{B\epsilon\eta^i}{1+AB\mu} \right]\left[b^{\bar{j}} + \frac{B\epsilon\eta^{\bar{j}}}{1+AB\mu} \right]}{\tau}, \quad (3.31)$$

$$\begin{aligned} H^{\bar{j}i} &= Aa^{\bar{j}i} + \left(\frac{B}{1+AB\mu} + \frac{B^2 C\epsilon^2}{\tau(1+AB\mu)^2} \right)\eta^i \eta^{\bar{j}} \\ &\quad + \frac{BC\epsilon}{1+AB\mu}\left(b^i \eta^{\bar{j}} + b^{\bar{j}} \eta^i \right) + \frac{C}{\tau} b^i b^{\bar{j}}, \end{aligned} \quad (3.32)$$

where,

$$\delta = 1+C\left[A\omega + \frac{B\epsilon^2}{1+AB\mu} \right],$$

and,

$$det\left[Aa_{i\bar{j}} + Bl_i l_{\bar{j}} + Cb_i b_{\bar{j}} \right] = \left[1+C\left(A\omega + \frac{B\epsilon^2}{1+AB\mu} \right) \right](1+AB\mu)\, Adet\left(a_{i\bar{j}} \right). \quad (3.33)$$

Step 3: We put

$$Q_{\bar{j}i} = Aa_{i\bar{j}} + Bl_i l_{\bar{j}} + Cb_i b_{\bar{j}}, \quad (3.34)$$

and

$$C_i = \sqrt{D}\eta_i,$$

clearly observe that and obtain

$$\begin{aligned} Q^{\bar{j}i} &= Aa^{\bar{j}i} + \left(\frac{B}{1+AB\mu} + \frac{CB^2\epsilon^2}{\tau(1+AB\mu)} \right)\eta^i \eta^{\bar{j}} \\ &\quad + \frac{BC\epsilon}{1+AB\mu}\left(b^i \eta^{\bar{j}} + b^{\bar{j}} \eta^i \right) + \frac{C}{1+AB\mu} b^i b^{\bar{j}}, \end{aligned} \quad (3.35)$$

and $C_i = X\eta^i + Yb^{\bar{j}}$, where

$$X=\left[1+\left(\frac{B}{1+B\mu}+\frac{B^2C\epsilon^2}{\delta(1+p\mu)^2}\right)\right]\mu+\frac{pq\epsilon}{\delta(1+B\mu)^3}, \tag{3.36}$$

$$Y=\frac{C}{\delta}+\frac{BC\epsilon\mu}{\delta(1+B\mu)}. \tag{3.37}$$

And

$$C^2=(X\mu+Y\epsilon)\sqrt{D},$$

$$1+C^2=\left[Aa^{\bar{j}i}+\left(\frac{B}{1+b\mu}+\frac{CB^2\epsilon^2}{\delta(1+B\mu)}+\frac{BC\epsilon}{1+B\mu}+\frac{C}{1+B\mu}\right)\right]\sqrt{D}\neq 0,$$

clearly, the matrix $H_{i\bar{j}}$ is invertible.

$$C^i=Aa^{\bar{j}i}+\left\{\frac{B\eta^i\eta^{\bar{j}}}{1+AB\mu}+\frac{C\left[b^i+\frac{B\epsilon\eta^i}{1+B\mu}\right]\left[b^{\bar{j}}+\frac{B\epsilon\eta^{\bar{j}}}{1+B\mu}\right]}{\delta}\right\}\eta_{\bar{j}},$$

and

$$C^{\bar{j}}=Aa^{\bar{j}i}+\left\{\frac{B\eta^i\eta^{\bar{j}}}{1+AB\mu}+\frac{C\left[b^i+\frac{B\epsilon\eta^i}{1+B\mu}\right]\left[b^{\bar{j}}+\frac{B\epsilon\eta^{\bar{j}}}{1+B\mu}\right]}{\delta}\right\}\eta_i,$$

where

$$C^iC^{\bar{j}}=X^2\eta^i\eta^{\bar{j}}+XY\left(\eta^ib^{\bar{j}}+\eta^{\bar{j}}b^i\right)+Y^2b^ib^{\bar{j}}.$$

Again by applying Proposition (2.1) we obtain the inverse of $H_{i\bar{j}}$ as:

$$\begin{aligned}H^{\bar{j}i}=&Aa^{\bar{j}i}+\left(\frac{\alpha FB}{\alpha^3+F\alpha B\mu}+\frac{C(\alpha FB\epsilon)^2}{\delta(\alpha^3+F\alpha B\mu)^2}\right)\eta^i\eta^{\bar{j}}+\frac{C}{\delta}b^ib^{\bar{j}}\\&+\frac{FBC\epsilon}{\delta(1+B\mu)}\left(b^i\eta^j+b^{\bar{j}}\eta^i\right)+\frac{X^2\eta^i\eta^{\bar{j}}+XY\eta^ib^{\bar{j}}+b^i\eta^{\bar{j}}+Y^2b^ib^{\bar{j}}}{L}.\end{aligned} \tag{3.38}$$

$$\begin{aligned}&det\left(Aa_{i\bar{j}}+Bl_il_{\bar{j}}+Cb_ib_{\bar{j}}+D\eta_i\eta_{\bar{j}}\right)\\&=\left[1+(X\mu+Y\epsilon)\sqrt{D}\right]\left[1+A\omega+\frac{B\epsilon^2}{1+B\mu}\right](1+B\mu)\,det\left(a_{i\bar{j}}\right).\end{aligned} \tag{3.39}$$

But $g_{i\bar{j}}=AH_{i\bar{j}}$, with $H_{i\bar{j}}$ from last step. Thus

$$g^{ji}=\frac{1}{A}H^{i\bar{j}}. \tag{3.40}$$

Therefore, from Equation (3.38) in Equation (3.40) and the Equation (3.39), then we obtained claims 1) and 2) are desired. □

4. Holomorphic Curvature of Complex Square Metric

The holomorphic curvature is the correspondent of the holomorphic sectional curvature in Complex Finsler geometry. Our goal is to find a notation of Complex Finsler spaces with square metric. By analogy with the naming from the real case [10], we shall call it the holomorphic flag curvature and we shall introduce it with respect to Chern-Finsler connection (c.n.c).

The holomorphic curvature $K_F(z,\eta)$ depends on the position $z \in M$ alone. In view of definition (2.1) we obtain the holomorphic curvature of Complex Finsler space with square metric if $R_{\bar{i}\bar{j}} = -g_{\bar{l}\bar{j}}\delta_{\bar{h}}\left(N_k^l\right)\bar{\eta}^h$, where, N_k^l is the Chern-Finsler connection coefficients.

To find Riemannian curvature $R_{\bar{j}k}$, we need the Chern Finsler connection (c.n.c) coefficients. Now, by direct computations, we get the Chern-Finsler (c.n.c) connection coefficients;

$$\begin{aligned} N_K^l = \frac{\alpha^4}{\left(\alpha+|\beta|\right)^2}\Bigg[& a^{\bar{j}i} + \left(\frac{\alpha FB}{\alpha^3 + F\alpha B\mu} + \frac{C\left(\alpha FB\epsilon\right)^2}{\delta\left(\alpha^3+\alpha FB\mu\right)^2}\right)\eta^i\eta^{\bar{j}} + \frac{C}{\delta}b^i b^{\bar{j}} \\ & + \frac{FBC\epsilon}{\delta(1+B\mu)}\left(b^i\eta^{\bar{j}} + b\bar{j}\eta^i\right) + \frac{M^2\eta^i\eta^{\bar{j}} + MN\left(\eta^i b^{\bar{j}} + b^i\eta^{\bar{j}} + N^2 b^i b^{\bar{j}}\right)}{L} \\ & \times \Bigg\{ \frac{\alpha^3 A_1 - A\left(A\left(\frac{3}{2}\alpha\eta^i\eta^{\bar{j}}\right)\right)}{\alpha^3} a_{\bar{i}\bar{j}} - F\left(A_2\right)l_i l_{\bar{j}} + 2\left(\frac{\alpha^4-\alpha^3 F}{\alpha}\right)A_3 b_i b_{\bar{j}} \\ & + \frac{\left(\alpha^4 - 2\alpha^3|\beta|\right)A_4' - a_4\left(A_5\right)}{\left(A_4\right)^2}\eta_i\eta_{\bar{j}}\Bigg\}\Bigg], \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} A = {} & 2\alpha^4 + 2|\beta|^4 + 2\alpha^2|\beta|^2 + 8\alpha^3|\beta| + 8\alpha|\beta|^3 - \alpha^5 + 3\alpha|\beta|^4 \\ & - 2\alpha^3|\beta|^2 - 3\alpha^4|\beta| - 4\alpha^2|\beta| + \alpha^5 \end{aligned}$$

$$\begin{aligned} A_1 = {} & \left(4\alpha^2 + 2|\beta|^2 + 12\alpha|\beta| + \frac{4|\beta|^3}{\alpha} - \frac{5\alpha^3}{2} + \frac{3}{2}\frac{|\beta|^4}{\alpha} - 3\alpha|\beta|^2 \right. \\ & \left. + 2\alpha^2|\beta| - |\beta|\right)\eta^i\eta^{\bar{j}} + \left(8|\beta|^3 + 4\alpha^2|\beta| + 8\alpha^3 + 24\alpha^2|\beta|^2 \right. \\ & \left. + 2\alpha|\beta|^3 - 4\alpha^3|\beta| - 3\alpha^4 - 4\alpha^2 + 5|\beta|^4\right)\eta^i \end{aligned}$$

$$\begin{aligned} A_2 = {} & \left(12\alpha^2 - 4\alpha + 3|\beta|^2 - \alpha|\beta| - 6\alpha^2|\beta| - 9|\beta| + \alpha + 2|\beta|\right)\eta^i\eta^{\bar{j}} \\ & + \left(12 - 2\alpha^2 - 4\alpha^3 - 4|\beta|\right)\eta^i \end{aligned}$$

$$A_3 = \left(3\alpha^3 + \alpha|\beta|^2 + 3|\beta|\right)\eta^i\eta^{\bar{j}} + 12\left(|\beta|^3 + 3\alpha^2|\beta| + 3\alpha|\beta|\right)\eta^i$$

$$\begin{aligned} A_4 = {} & 4\alpha^7 - 2\alpha^4 + 4\alpha^5|\beta|^2 + 12\alpha^3|\beta|^3 + 8\alpha^6|\beta| + 16\alpha^4|\beta|^2 \\ & + 2\alpha^2|\beta|^4 + 4\alpha^5|\beta| - 2\alpha^4|\beta| + 8\alpha|\beta|^4 \end{aligned}$$

$$A_4' = \left(24\alpha^6 + 14\alpha^5 - 4\alpha^2 + 2|\beta|^4 + 4\frac{|\beta|^4}{\alpha} + |\beta| + 10\alpha^3|\beta|^2 \right.$$

$$\left. + 18\alpha|\beta|^3 2\alpha^2|\beta|^2 + 20\alpha^3|\beta| + 4\alpha^2|\beta| \right)\eta^i\eta^{\bar{j}}$$

$$+\left(8\alpha^6 + 12\alpha^5 + 4\alpha + 2\alpha^4 + 32|\beta|^3 + 10|\beta|^4 + 36\alpha^3|\beta|^2\right)\eta^i$$

$$A_5 = 2\alpha^3\eta^i + 3\alpha|\beta|\eta^i\eta^{\bar{j}}.$$

Observed that the Equation (4.1) can be expressed by the identity as:

$$N_k^l = Re\left(N_k^l\right) + Img\left(N_k^l\right), \tag{4.2}$$

where,

$$Re\left(N_k^l\right) = \frac{\alpha^4}{\left(\alpha+|\beta|\right)^2}\left[\left(\frac{\alpha FB}{\alpha^3 + F\alpha B\mu} + \frac{C\left(\alpha FB\epsilon\right)^2}{\delta\left(\alpha^3 + \alpha FB\mu\right)^2}\right)\eta^i\eta^{\bar{j}}\right.$$

$$+\frac{M^2}{L}\left\{\frac{\alpha^3 A_1 - A\left(A\left(\frac{3}{2}\alpha\eta^i\eta^{\bar{j}}\right)\right)}{\alpha^3}a_{i\bar{j}}\right\}\eta^i\eta^{\bar{j}}$$

$$\left.+\frac{M^2}{L}\eta_i\eta_{\bar{j}}\left\{\frac{\alpha^4 - 2\alpha^3|\beta|A_4' - A_4\left(A_5\right)}{\left(A_4\right)^2}\right\}\eta^i\eta^{\bar{j}}\right]$$

$$Img\left(N_k^l\right) = \frac{\alpha^4}{\left(\alpha+|\beta|\right)^2}\left[a_{a_{i\bar{j}}+\frac{C}{\delta}b^i b^{\bar{j}}} + \frac{FBC}{\delta\left(1+B\mu\right)}\left(b^i\eta^{\bar{j}} + b_{\bar{j}}\eta^i\right)\right.$$

$$+\frac{MN}{L}\left(\eta^i b^{\bar{j}} + b^i\eta^{\bar{j}} + N^2 b^i b^{\bar{j}}\right) - F\left(A_2\right)l_i l_{\bar{j}} \quad .$$

$$\left.+2\left(\frac{\alpha^4 - \alpha^3 F}{\alpha}\right)A_3 b_i b_{\bar{j}}\right]$$

Now using Equation (4.1) and ($g^{\bar{j}i}$) on $R_{\bar{j}k}$ (see definition (2.1)) we get the Riemann curvature tensor $R_{\bar{j}k}$ as,

$$R_{\bar{j}k} = \frac{\alpha^4}{\left(\alpha+|\beta|\right)^2}\left\{D_{17}+D_{18}+D_{19}+D_{20}\right\}\left(b^i b^{\bar{j}} l_i l_{\bar{j}}\right)\bar{\eta}^h + \left\{D_{41}-D_{42}-D_{43}\right.$$

$$+D_{44}+D_{45}+D_{46}-D_{47}+D_{48}+D_{53}+D_{54}+D_{55}+D_{56}+D_{57}+D_{58}$$

$$\left.+D_{59}+D_{60}+D_{91}+D_{92}+D_{93}+D_{167}+D_{168}+D_{169}+D_{170}\right\}\left(\eta^i\eta^{\bar{j}} l_i l_{\bar{j}}\right)\bar{\eta}^h$$

$$+\left\{D_{21}+D_{95}+D_{173}\right\}\left(\eta^i b^{\bar{j}} l_i l_{\bar{j}}\right)\bar{\eta}^h + \left\{D_{120}+D_{121}+D_{231}-D_{232}-D_{233}\right.$$

$$+D_{234}+D_{235}+D_{236}-D_{237}+D_{238}-D_{241}+D_{242}+D_{243}+D_{244}+D_{245}$$

$$+D_{246}+D_{247}+D_{248}+D_{249}+D_{253}+D_{254}+D_{255}+D_{256}+D_{257}+D_{258} \tag{4.3}$$

$$\left.+D_{259}+D_{260}+D_{261}+D_{262}+D_{263}\right\}\left(\eta^i b^{\bar{j}} + b^i\eta^{\bar{j}}\right)\bar{\eta}^h,$$

where

$$D_{17} = CFA_2,\ D_{18} = N^2 FA_2,\ D_{19} = \frac{F^2 A_2 BC\epsilon}{\delta(1+B\mu)},\ D_{20} = \frac{XYFA_2}{L},$$

$$D_{41} = \frac{3ABCF^2\epsilon}{2\delta\alpha^4\left(\alpha^2 + B\mu\alpha^2 + FB\mu + FB^2\mu^2\right)},\ D_{42} = \frac{\alpha^3 ABX^2 F}{L\left(\alpha^3 + F\alpha B\mu\right)},$$

$$D_{43} = \frac{3X^2 AB}{2\left(\alpha^3 + F\alpha B\mu\right)},\ D_{44} = \frac{3XYAB}{2\left(\alpha^3 + F\alpha B\mu\right)},$$

$$D_{45} = \frac{A_4'\left(\alpha^5 - 2\alpha^4|\beta|\right) - A_4 A_5}{A_4^2\left(\alpha^3 + F\alpha B\mu\right)},\ D_{46} = \frac{3Y^2 ABF}{2\left(\alpha^3 + F\alpha B\mu\right)},$$

$$D_{47} = \frac{XYA_1 FB}{\alpha^3 + F\alpha + F\alpha B\mu},\ D_{48} = \frac{ACBF}{\delta\left(\alpha^3 + F\alpha B\mu\right)},$$

$$D_{53} = \frac{CF^2\alpha A_2}{\delta\left(\alpha^3 + F\alpha B\mu\right)},\ D_{54} = \frac{N^2 F^2 A_2 B}{\alpha_F^3\alpha B\mu},\ D_{57} = \frac{XY\alpha F^2 A_2 B}{L\left(\alpha^3 + F\alpha B\mu\right)},$$

$$D_{58} = \frac{2C\alpha^5 F^3 B^3 \epsilon A_3^2(\alpha - F)}{\delta\left(\alpha^3 + F\alpha B\mu\right)^3},\ D_{59} = \frac{2\left(\alpha^4 F^2 B^2 - \alpha^3 B^3 F^3\right)}{\left(\alpha^3 + F\alpha B\mu\right)^2},$$

$$D_{60} = 2A_3\alpha^2(\alpha - F),\ D_{91} = \frac{C^2 FA_2}{\delta^2},\ D_{92} = \frac{Y^2 FA_2 C}{\delta},$$

$$D_{93} = \frac{F^2 A_2 BC^2\epsilon}{\delta(1+B\mu)},\ D_{167} = \frac{C(\alpha FBX\epsilon)^2 FA_2}{\delta L\left(\alpha^3 + F\alpha B\mu\right)^2},\ D_{168} - \frac{X^4 FA_2}{L^2},$$

$$D_{169} = \frac{CX^2 FA_2}{\delta L},\ D_{170} = \frac{X^2 Y^2 FA_2}{L},\ D_{96} = \frac{2C^2\alpha^2(\alpha FB\epsilon)^2(\alpha - F)}{\delta^2\left(\alpha^3 + F\alpha B\mu\right)^2} A_3,$$

$$D_{21} = \frac{XYFA_2}{L},\ D_{95} = \frac{XYFCA_2}{L\delta},\ D_{173} = \frac{X^3 YFA_2}{L^2},\ D_{120} = \frac{3XYAFBC\epsilon}{2\alpha\delta(1+B\mu)},$$

$$D_{121} = \frac{\left(\left(\alpha^4 - 2\alpha^3|\beta|\right)A_4' - A_4 A_5\right)FBC\epsilon}{\delta(A_4)^2(1+B\mu)},\ D_{237} = \frac{A_1(FBXY)}{L\alpha(\alpha^2 + FB\mu)},$$

$$D_{232} = \frac{3ACXY(FB\epsilon)^2}{2\alpha\delta L\left(\alpha^3 + f\alpha B\mu\right)^2},\ D_{233} = \frac{3CAXY}{2\alpha^2 L},\ D_{234} = \frac{3FBCXYA\epsilon}{\delta L(1+B\mu)2\alpha^3},$$

$$D_{235} = \frac{\alpha^2 AX^3 Y}{L^2},\ D_{236} = \frac{3X^3 YA}{2\alpha L},\ D_{237} = \frac{3X^2 Y^2 A}{2\alpha L},$$

$$D_{238} = \frac{XY\left(\alpha^4 - 2\alpha^3|\beta|\right)A_4' - A_4 A_5}{(A_4)^2},\ D_{241} = \frac{A_1 CXY}{\delta\alpha L},$$

$$D_{242} = \frac{XY^2 A}{L},\ D_{243} = \frac{\alpha F^2 BA_2 XY}{\alpha^3 + F\alpha B\mu},\ D_{244} = \frac{CXY(\alpha FB\epsilon)^F A_2}{\delta L\left(\alpha^3 + F\alpha B\mu\right)^2},$$

$$D_{245} = \frac{M^3 NFA_2}{L^2},\ D_{246} = \frac{X^2 Y^2 FA_2}{L^2},\ D_{247} = \frac{2CXY\left((\alpha FB\epsilon)^2\alpha^2(\alpha - F)\right)}{\delta L\left(\alpha^3 + F\alpha B\mu\right)^2},$$

$$D_{253} = \frac{2X^2 Y^2(\alpha - F)}{L^2},\ D_{254} = \frac{\left(\alpha^4 - 2\alpha^3|\beta|A_4'\right) - A_4 A_5}{L(A_4)^2},$$

$$D_{255}=\frac{CXY\left(F\alpha B\epsilon\right)^{2}}{\delta L\left(\alpha^{3}+F\alpha B\mu\right)^{2}\left(A_{4}\right)^{2}}\left(\alpha^{4}-2\alpha^{3}\left|\beta\right|\right)A_{4}'-A_{4}A_{5},$$

$$D_{256}=\frac{FBC\epsilon XY}{\delta L\left(\alpha^{3}+F\alpha B\mu\right)\left(a_{4}\right)^{2}}\left(\alpha^{4}-2\alpha^{3}\left|\beta\right|\right)A_{4}'-A_{4}A_{5},$$

$$D_{257}=\frac{X^{2}Y}{L^{2}\left(A_{4}\right)^{2}}\left(\alpha^{4}-2\alpha^{3}\left|\beta\right|\right)A_{4}'-A_{4}A_{5}.$$

Notice that, on contracting with $(\eta)^{h}$ in $g^{\bar{j}i}$. We get the above coefficients D-tensor.

Again, by using Chern-Finsler connection coefficients, we get the coefficients of torsion.

$$\begin{aligned}T_{jk}^{i}=\;&D_{1}a_{i\bar{j}}+\{D_{28}+D_{29}\}a^{i\bar{j}}b^{i}\eta^{j}+\{D_{30}+D_{31}\}a^{i\bar{j}}b^{j}\eta^{i}\delta_{i}^{j}\\&+\{D_{32}+D_{33}\}^{i\bar{j}}b^{i}\eta^{j}\delta_{j}^{i}+\{D_{56}+D_{57}\}a^{i\bar{j}}b^{i}\eta^{j}\delta_{j}^{i}+\{D_{60}\}\eta^{i}b^{j}l_{i}j_{\bar{j}}\eta^{i}\bar{\eta}^{j}\\&+\{D_{73}+D_{74}+D_{185}+D_{186}\}\delta_{i}^{j}b^{i}\eta^{j}+\{D_{88}\}\eta^{i}+\{D_{96}+D_{97}\}b^{i}\eta^{j}l_{i}l_{\bar{j}}\\&+\{D_{98}\}\eta^{i}b^{j}l_{i}l_{\bar{j}}+\{D_{116}+D_{229}\}\eta^{i}b^{j}+\{D_{115}+D_{117}\}b^{i}\eta^{j}\\&+\{D_{126}\}\eta^{i}b^{j}b^{i}b^{\bar{j}}+\{D_{127}\}\eta^{i}b^{j}b^{\bar{j}}\eta^{i}+\{D_{128}\}b^{i}b^{\bar{j}}a_{i\bar{j}}b^{i}\eta^{j}\\&+\{D_{129}\}b^{i}b^{\bar{j}}a_{i\bar{j}}b^{j}\eta^{i}+\{D_{133}+D_{134}\}\left(\eta^{i}b_{j}b_{i}\eta^{j}+\eta^{i}\eta^{j}b_{i}\eta^{i}\right)\delta_{i}^{j}\\&+\{D_{135}\}\left(\eta^{i}b_{j}l_{i}l_{\bar{j}}b^{i}\eta^{j}+\eta^{i}b^{j}l_{i}l_{\bar{j}}b^{j}\eta^{i}\right)+\{D_{142}\}\left(\eta^{j}b_{j}b_{i}\eta^{j}+\eta^{i}\eta^{j}\delta_{i}^{j}\right)\\&+\{D_{149}\}\delta_{i}^{j}\left(b^{j}\eta^{i}+b^{j}\eta_{j}b^{j}\eta^{i}\right)+\{D_{150}+D_{151}\}\delta_{i}^{j}\left(b^{i}\eta_{i}b^{i}\eta^{j}+b^{i}b^{j}\right)\\&+\{D_{230}\}\eta^{i}\eta^{j}\eta^{i}b^{j}+\{D_{231}\}b^{i}\eta^{j}+\{D_{232}\}\eta^{i}\eta^{j}b^{i}\eta^{j}\\&+\{D_{248}\}\eta^{i}b^{j}l_{i}l_{\bar{j}}\left(\eta^{i}b^{j}+b^{i}\eta^{j}\right)+\{D_{251}\}a^{\bar{j}i}b_{i}b_{\bar{j}}\left(\eta^{i}b^{j}+b^{i}\eta^{j}\right)\\&+\{D_{260}+D_{261}\}\delta_{i}^{j}\left(b^{i}\eta_{j}\eta^{i}b^{j}+b^{j}\eta^{i}\right),\end{aligned}\tag{4.4}$$

where,

$$D_{1}=\frac{3A}{2\alpha^{3}},\ D_{28}=D_{29}=\frac{M^{2}}{L}\left(\frac{\alpha^{4}-2\alpha\left|\beta\right|A_{4}'-A_{4}A_{5}}{A_{4}^{2}}\right),$$

$$D_{30}=D_{31}=\frac{FBC\epsilon}{\delta\left(1+B\mu\right)}\left(\frac{\alpha^{4}-2\alpha\left|\beta\right|A_{4}'-A_{4}A_{5}}{A_{4}^{2}}\right),$$

$$D_{32}=D_{33}=\frac{MN}{L}\left(\frac{\alpha^{4}-2\alpha\left|\beta\right|A_{4}'-A_{4}A_{5}}{A_{4}^{2}}\right),$$

$$D_{56}=D_{57}=\frac{MNF^{2}A_{2}\alpha B}{L\left(\alpha^{3}+F\alpha B\mu\right)},\ D_{60}=2A_{3}\left(\frac{\alpha^{4}-\alpha^{3}F}{\alpha}\right),$$

$$D_{73}=\frac{N^{2}\alpha FB\left(\alpha^{3}-2\alpha^{3}\left|\beta\right|A_{4}'-A_{4}A_{5}\right)}{L\left(\alpha^{3}+F\alpha B\mu\right)A_{4}^{2}},\ D_{74}=\frac{A_{1}C}{\delta\alpha},$$

$$D_{185}=\frac{M^{2}N}{L^{2}}\left(\frac{\alpha^{4}A_{4}'-2\alpha\left|\beta\right|A_{4}'-A_{4}A_{5}}{\left(A_{4}\right)^{2}}\right),$$

$$D_{186}=\frac{CM^{2}}{\delta L}\left(\frac{\alpha^{4}A_{4}-2\alpha^{3}\left|\beta\right|A_{4}'-A_{4}A_{5}}{\left(A_{4}\right)^{2}}\right),\ D_{88}=\frac{\alpha F^{2}BCA_{2}}{\delta\left(\alpha^{3}+F\alpha B\mu\right)},$$

$$D_{96}=\frac{2C^2\left(\alpha FB\epsilon\right)^2\left(\alpha^2-\alpha^3F\right)}{\delta^2\alpha\left(\alpha^3+F\alpha B\mu\right)^2},\ D_{98}=\frac{2C\left(\alpha^4-\alpha^3F\right)}{\delta\alpha}A_3,$$

$$D_{97}=2\frac{\left(\alpha^4-\alpha^3F\right)FB}{\delta\alpha\left(\alpha^3+F\alpha B\mu\right)^2}A_3,\ D_{116}=\frac{3AFBC\epsilon}{2\alpha63\delta\left(1+B\mu\right)},\ D_{229}=\frac{3AMN}{2\alpha^3L},$$

$$D_{115}=\frac{3ABC\epsilon F}{2\alpha^3\delta\left(1+B\mu\right)},\ D_{117}=\frac{A_1FBC\epsilon}{2\alpha(1+B\mu)},\ D_{126}=\frac{3N^2AFBC\epsilon}{2\alpha\delta\left(1+B\mu\right)},$$

$$D_{127}=\frac{MNA_1FBC\epsilon}{L\alpha\delta\left(1+B\mu\right)},D_{129}=\frac{A_1C^2FB\epsilon}{\delta^2\alpha(1+B\mu)},\ D_{133}=\frac{N^2F^2A_2BC\epsilon}{\delta(1+B\mu)},$$

$$D_{134}=\frac{F^2A_2\left(BC\epsilon\right)^2}{\delta^2\left(1+B\mu\right)},\ D_{135}=\frac{MNF^2A_2BC\epsilon}{L\delta\left(1+B\mu\right)},$$

$$D_{142}=\frac{2\left(FBC\epsilon\right)^2\left(\alpha^2-\alpha^3F\right)}{\delta\alpha\left(1+B\mu\right)^2}A_3,\ D_{149}=\frac{FBC\epsilon^2\left(\alpha^4-2\alpha^3\left|\beta\right|\right)A_4'-A_4A_5}{\delta^2\left(1+B\mu\right)^2\left(A_4\right)^2},$$

$$D_{150}=\frac{MNFBC\epsilon\left(\alpha^4A_4-2\alpha^3\left|\beta\right|\right)A_4'-A_4A_5}{L\delta\left(1+B\mu\right)\left(A_4\right)^2},$$

$$D_{151}=\frac{\left(FBC\epsilon\right)^2\left(\alpha^4A_4'-2\alpha^3\left|\beta\right|\right)A_4'-A_4A_5}{\left(1+B\mu\right)^2\left(A_4\right)^2},\ D_{230}=\frac{A_1MN}{L\alpha},$$

$$D_{231}=\frac{3AMN}{2\alpha^3L},\ D_{232}=\frac{A_1\left(FBMN\right)}{L\alpha\left(\alpha^2+FB\mu\right)},$$

$$D_{238}=\frac{2CMN\left(\alpha FB\epsilon\right)^2\left(\alpha^4-\alpha^3F\right)}{\delta L\alpha\left(\alpha^3+F\alpha B\mu\right)^2}A_3,\ D_{251}=\frac{2CMN\left(\alpha^4-\alpha^3F\right)}{\delta^3\alpha}A_3,$$

$$D_{260}=\frac{M^2N^2\left(\alpha^4-2\alpha^3\left|\beta\right|\right)A_4'-A_4A_5}{L^2\left(A_4\right)^2},D_{261}=\frac{FBC\epsilon MN\left(\alpha^4-2\alpha^3\left|\beta\right|\right)A_4'-A_4A_5}{\delta L\left(1+B\mu\right)\left(A_4\right)^2}.$$

Theorem 4.3. *The holomorphic flag curvature of Complex Square metric* $F=\left(\frac{\left(\alpha+\left|\beta\right|\right)^2}{\alpha}\right)$ *is given by,*

$$K_F\left(Z,\eta\right)=\alpha^2\left(\alpha+\left|\beta\right|\right)^2\bar{\eta}^j\eta^kRe\left(N_k^l\right),\tag{4.5}$$

where,

$$\begin{aligned}N_k^l=\frac{\alpha^4}{\left(\alpha+\left|\beta\right|\right)^2}\Bigg[&\left(\frac{\alpha FB}{\alpha^3+F\alpha B\mu}+\frac{C\left(\alpha FB\epsilon\right)^2}{\delta\left(\alpha^3+\alpha FB\mu\right)^2}\right)\eta^i\eta^{\bar{j}}\\&+\frac{M^2}{L}\left\{\frac{\alpha^3A_1-A\left(A\left(\frac{3}{2}\alpha\eta^i\eta^{\bar{j}}\right)\right)}{\alpha^3}a_{i\bar{j}}\right\}\eta^i\eta^{\bar{j}}\\&+\frac{M^2}{L}\eta_i\eta_{\bar{j}}\left\{\frac{\alpha^4-2\alpha^3\left|\beta\right|A_4'-A_4\left(A_5\right)}{\left(A_4\right)^2}\right\}\eta^i\eta^{\bar{j}}\Bigg]\end{aligned}$$

Proof. From Equation (4.2) plugging into (2.1), it yields.

$$K_F(z,\eta)=\frac{2}{L^2}(z,\eta)g_{i\bar{j}}\left[\delta_{\bar{h}}\left(N_k^l\right)\bar{\eta}^h\right]\eta^j\eta^k, \tag{4.6}$$

where, N_k^l is in Equation (4.1).

Then, comparing (4.6) with (4.2) we get (4.5) as desired. □

Proposition 4.4. *If* $F=\left(\frac{\left(\alpha+|\beta|\right)^2}{\alpha}\right)$ *be Complex Square metric of dimension* $n\geq 2$ *with non-zero* $K_F(z,\eta)$, *then it is not a Kähler and not a weakly Kähler.*

Proof. Observing Equation (4.3) T_{jk}^i is non zero and since by definition it is not a Kähler. Further, on contracting T_{jk}^i by η^j, it yields

$$T_{jk}^i\eta^j\neq 0.$$

Therefore, it is not a weakly Kähler. □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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