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Dissipative Properties of ω-Order Preserving Partial Contraction Mapping in Semigroup of Linear Operator

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Abstract

This paper consists of dissipative properties and results of dissipation on infinitesimal generator of a C_0-semigroup of ω-order preserving partial contraction mapping (ω-OCP_n) in semigroup of linear operator. The purpose of this paper is to establish some dissipative properties on ω-OCP_n which have been obtained in the various theorems (research results) and were proved.

Keywords

Semigroup, Linear Operator, Dissipative Operator, Contraction Mapping and Resolvent

1. Introduction

Suppose X is a Banach space, $X_n \subseteq X$ a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup that is strongly continuous one-parameter semigroup of bounded linear operator in X. Let ω-OCP_n be ω-order-preserving partial contraction mapping (semigroup of linear operator) which is an example of C_0-semigroup. Furthermore, let $Mm(\mathbb{N})$ be a matrix, $L(X)$ a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $F(x)$ a duality mapping on X and A is a generator of C_0-semigroup. Taking the importance of the dissipative operator in a semigroup of linear operators into cognizance, dissipative properties characterized the generator of a semigroup of linear operator which does not require the explicit knowledge of the resolvent.

This paper will focus on results of dissipative operator on ω-OCP_n on Banach space as an example of a semigroup of linear operator called C_0-semigroup.

Yosida [1] proved some results on differentiability and representation of one-parameter semigroup of linear operators. Miyadera [2], generated some strongly continuous semigroups of operators. Feller [3], also obtained an unbounded semigroup of bounded linear operators. Balakrishnan [4] introduced fractional powers of closed operators and semigroups generated by them. Lumer and Phillips [5], established dissipative operators in a Banach space and Hille & Philips [6] emphasized the theory required in the inclusion of an elaborate introduction to modern functional analysis with special emphasis on functional theory in Banach spaces and algebras. Batty [7] obtained asymptotic behaviour of semigroup of operator in Banach space. More relevant work and results on dissipative properties of ω-Order preserving partial contraction mapping in semigroup of linear operator could be seen in Engel and Nagel [8], Vrabie [9], Laradji and Umar [10], Rauf and Akinyele [11] and Rauf *et al.* [12].

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [9]

C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [11]

Transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in \mathrm{Dom}\,\alpha : x \le y \Rightarrow \alpha x \le \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Subspace Semigroup) [8]

A subspace semigroup is the part of A in Y which is the operator A_* defined by $A_* y = Ay$ with domain $D(A_*) = \{y \in D(A) \cap Y : Ay \in Y\}$.

Definition 2.4 (Duality set)

Let X be a Banach space, for every $x \in X$, a nonempty set defined by $F(x) = \left\{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\right\}$ is called the duality set.

Definition 2.5 (Dissipative) [9]

A linear operator $(A, D(A))$ is dissipative if each $x \in X$, there exists $x^* \in F(x)$ such that $Re(Ax, x^*) \le 0$.

2.1. Properties of Dissipative Operator

For dissipative operator $A : D(A) \subseteq X \to X$, the following properties hold:

a) $\lambda - A$ is injective for all $\lambda > 0$ and

$$\left\|(\lambda - A)^{-1}\right\| \le 1/\lambda \|y\| \tag{2.1}$$

for all y in the range $\mathrm{rg}(\lambda - A) = (\lambda - A)D(A)$.

b) $\lambda - A$ is surjective for some $\lambda > 0$ if and only if it is surjective for each $\lambda > 0$. In that case, we have $(0, \infty) \subset \rho(A)$, where $\rho(A)$ is the resolvent of the generator A.

c) A is closed if and only if the range $\mathrm{rg}(\lambda - A)$ is closed for some $\lambda > 0$.

d) If $\mathrm{rg}(A) \subseteq D(A)$, that is if A is densely defined, then A is closable. its closure A is again dissipative and satisfies $\mathrm{rg}(\lambda - A) = \mathrm{rg}(\lambda - A)$ for all $\lambda > 0$.

Example 1

2×2 matrix $\left[M_m\left(\mathbb{N}\cup\{0\}\right)\right]$

Suppose

$$A=\begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}$$

and let $T(t)=e^{tA}$, then

$$e^{tA}=\begin{pmatrix}e^{t} & e^{2t}\\ e^{2t} & e^{2t}\end{pmatrix}$$

3×3 matrix $\left[M_m\left(\mathbb{N}\cup\{0\}\right)\right]$

Suppose

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 2\\ - & 2 & 3\end{pmatrix}$$

and let $T(t)=e^{tA}$, then

$$e^{tA}=\begin{pmatrix}e^{t} & e^{2t} & e^{3t}\\ e^{t} & e^{2t} & e^{2t}\\ I & e^{2t} & e^{3t}\end{pmatrix}$$

Example 2

In any 2×2 matrix $\left[M_m(\mathbb{C})\right]$, and for each $\lambda>0$ such that $\lambda\in\rho(A)$ where $\rho(A)$ is a resolvent set on *X*.

Also, suppose

$$A=\begin{pmatrix}1 & 2\\ - & 2\end{pmatrix}$$

and let $T(t)=e^{tA_\lambda}$, then

$$e^{tA_\lambda}=\begin{pmatrix}e^{t\lambda} & e^{2t\lambda}\\ I & e^{2t\lambda}\end{pmatrix}$$

Example 3

Let $X=C_{ub}\left(\mathbb{N}\cup\{0\}\right)$ be the space of all bounded and uniformly continuous function from $\mathbb{N}\cup\{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\|\cdot\|_\infty$ and let $\{T(t);t\geq 0\}\subseteq L(X)$ be defined by

$$\left[T(t)f\right](s)=f(t+s)$$

For each $f\in X$ and each $t,s\in\mathbb{R}_+$, it is easily verified that $\{T(t);t\geq 0\}$ satisfies Examples 1 and 2 above.

Example 4

Let $X=C[0,1]$ and consider the operator $Af=-f'$ with domain $D(A)=\{f\in C'[0,1]:f(0)=0\}$. It is a closed operator whose domain is not dense. However, it is dissipative, since its resolvent can be computed explicitly as

$$R(\lambda,A)f(t)=\int_0^t e^{-\lambda(t-s)}f(s)\,\mathrm{d}s$$

for $t \in [0,1]$, $f \in C[0,1]$. Moreover, $\|R(\lambda, A)\| \le \frac{1}{\lambda}$ for all $\lambda > 0$. Therefore $(A, D(A))$ is dissipative.

2.2. Theorem (Hille-Yoshida [9])

A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

1) A is densely defined and closed,

2) $(0,+\infty) \subseteq \rho(A)$ and for each $\lambda > 0$

$$\|R(\lambda, A)\|_{L(X)} \le \frac{1}{\lambda} \tag{2.2}$$

2.3. Theorem (Lumer-Phillips [5])

Let X be a real, or complex Banach space with norm $\|\cdot\|$, and let us recall that the duality mapping $F : X \to 2^{x}$ is defined by

$$F(x) = \left\{x^* \in X^*; (x, x^*) = \|x\|^2 = \|x^*\|^2\right\} \tag{2.3}$$

for each $x \in X$. In view of Hahn-Banach theorem, it follows that, for each $x \in X$, $F(x)$ is nonempty.

2.4. Theorem (Hahn-Banach Theorem [2])

Let V be a real vector space. Suppose $p : V \in [0,+\infty]$ is mapping satisfying the following conditions:

1) $p(0) = 0$;

2) $p(tx) = tp(x)$ for all $x \in V$ and real of $t \ge 0$; and

3) $p(x+y) \le p(x) + p(y)$ for every $x, y \in v$.

Assume, furthermore that for each $x \in V$, either both $p(x)$ and $p(-x)$ are ∞ or that both are finite.

3. Main Results

In this section, dissipative results on ω-OCP_n as a semigroup of linear operator were established and the research results(Theorems) were given and proved appropriately:

Theorem 3.1

Let $A \in w\text{-}OCP_n$ where $A : D(A) \subseteq X \to X$ is a dissipative operator on a Banach space X such that $\lambda - A$ is surjective for some $\lambda > 0$. Then

1) the part A, of A in the subspace $X_0 = \overline{D(A)}$ is densely defined and generates a constrain semigroup in X_0, and

2) considering X to be a reflexive, A is densely defined and generates a contraction semigroup.

Proof

We recall from Definition 2.3 that

$$A_* x = Ax \tag{3.1}$$

for

$$x \in D(A_*) = \{x \in x \in D(A) : Ax \in X_0\} = R(\lambda, A) X_0 \tag{3.2}$$

Since $R(\lambda, A)$ exists for $\lambda > 0$, this implies that $R(\lambda, A)_* = R(\lambda, A_*)$, hence

$$(0, \infty) \subset \rho(A_*)$$

we need to show that $D(A_*)$ is dense in X_0.

Take $x \in D(A)$ and set $x_n = nR(n, A)x$. Then $x_n \in D(A)$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} R(n, A) Ax + x = x,$$

since $\|R(n, A)\| \le \frac{1}{n}$. Therefore the operators $nR(n, A)$ converge pointwise on $D(A)$ to the identity. Since $\|nR(n, A)\| \le 1$ for all $n \in \mathbb{N}$, we obtain the convergence of $y_n = nR(n, A)y \to y$ for all $y \in X_0$. If for each y_n in $D(A_*)$, the density of $D(A_*)$ in X_0 is shown which proved (i).

To prove (ii), we need to obtain the density of $D(A)$.

Let $x \in X$ and define $x_n = nR(n, A)x \in D(A)$. The element $y = nR(1, A)x$, also belongs to $D(A)$. Moreover, by the proof of (i) the operators $nR(n, A)$ converges towards the identity pointwise on $X_0 = \overline{D(A)}$. It follows that

$$y_n = R(1, A)x_n = nR(n, A)R(1, A)x \to y \quad \text{for } n \to \infty$$

Since X is reflexive and $\{x_n : n \in \mathbb{N}\}$ is bounded, there exists a subsequence, still denoted by $(x_n)_{(n\in\mathbb{N})}$, that converges weakly to some $z \in X$. Since $x_n \in D(A)$, implies that $z \in \overline{D(A)}$.

On the other hand, the elements $x_n = (1 - A)y_n$ converges weakly to *z*, so the weak closedness of A implies that $y \in D(A)$ and $x = (1 - A)y = z \in \overline{D(A)}$ which proved (ii).

Theorem 3.2

The linear operator $A : D(A) \subseteq X \to X$ is a dissipative if and only if for each $x \in D(A)$ and $\lambda > 0$, where $A \in \omega\text{-}OCP_n$, then we have

$$\|(\lambda_1 - A)x\| \ge \lambda \|x\| \tag{3.3}$$

Proof

Suppose A is dissipative, then, for each $x \in D(A)$ and $\lambda > 0$, there exists $x^* \in F(x)$ such that $Re(\lambda x - Ax, x^*) \le 0$. Therefore

$$\|x\| \|\lambda x - Ax\| \ge |(\lambda x - Ax, x)| \ge Re(\lambda x - Ax, x) \ge \lambda \|x\|^2$$

and this completes the proof. Next, let $x \in D(A)$ and $\lambda > 0$.

Let $y_\lambda^* \in F(\lambda x - Ax)$ and let us observe that, by virtue of (3.3), $\lambda x - Ax = 0$ $\Rightarrow$ $x = 0$.

So, in this case, we clearly have $Re(x^*, \lambda x - Ax) = 0$. Therefore, by assuming that $\lambda x - Ax \ne 0$. As a consequence, $y_\lambda^* \ne 0$, and thus

$$z_\lambda^* = \frac{y_\lambda^*}{\|y_\lambda^*\|}$$

lies on the unit ball, *i.e.* $\|z_\lambda^*\| = 1$. We have $(\lambda x - Ax, z_\lambda^*) = \|\lambda x - Ax\| \geq \lambda\|x\| \Rightarrow Re(x, z_\lambda^*) - Re(Ax, z_\lambda^*) \leq \lambda\|x\| - Re(Ax, z_\lambda^*)$ hence

$$Re(Ax, z_\lambda^*) \leq 0$$

and $Re(z_\lambda^*, x) \geq \|x\| - \frac{1}{\lambda}\|Ax\|$. Now, let us recall that the closed unit ball in X^* is weakly-star compact. Thus, the net $(z_\lambda^*)_{\lambda>0}$ has at least one weak-star cluster point $z^* \in X^*$ with

$$\|z^*\| \leq 1 \tag{3.4}$$

From (3.4), it follows that $Re(Ax, z^*) \leq 0$ and $Re(x, z^*) \geq \|x\|$. Since $Re(x, z^*) \leq |(x, z^*)| \leq \|x\|$, it follows that $(x, z^*) = \|x\|$. Hence $x^* = \|x\|z^* \in F(x)$ and $Re(Ax, x^*) \leq 0$ and this completes the proof.

Proposition 3.3

Let $A: D(A) \subseteq X \to X$ be infinitesimal generator of a C_0-semigroup of contraction and $A \in \omega\text{-}OCP_n$. Suppose $X_* = D(A)$ is endowed with the graph-norm $|\cdot|_{D(A)}: X_* \to \mathbb{N} \cup \{0\}$ defined by $|u|_{D(A)} = \|u - Au\|$ for $u \in X_*$. Then operator $A_*: D(A_*) \subseteq X_* \to X_*$ defined by

$$\begin{cases} D(A_*) = \{x \in X_*;\ Ax \in X_*\} \\ A_*x = Ax, \quad \text{for } x \in D(X_*) \end{cases}$$

is the infinitesimal generator of a C_0-semigroup of contractions on X_*.

Proof

Let $\lambda > 0$ and $f \in X_*$ and let us consider the equation $\lambda u - Au = F$ Since A generates a C_0-semigroup of contraction [6], it follows that this equation has a unique solution $u \in D(A)$.

Since $f \in X_*$, we conclude that $Au \in D(A)$ and thus $u \in D(A_*)$.

Thus $\lambda u - A_*u = f$. On the other hand, we have

$$\begin{aligned} \left|(\lambda I - A_*)^{-1} f\right|_{D(A)} &= \left\|(I - A)(\lambda I - A)^{-1} f\right\| \\ &= \left\|(\lambda I - A)^{-1}(I - A) f\right\| \leq \frac{1}{\lambda}\|f - Af\| = \frac{1}{\lambda}|f|_{D(A)} \end{aligned} \tag{3.5}$$

which shows that A_* satisfies condition (ii) in Theorem 2.2. Moreover, it follows that A_* is closed in X_*.

Indeed, as $(\lambda I - A)^{-1} \in L(X_*)$, it is closed, and consequently $\lambda I - A_*$ enjoys the same property which proves that A_* is closed.

Now, let $x \in X_*$, $\lambda > 0$, $A \in \omega\text{-}OCP_n$ and let $x_\lambda = \lambda x - A_*x$. Clearly $x_\lambda \in D(A_*)$, and in addition $\lim_{\lambda\to\infty} |x_\lambda - x|_{D(A)} = 0$ Thus, $D(A_*)$ is dense in X_* by virtue of Theorem 2.2, A_* generates a C_0-semigroup of contraction on X_*. Hence the proof.

4. Conclusion

In this paper, it has been established that $\omega\text{-}OCP_n$ possesses the properties of dissipative operators as a semigroup of linear operator, and obtaining some dissipative results on ω-OCP_n.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Yosida, K. (1948) On the Differentiability and Representation of One-Parameter Semigroups of Linear Operators. *Journal of the Mathematical Society of Japan*, **1**, 15-21. https://doi.org/10.2969/jmsj/00110015

[2] Miyadera, I. (1952) Generation of Strongly Continuous Semigroups Operators. *Tohoku Mathematical Journal*, **4**, 109-114. https://doi.org/10.2748/tmj/1178245412

[3] Feller, W. (1953) On the Generation of Unbounded Semigroup of Bounded Linear Operators. *Annals of Mathematics*, **58**, 166-174. https://doi.org/10.2307/1969826

[4] Balakrishnan, A.V. (1960) Fractional Powers of Closed Operators and Semigroups Generated by Them. *Pacific Journal of Mathematics*, **10**, 419-437. https://doi.org/10.2140/pjm.1960.10.419

[5] Lumer, G. and Phillips, R.S. (1961) Dissipative Operators in a Banach Space. *Pacific Journal of Mathematics*, **11**, 679-698. https://doi.org/10.2140/pjm.1961.11.679

[6] Hille, E. and Phillips, R.S. (1981) Functional Analysis and Semigroups. American Mathematical Society, Providence, Colloquium Publications Vol. 31.

[7] Batty, C.J.K. (1994) Asymptotic Behaviour of Semigroup of Operators. *Banach Center Publications*, **30**, 35-52. https://doi.org/10.4064/-30-1-35-52

[8] Engel, K. and Nagel, R. (1999) One-Parameter Semigroup for Linear Evolution Equations. Graduate Texts in Mathematics Vol. 194, Springer, New York.

[9] Vrabie, I.I. (2003) C_0-Semigroup and Application. Mathematics Studies Vol. 191, Elsevier, North-Holland.

[10] Laradji, A. and Umar, A. (2004) Combinatorial Results for Semigroups of Order Preserving Partial Transformations. *Journal of Algebra*, **278**, 342-359. https://doi.org/10.1016/j.jalgebra.2003.10.023

[11] Rauf, K. and Akinyele, A.Y. (2019) Properties of ω-Order-Preserving Partial Contraction Mapping and Its Relation to C_0-Semigroup. *International Journal of Mathematics and Computer Science*, **14**, 61-68.

[12] Rauf, K., Akinyele, A.Y., Etuk, M.O., Zubair, R.O. and Aasa, M.A. (2019) Some Results of Stability and Spectra Properties on Semigroupn of Linear Operator. *Advances of Pure Mathematics*, **9**, 43-51. https://doi.org/10.4236/apm.2019.91003