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Travelling Solitary Wave Solutions to Higher Order Korteweg-de Vries Equation

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Abstract

The travelling solitary wave solutions to the higher order Korteweg-de Vries equation are obtained by using tanh-polynomial method. The method is effective and concise, which is also applied to various partial differential equations to obtain traveling wave solutions. The numerical simulation of the solutions is given for completeness. Numerical results show that the tanh-polynomial method works quite well.

Keywords

Higher Order Korteweg-de Vries Equation, Travelling Wave Solutions, Solitary Wave

1. Introduction

Nonlinear wave phenomena appear in various scientific and engineering fields [1] [2] [3] [4], such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. The higher order Korteweg-de Vries equation is one of important equations for description localized structures in the modern physics, such as two-layer fluid [5], steady-state solitary waves in a fluid [6], a three-layer fluid with a constant buoyancy frequency in an each layer [7] [8], which means the investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role. The Korteweg-de Vries (KdV) equation [9] [10] is first used to describe the nonlinear long internal waves in a fluid stratified by both density and current. The higher-order Korteweg-de Vries equation is investigated with different methods [11] [12] [13], which is written as follows.

$$u_t + u_{xxxxx} + \gamma u u_{xxx} + \beta u_x u_{xx} + \alpha u^2 u_x = 0 \tag{1}$$

where the subscripts denote the partial derivatives of x and t. The γ, β and α are constant parameters; u represents a real scalar function $u(x, t)$.

2. The Tanh-Function Method

Consider a given evolution equation with independent variables x and t in the form are given as

$$P\left(u,u_x,u_t,u_{xt},u_{xx},u_{tt},\cdots\right)=0\,. \tag{2}$$

By using the wave variable $\xi = x+\kappa t$ and substituting into Equation (2), we obtain the following ordinary differential equation

$$Q\left(u,u_\xi,u_{\xi\xi},u_{\xi\xi\xi},\cdots\right)=0\,. \tag{3}$$

where $u_\xi,u_{\xi\xi},\cdots$ denotes the derivative with respect to the same sole variable ξ.

The fact that the solutions of many nonlinear equations can be expressed as a finite series of tanh-function motivates us to seek for the solutions

$$u\left(x,t\right)=U\left(\xi\right)=\sum_{i=0}^{m}a_i\tanh\left(\xi\right)^i \tag{4}$$

where ξ is a function about x and t, $a_0,a_1,a_2,\cdots$ are constant parameters. m can be obtained by balancing the derivative term of the highest order with the nonlinear term as the follow

$$\mathrm{O}\left(\frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right)=m+p,\quad p=1,2,3,\cdots$$

and

$$\mathrm{O}\left(u^q\frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right)=\left(q+1\right)m+p,\quad q=0,1,2,\cdots;p=1,2,3,\cdots$$

Usually m is a positive integer, however, once in a while, the value of m is a negative or a fraction, the other kinds of expression will introduced. In the following, we illustrate the method by using it to solve the higher-order Korteweg-de Vries equations.

3. The Tanh-Function Method Solutions for Higher-Order Korteweg-de Vries Equation

Putting the variable $\xi = x+\kappa t$ into Equation (1) and we find

$$ku_\xi+u_{\xi\xi\xi\xi\xi}+\gamma uu_{\xi\xi\xi}+\beta u_\xi u_{\xi\xi}+\alpha u^2u_\xi=0\,; \tag{5}$$

which is an ordinary differential equation. By using the above method, $m = 2$ is obtained by balancing the derivative term of the highest order with the nonlinear term.

$$u\left(\xi\right)=a_0+a_1\tanh\left(\xi\right)+a_2\tanh\left(\xi\right)^2 \tag{6}$$

substituting (6) into (5) yields a set of algebraic equations for a_0, a_1, a_2, γ, β and α. Collecting all terms with the same power of $\tanh\left(\xi\right)$ together, equating each coefficient to zero, we obtain a set of simultaneous algebraic equations as follows:

$$16a_1 + \alpha a_0^2 a_1 + 2a_1 a_2 \beta - 2a_1 a_0 \beta + a_1 \kappa = 0$$

$$2\alpha a_1^2 a_0 + 272a_2 + 2\alpha a_0^2 a_2 - 2a_1^2 \beta + 4a_2^2 \beta - 2a_1^2 \gamma - 16a_0 a_2 \gamma + 2a_2 \kappa = 0$$

$$-136a_1 - \alpha a_0^2 a_1 + \alpha a_1^3 + 6\alpha a_0 a_1 a_2 - 14a_1 a_2 \beta + 8a_0 a_1 \gamma - 18a_1 a_2 \gamma - a_1 \kappa = 0$$

$$\begin{aligned} &-2\alpha a_1^2 a_0 - 1232a_2 - 2\alpha a_0^2 a_2 + 4\alpha a_1^2 a_2 + 4\alpha a_2^2 a_0 + 4a_1^2 \beta \\ &-20a_2^2 \beta + 8a_1^2 \gamma + 40a_0 a_2 \gamma - 16a_0 a_2 \gamma - 2a_2 \kappa = 0 \end{aligned}$$

$$240a_1 - \alpha a_1^3 - 6\alpha a_0 a_1 a_2 + 5\alpha a_2^2 a_1 + 22a_1 a_2 \beta - 6a_0 a_1 \gamma + 48a_1 a_2 \gamma = 0$$

$$\begin{aligned} &1680a_2 - 4\alpha a_1^2 a_2 - 4\alpha a_2^2 a_0 + 2\alpha a_2^3 - 2a_1^2 \beta + 28a_2^2 \beta \\ &-6a_1^2 \gamma - 24a_0 a_2 \gamma + 40a_2^2 \gamma = 0 \end{aligned}$$

$$-120a_1 - 5\alpha a_2^2 a_1 - 10a_1 a_2 \beta - 30a_1 a_2 \gamma = 0$$

$$-720a_2 - 2\alpha a_2^3 - 12a_2^2 \beta - 24a_2^2 \gamma = 0$$

Solving the algebraic equations, we get the results:

1) The first case $a_1 = 0$

$$a_0 = \frac{4\gamma - \sqrt{-136\alpha - 2\alpha a_2 \beta + 16\gamma^2 - \alpha\kappa}}{\alpha}, \quad a_2 = \frac{-6\beta - 12\gamma - \sqrt{(6\beta + 12\gamma)^2 - 1440\alpha}}{2\alpha},$$

$$u_1(x,t) = a_0 + a_2 \tanh(x + \kappa t)^2 \tag{7}$$

$$a_0 = \frac{4\gamma + \sqrt{-136\alpha - 2\alpha a_2 \beta + 16\gamma^2 - \alpha\kappa}}{\alpha}, \quad a_2 = \frac{-6\beta - 12\gamma + \sqrt{(6\beta + 12\gamma)^2 - 1440\alpha}}{2\alpha}$$

$$u_2(x,t) = a_0 + a_2 \tanh(x + \kappa t)^2 \tag{8}$$

$$a_0 = \frac{4\gamma - \sqrt{-136\alpha - 2\alpha a_2 \beta + 16\gamma^2 - \alpha\kappa}}{\alpha}, \quad a_2 = \frac{-6\beta - 12\gamma + \sqrt{(6\beta + 12\gamma)^2 - 1440\alpha}}{2\alpha}$$

$$u_3(x,t) = a_0 + a_2 \tanh(x + \kappa t)^2 \tag{9}$$

$$a_0 = \frac{4\gamma + \sqrt{-136\alpha - 2\alpha a_2 \beta + 16\gamma^2 - \alpha\kappa}}{\alpha}, \quad a_2 = \frac{-6\beta - 12\gamma - \sqrt{(6\beta + 12\gamma)^2 - 1440\alpha}}{2\alpha},$$

$$u_4(x,t) = a_0 + a_2 \tanh(x + \kappa t)^2 \tag{10}$$

2) The second case

$$a_0 = \frac{840 + 10a_2\beta + 15a_2\gamma}{16\gamma}, \quad a_1 = \sqrt{\frac{10a_0\gamma + 6a_2(2\beta + 3\gamma - \alpha a_0) - 1320}{\alpha}}, \quad a_2 = \frac{-3\gamma + 3\sqrt{\gamma^2 - 64\alpha}}{2\alpha}$$

$$u_5(x,t) = a_0 + a_1 \tanh(x + \kappa t) + a_2 \tanh(x + \kappa t)^2 \tag{11}$$

$$a_0 = \frac{840 + 10a_2\beta + 15a_2\gamma}{16\gamma}, \quad a_1 = -\sqrt{\frac{10a_0\gamma + 6a_2(2\beta + 3\gamma - \alpha a_0) - 1320}{\alpha}}, \quad a_2 = \frac{-3\gamma - 3\sqrt{\gamma^2 - 64\alpha}}{2\alpha}$$

$$u_6(x,t) = a_0 + a_1 \tanh(x + \kappa t) + a_2 \tanh(x + \kappa t)^2 \tag{12}$$

$$a_0=\frac{840+10a_2\beta+15a_2\gamma}{16\gamma},\quad a_1=\sqrt{\frac{10a_0\gamma+6a_2\left(2\beta+3\gamma-\alpha a_0\right)-1320}{\alpha}},\quad a_2=\frac{-3\gamma-3\sqrt{\gamma^2-64\alpha}}{2\alpha}$$

$$u_7\left(x,t\right)=a_0+a_1\tanh\left(x+\kappa t\right)+a_2\tanh\left(x+\kappa t\right)^2 \tag{13}$$

$$a_0=\frac{840+10a_2\beta+15a_2\gamma}{16\gamma},\quad a_1=-\sqrt{\frac{10a_0\gamma+6a_2\left(2\beta+3\gamma-\alpha a_0\right)-1320}{\alpha}},\quad a_2=\frac{-3\gamma+3\sqrt{\gamma^2-64\alpha}}{2\alpha}$$

$$u_8\left(x,t\right)=a_0+a_1\tanh\left(x+\kappa t\right)+a_2\tanh\left(x+\kappa t\right)^2 \tag{14}$$

Equations (7)-(14) are the solutions in the $\tanh\left(x+\kappa t\right)$ form for higher order Korteweg-de Vries Equation (1). The numerical simulations $u_1\left(x,t\right)$ and $u_2\left(x,t\right)$ are shown with $\alpha=-1$, $\beta=2$, $\gamma=0.4$, $\kappa=1$, the range of x and t are $x\in\left[-1,1\right]$ and $t\in\left[-1,1\right]$ in Figure 1 and Figure 2, respectively. Simulation $u_5\left(x,t\right)$ of Equation (11) with $\alpha=0.01$, $\beta=-8$, $\gamma=3$, $\kappa=1$, the range of x and t are $x\in\left[-10,10\right]$ and $t\in\left[-10,10\right]$, which is shown in Figure 3. Simulation $u_6\left(x,t\right)$ for Equation (12), with $\alpha=0.01$, $\beta=-5$, $\gamma=3$, $\kappa=1$. The range of x and t are $x\in\left[-10,10\right]$ and $t\in\left[-10,10\right]$, which is shown in Figure 4. From the figures, we know that the amplitude of the traveling wave solutions changes with time. All figures are smooth and no singularity in the given time and given field. These solutions help us find the peak and the deep location in physics system.

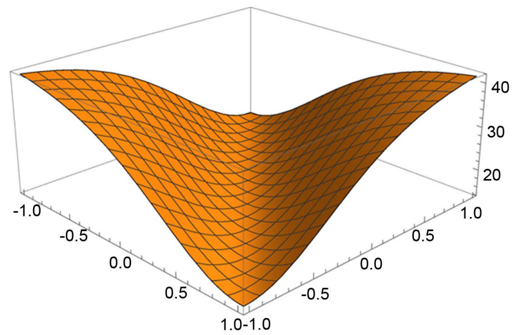

Figure 1. Simulation $u_1\left(x,t\right)$ of Equation (7) with $\alpha=-1$, $\beta=2$, $\gamma=0.4$, $\kappa=1$. The range of x and t are $x\in\left[-1,1\right]$ and $t\in\left[-1,1\right]$. The valley appears in the middle.

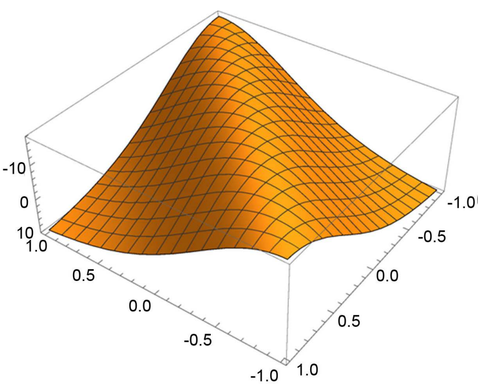

Figure 2. Simulation $u_2\left(x,t\right)$ for Equation (8), with $\alpha=-1$, $\beta=2$, $\gamma=0.4$, $\kappa=1$. The range of x and t are $x\in\left[-1,1\right]$ and $t\in\left[-1,1\right]$. The peak appears in the middle.

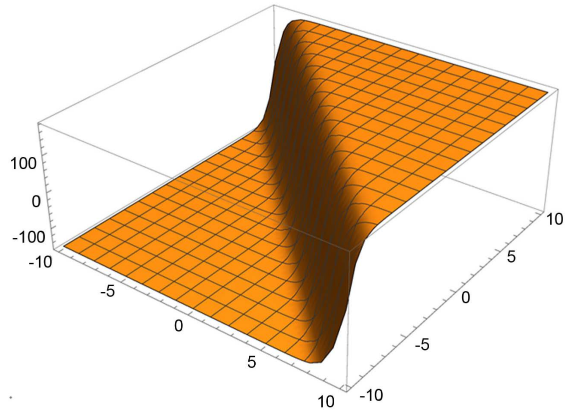

Figure 3. Simulation $u_5(x,t)$ of Equation (11) with $\alpha = 0.01$, $\beta = -8$, $\gamma = 3$, $\kappa = 1$. The range of x and t are $x \in [-10,10]$ and $t \in [-10,10]$. The jump appears in the middle.

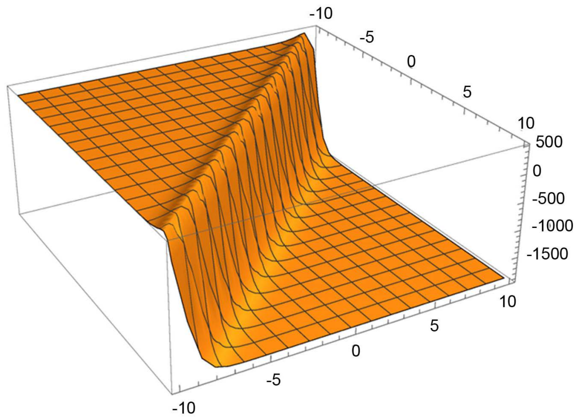

Figure 4. Simulation $u_6(x,t)$ for Equation (12), with $\alpha = 0.01$, $\beta = -5$, $\gamma = 3$, $\kappa = 1$. The range of x and t are $x \in [-10,10]$ and $t \in [-10,10]$. The jump appears in the middle.

4. Summary

Some new analytical solutions of the higher-order Korteweg-de Vries Equation (1) are obtained by successfully employing tanh-function method in this paper, which can be employed to discuss some interest physical phenomena, such as two-layer fluid, steady-state solitary waves in a fluid, three-layer fluid with a constant buoyancy frequency in an each layer. This tanh-function method is based on a previous work [14] [15] [16] [17] [18]. It is very interest to find these novel analytical evolution solutions, which is useful in establishing the connection between the longstanding rheological mechanics energy conservation problem and two-layer fluid, steady-state solitary waves in a fluid.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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