

Optimal Excess-of-Loss Reinsurance and Investment Problem for Insurers with Loss Aversion

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Abstract

This paper studies an optimal reinsurance and investment problem for a loss-averse insurer. The insurer's goal is to choose the optimal strategy to maximize the expected S-shaped utility from the terminal wealth. The surplus process of the insurer is assumed to follow a classical Cramér-Lundberg (C-L) model and the insurer is allowed to purchase excess-of-loss reinsurance. Moreover, the insurer can invest in a risk-free asset and a risky asset. The dynamic problem is transformed into an equivalent static optimization problem via martingale approach and then we derive the optimal strategy in closed-form. Finally, we present some numerical simulation to illustrate the effects of market parameters on the optimal terminal wealth and the optimal strategy, and explain some economic phenomena from these results.

Keywords

Loss Aversion, Excess-of-Loss Reinsurance, Martingale Approach, Investment, S-Shaped Utility

1. Introduction

Recently, optimal reinsurance and investment problems for insurers have attracted increasing attention from academics and industries. By purchasing reinsurance and investing in the financial market, insurance companies reduce their exposure risk and gain profits from investment. There are many literatures in this field. Browne [1] considered a diffusion risk model and found the closed-form of optimal investment strategies for exponential utility maximization of terminal wealth. Yang and Zhang [2] studied the same investment problem for an insurer under the assumption that the risk process is compound Poisson

process. In Wang *et al.* [3], closed-form strategies were obtained for an insurer under the mean-variance criterion as well as the expected constant absolute risk aversion (CARA) utility maximization through the martingale approach. Zhao and Rong [4] focused on the constant elasticity variance (CEV) model in portfolio selection problem of utility maximization. Liang and Young [5] considered the per-loss optimal reinsurance and derived the optimal investment and reinsurance strategy with the criteria that minimization the ruin probability. Bai and Guo [6] considered an optimal proportional reinsurance and investment problem with multiple risky assets, and showed that the optimal strategies are equivalent for maximizing the expected exponential utility and minimizing the probability of ruin in some special cases. In addition, some scholars have recently studied the optimal investment and reinsurance strategies for insurers under the mean-variance criterion proposed by Markowitz [7], see, for example, Li *et al.* [8] studied an optimal investment problem for the DC plan with default risk under the CEV model in a mean-variance framework and derived the explicit expressions of the equilibrium investment strategy by solving extended *Hamilton-Jacobi-Bellman* (HJB) equation in the post-default and the pre-default cases. Wang *et al.* [9] studied a mean-variance problem under the time-consistent condition, and obtained investment-reinsurance equilibrium strategy and the corresponding efficient frontier in explicit form using two systems of backward stochastic differential equations (BSDEs).

Generally, the above mentioned researches assume that investors are rational and risk averse. However, rationality hypothesis cannot be suit due to the investors' psychological effects. Based on the experiment and relative results, Kahneman and Tversky [10] for the first time studies the decision-making behavior of investors from the perspective of cognitive psychology, proposed the concept of loss aversion, and put forward the prospect theory, which was a milestone in the history of behavioral finance. Loss aversion can be expressed by the S-shaped utility function which is concave for gains and convex for losses, and steeper for losses than for gains. Afterwards, many scholars, based on loss aversion in prospect theory, introduced it into portfolio theory to analyze portfolio selection problems, and more and more literature study the implications of loss aversion.

After Kahneman and Tversky [10], Cox and Huang [11] considered a consumption-portfolio problem in continuous time under uncertainty, and they proposed the martingale approach to solve the optimal consumption-portfolio problem for hyperbolic absolute risk aversion utility functions when the asset prices follow a geometric Brownian motion. Berkelaar [12] derived closed-form solutions for the optimal portfolio choice under loss aversion by considering a specific two-piece power utility function in a continuous-time complete market setting. Guo [13] investigated an optimal portfolio selection problem for the insurer and used the Lévy process to describe the insurer's surplus process. Song *et al.* [14] investigated the optimal portfolio and consumption problem with the downside consumption constraints under loss aversion in an infinite horizon.

Curatola [15] investigated a consumption-investment problem for loss-averse investors, and in the s-shaped utility function, the reference level was relative to the consumption and changed over time. Guan and Liang [16] derived the optimal investment strategies for DC pension plan under loss aversion and Value-at-Risk (VaR) constraints, of which the sensitivity analysis showed that the loss aversion pension manager has a complex behavior and may invest more or less on the risk assets based on the reference point. Chen *et al.* [17] further studied the same investment problem for DC pension under loss aversion, which paid close attention to inflation and longevity risk and constructed a minimum performance constraint to guarantee the elementary needs of the member after retirement. Based on the Chen *et al.* [17], Dong and Zheng [18] added the short-selling constraints to the DC plan, then the market become incomplete and the martingale method was not applicable, so they used dual control method and HJB equation to solve the problem and derive the explicit expressions of the optimal wealth process and optimal strategies. Du *et al.* [19] considered a one-period two-echelon supply chain composed of a loss-averse supplier with yield randomness and a loss-averse retailer with demand uncertainty. They derived the optimal ordering policy of the loss-averse retailer and the optimal production policy of the loss-averse supplier under these conditions, discussed the effect of loss aversion on both parties' decision making, and showed how loss aversion contributes to decision bias.

To the best of our knowledge, there is few work incorporating loss aversion into the optimal reinsurance and investment problem. This paper adopts the S-shaped utility function to describe the insurer preference, and the insurer is allowed to invest in a risk-free asset and a risky asset. Moreover, the insurer can purchase excess-of-loss reinsurance, which is more practical in reality. Typically, three types of risk models are commonly considered in reinsurance and investment problems, the Cramér-Lundberg model (see Zeng *et al.* [20], *et al.*), the diffusion risk model (Chen and Li [21], *et al.*) and the jump-diffusion risk model (Gu *et al.* [22], Zeng *et al.* [23], *et al.*). In this paper we adopt the classical C-L model, and define a complete financial market. By using martingale approach and the Lagrange duality method, the closed-form solutions of the optimal investment strategy and the optimal wealth process are given. The legitimacy of the martingale approach follows from the completeness of the market model, which is a key assumption for the derivation of explicit optimal solutions by the martingale approach.

This paper is related to Guo [13], who studied the optimal investment strategies for an insurer with loss aversion. Although we employ similar martingale approach as Guo [13], this paper is different from theirs at least in two aspects. Firstly, we extend their models by considering a reinsurance market and allowing the insurer to purchase excess-of-loss reinsurance, which leads our model to be more complicated than theirs. So we define a function, which is similar to pricing kernel, and construct a martingale process to solve the problem. Second-

ly, we analyze the properties of the optimal strategy and present numerical examples to illustrate our results.

The main contribution of this paper is as follows: 1) the optimal reinsurance and investment strategy with loss aversion is studied and the closed-form expression of the optimal strategy is derived; 2) we define a quasi-pricing kernel and construct a martingale process to solve the problem. We find that the optimal terminal wealth is piecewise function. In good states of market, the optimal wealths is of the same form with the smooth CRRA utility function case, on the contrary the optimal wealth approaches 0 in bad states of market. Similarly, the optimal investment and reinsurance strategy are also divided into two cases respectively. When the market deteriorates, the insurer will stop investing in the risky asset and purchasing reinvestment strategy.

The rest of this paper is organized as follows: The financial market and insurance model are described in Section 2. In Section 3, we establish the optimal reinsurance-investment problem, and the optimal strategy is derived by using Lagrangian duality and martingale method. Section 4 presents numerical illustrations to demonstrate our results. Section 5 concludes the paper and provides further discussion.

2. Model Formulation

We impose the following standard assumptions: the insurer can trade in the financial market and in the insurance market continuously over time, no transaction costs or taxes are involved in trading. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, 0 \leq t \leq T\}, P)$ be a filtered, complete probability space satisfying the usual conditions, in which $T > 0$ is a finite time horizon. All stochastic processes introduced below are assumed to be adapted processes in this space.

2.1. Surplus Process

Assume that an insurer's basic surplus process is described by the classical Cramér-Lundberg (C-L) model: without reinsurance and investment, the insurer's surplus U is given by:

$$dU(t) = cdt - d \sum_{i=1}^{N(t)} Z_i, U(0) > 0, \quad (1)$$

where $U(0)$ is the initial surplus, $c > 0$ is the premium rate, $\{N(t), 0 \leq t \leq T\}$ is a homogeneous Poisson process with intensity $\lambda > 0$, Z_i represents the size of the i th claim and the claim sizes $Z_1, Z_2, \dots$ are assumed to be independent and identically distributed (i.i.d.), non-negative random variables with common distribution F having finite first-order moment μ_∞ , second-order moment σ_∞^2 , and $F(z) = Pr\{Z \leq z\}$. Consequently, $\sum_{i=1}^{N(t)} Z_i$ is a compound Poisson process representing the cumulative amount of claims in time interval $[0, t]$.

According to Gu *et al.* [24] and Li *et al.* [25], we introduce a Poisson random

measure $N(dz, dt)$ defined on $\Omega \times \mathcal{R}^+ \times [0, T]$, therefore $\sum_{i=1}^{N(t)} Z_i$ of discrete time can be converted into continuous time. $N(dz, dt)$ represents the number of insurance claims of size $(z, z + dz)$ within the time period $(t, t + dt)$, and $Z(t)$ stands for the claim at time $t \in [0, T]$. We assume that the premium rate c is calculated according to the expected value principle, i.e., $c = (1 + \eta)\lambda\mu_\infty$, and we denote the $\nu(dz) = \lambda dF(z)$, i.e.,

$$c = (1 + \eta) \int_0^\infty z \nu(dz),$$

where $\eta > 0$ is the insurer's relative safety loading, ν is a Lévy measure such that $\int_0^\infty z \nu(dz) < \infty$, $\nu(dz)$ represents the expected number of insurance claims of size $(z, z + dz)$ within a unit time interval, and denotes the compensated measure of $\tilde{N}(dz, dt)$ by $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$. Putting it all together, the insurer's surplus U without reinsurance is governed by

$$dU(t) = cdt - d \sum_i^{N(t)} Z_i = (1 + \eta) \int_0^\infty z \nu(dz)dt - \int_0^\infty z N(dz, dt).$$

In this paper, the insurer can purchase a reinsurance strategy with retained claim $\{l(t), 0 \leq t \leq T\}$, with the only restriction $0 \leq l(t) \leq Z(t)$ when the claim equals $Z(t)$ at time $t \in [0, T]$. Note that the reinsurer covers the excess loss $Z(t) - l(t)$. We will look for a reinsurance strategy given in feedback form by $l(t) = l(Z(t), t)$, in which we slightly abuse notation by using l on both sides of this equation.

Here, we assume that reinsurance is not inexpensive, i.e., the safety loading of the reinsurer θ is greater than the safety loading of the insurer η . Assuming we use the expected value principle again for the reinsurer, the reinsurance premium rate calculates as

$$(1 + \theta) \int_0^\infty (z - l(t)) \nu(dz).$$

Under the retention $l(t)$, the dynamics of the surplus process is governed by

$$\begin{aligned} dU(t) &= cdt - (1 + \theta) \int_0^\infty (z - l(t)) \nu(dz)dt - \int_0^\infty l(t) N(dz, dt) \\ &= \int_0^\infty [(\eta - \theta)z + \theta l(t)] \nu(dz)dt - \int_0^\infty l(t) \tilde{N}(dz, dt). \end{aligned}$$

Remark 2.1. According to the expected value principle, $c = (1 + \eta)\lambda\mu_\infty = (1 + \eta) \int_0^\infty z \nu(dz)$, therefore $\int_0^\infty z \nu(dz) = \lambda\mu_\infty$.

2.2. Wealth Process

Assume that the financial market consists of one risk-free asset and one risky asset. The price process of the risk-free asset price solves

$$dS_0(t) = rS_0(t)dt,$$

in which we assume the risk-free interest rate $r > 0$ is constant, and the price process of the risky asset is described by the geometric Brownian motion with dynamics

$$\frac{dS(t)}{S(t)} = \mu_s dt + \sigma_s dW(t),$$

in which $\mu_s > r$, $\sigma_s > 0$, and $\{W(t), t \geq 0\}$ is a standard Brownian motion, independent of the $N(dz, dt)$.

Therefore, the market price of risk process can be defined by

$$\lambda_s = \frac{\mu_s - r}{\sigma_s},$$

and λ_s is bounded.

During the time horizon $[0, T]$, the insurer is allowed to dynamically purchase excess-of-loss reinsurance and invest in the financial market. Let $\pi(t)$ be the total amount of money invested in the risky asset at time t . An reinsurance-investment strategy is described by $\phi(t) = (\pi(t), l(t))$, and the amount invested in the risk-free asset $S_0(t)$ is $X(t) - \pi(t)$, where $X(t)$ is the wealth process associated with the strategy. Then $X(t)$ is a solution to the following stochastic differential equation (SDE):

$$\begin{aligned} dX(t) &= (X(t) - \pi(t)) \frac{dS_0(t)}{S_0(t)} + \pi(t) \frac{dS(t)}{S(t)} + dU(t) \\ &= \left[rX(t) + (\mu_s - r)\pi(t) + \int_0^\infty ((\eta - \theta)z + \theta l(t)) \nu(dz) \right] dt \\ &\quad + \pi(t) \sigma_s dW(t) - \int_0^\infty l(t) \tilde{N}(dz, dt), \\ X(0) &= x_0. \end{aligned} \quad (2)$$

2.3. Loss Aversion

Kahneman and Tversky [10] (1979) conducted experiments to observe how people make decisions under uncertainty and proposed an alternative framework, which is known as prospect theory. The experiments demonstrated that the negative feeling associated with a loss is typically larger than the pleasure associated with an equivalent gain, therefore the majority of investors are loss-averse who were more sensitive to losses than to gains. At the same time, the experiments also demonstrated that most investors are risk-averse towards gains, but they will change to be risk-seeking when they have to make a decision about potential losses.

Based on the experiments and relative results, Kahneman and Tversky proposed a utility function, which is defined over gains and losses relative to the reference point ξ as follows:

$$U(x) = \begin{cases} \alpha(x - \xi)^{\gamma_1}, & x > \xi, \\ -\beta(\xi - x)^{\gamma_2}, & x \leq \xi, \end{cases} \quad (3)$$

$\alpha > 0$ and $\beta > 0$ are required to ensure that $U(x)$ is an increasing function, $\beta > \alpha$ holds for loss aversion, γ_1 and γ_2 are the curvature parameters for gains and losses, and $0 < \gamma_1 < 1$, $0 < \gamma_2 < 1$ for the convex-concave shape (Figure 1).

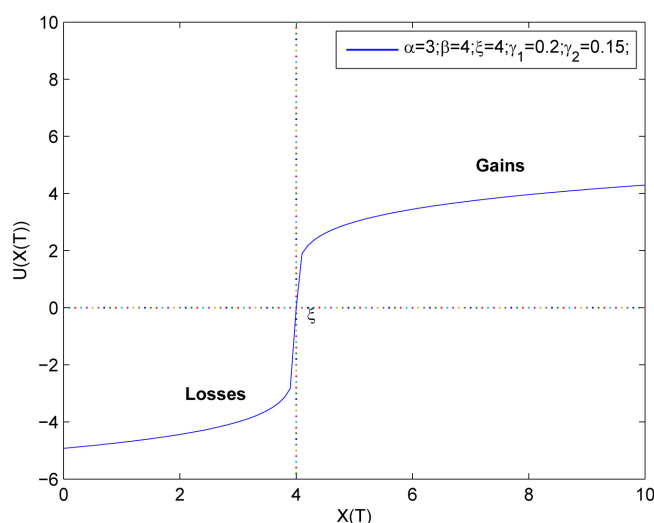


Figure 1. S-shaped utility function.

3. Optimal Strategy

Definition 3.1. A strategy $\phi(t) = (\pi(t), l(t))$, $t \in [0, T]$ is called admissible if it satisfies the following conditions:

- 1) $\phi(t)$ is $\{\mathcal{F}_t, 0 \leq t \leq T\}$ -progressively measurable;
- 2) $Z(t) \geq 0$, $0 \leq l(t) \leq Z(t)$, and $t \in [0, T]$;
- 3) For all $(X(t), t) \in \mathbb{R} \times [0, T]$, the stochastic differential Equation (2) has a unique solution.

Note that $\phi(t)$ is the admissible strategy and Φ is the admissible space. Following utility maximization criterion, the problem of choosing an optimal portfolio can be formulated as follows:

$$\begin{aligned} & \max_{\phi(\cdot)} E[U(X(T))] \\ & \text{s.t.} \begin{cases} (X(t), \phi(t)) \text{ satisfies (2),} \\ \phi(t) \in \Phi. \end{cases} \end{aligned} \quad (4)$$

In order to facilitate the solution of this problem, markets are assumed to be complete, which implies the existence unique state pricing kernel. Since the S-shaped utility is convex-concave, the stochastic optimal control approach can not be feasible. In this case, martingale approach proposed by Cox and Huang [11] becomes the important means in applying S-shaped utility. Moreover, due to the consideration of excess-of-loss reinsurance, the problem is more complicated. In order to get the optimal strategy, we define a quasi-pricing kernel $H(t)$

$$H(t) = \exp \left\{ -rt - \frac{1}{2} \lambda_s^2 t - \lambda_s W(t) + \int_0^t \int_0^\infty \ln(1+\theta) N(dz, ds) - \int_0^t \int_0^\infty \theta v(dz) ds \right\}, \quad (5)$$

and construct a martingale process, see Proposition 3.1.

Proposition 3.1. If $H(t)$ is defined by (5) for $[0, T]$, $H(t)X(t) - (\eta - \theta) \int_0^t z v(dz) \int_0^s H(s) ds$ is a martingale.

Proof. Consider a lévy-type stochastic integral of the form

$$Y^i(t) = Y^i(0) + Y_c^i(t) + \int_0^t \int_A J^i(s, x) N(ds, dx),$$

where

$$Y_c^i(t) = \int_0^t G^i(s) ds + \int_0^t F_j^i(s) dB^j(s),$$

Itô formula for lévy-type stochastic integrals can be written as

$$\begin{aligned} & f(Y(t)) - f(Y(0)) \\ &= \int_0^t \partial_i f(Y(s-)) dY_c^i(s) + \frac{1}{2} \int_0^t \partial_i \partial_j f(Y(s-)) d[Y_c^i, Y_c^j](s) \\ &+ \int_0^t \int_A [f(Y(s-) + J(s, x)) - f(Y(s-))] N(ds, dx), \end{aligned} \quad (6)$$

for each $f \in C(\mathbb{R}^d)$, $t > 0$, $0 \leq i \leq d$.

For more information about Lévy processes, please see the Lévy Process and Stochastic Calculus [26].

Using the Itô formula for lévy-type stochastic integrals, we find that

$$dH(t) = H(t) \left(-r dt - \lambda_s dW(t) + \int_0^\infty \theta \tilde{N}(dz, dt) \right), \quad (7)$$

and

$$\begin{aligned} d(H(t)X(t)) &= H(t)dX(t) + X(t)dH(t) + d[H, X](t) \\ &= H(t) \left[rX(t) + (\mu_s - r)\pi(t) + \int_0^\infty ((\eta - \theta)z + \theta l(t)) \nu(dz) dt \right. \\ &\quad \left. + \pi(t)\sigma_s dW(t) - \int_0^\infty l(t)\tilde{N}(dz, dt) \right] + X(t)H(t) \left(-r dt - \lambda_s dW(t) \right. \\ &\quad \left. + \int_0^\infty \theta \tilde{N}(dz, dt) \right) + H(t) \left(-\lambda_s \pi(t)\sigma_s dt - \int_0^\infty \theta l(t)N(dz, dt) \right) \\ &= H(t)(\eta - \theta) \left[\int_0^\infty z \nu(dz) \right] dt + H(t)(\pi(t)\sigma_s - \lambda_s X(t)) dW(t) \\ &\quad + H(t) \left[X(t) \int_0^\infty \theta - \int_0^\infty (\theta + 1)l(t) \right] \tilde{N}(dz, dt), \end{aligned} \quad (8)$$

Therefore, $H(t)X(t) - (\eta - \theta) \int_0^t z \nu(dz) \int_0^s H(s) ds$ is a martingale. $\square$

Now, the dynamic maximization problem (4) can be converted into the following equivalent static optimization problem with constraint:

$$\begin{aligned} & \max_{X(T)} E[U(X(T))] \\ \text{s.t.} & \begin{cases} E \left[H(T)X(T) - (\eta - \theta) \int_0^\infty z \nu(dz) \int_0^t H(s) ds \right] \leq x_0, \\ X(T) \geq 0. \end{cases} \end{aligned} \quad (9)$$

Theorem 3.1. *The optimal terminal wealth for the loss-averse member in the dynamic problem (4) is*

$$X^{*,y^*}(T) = \begin{cases} \xi + \left(\frac{\alpha \gamma_1}{y^* H(T)} \right)^{\frac{1}{1-\gamma_1}}, & H(T) < \bar{H}, \\ 0, & H(T) \geq \bar{H}, \end{cases} \quad (10)$$

where $\bar{H}$ satisfies $f(\bar{H})=0$ with

$$f(x) = \frac{1-\gamma_1}{\gamma_1} y^* x \left(\frac{\alpha\gamma_1}{y^* x} \right)^{\frac{1}{1-\gamma_1}} - y^* x \xi + \beta \xi^{\gamma_2},$$

$y^* > 0$ is a Lagrange multiplier, satisfies

$$E \left[H(T) X^{*,y^*}(T) - (\eta - \theta) \int_0^\infty z v(dz) \int_0^T H(s) ds \right] = x_0.$$

Proof. First we define the Lagrangian function of problem (9) as follows:

$$\begin{aligned} L(X(T), y) = & E[U(X(T))] - yE[H(T)X(T)] \\ & + yE \left[(\eta - \theta) \int_0^\infty z v(dz) \int_0^T H(s) ds \right] + yx_0, \end{aligned} \quad (11)$$

where y is the Lagrangian multiplier. According to lagrange dual theory, we can get the solution of the optimal $X^{*,y}(T)$ with fixed parameter y , and then figure out the optimal parameters y^* . When KKT condition is satisfied, the optimal solution of the original problem and the dual problem is equal.

Hence, the equivalent problem of the original problem (9) can be written as:

$$\begin{cases} \min \max_{y>0, X(T)} L(X(T), y), \\ X(T) \geq 0. \end{cases} \quad (12)$$

When we find the optimal $X^{*,y}(T)$ with fixed parameter y , we can only focus on the part of $X(T)$ in (11) and ignore irrelevant items that only influence the values of the Lagrangian multiplier. In this case, the problem (12) turns into the following problem:

$$\begin{cases} \max_{X(T)} \{ E[U(X(T))] - yE[H(T)X(T)] \}, \\ X(T) \geq 0. \end{cases} \quad (13)$$

Denote $U_1(X) = \alpha(X - \xi)^{\gamma_1}$, and $U_2(X) = -\beta(\xi - X)^{\gamma_2}$. If $X > \xi$, the utility function $U_1(X)$ is concave and we denote another Lagrangian function $L_2 = U(X) - yH(T)X + \zeta X$ where ζ is Lagrange multiplier. The maximum $X_1^{*,y}$ satisfies the KKT conditions:

$$\begin{cases} U'(X) - yH(T) + \zeta = 0, & X_1^* \geq 0, \\ \zeta X_1^{*,y} = 0, & \zeta \geq 0. \end{cases} \quad (14)$$

Solving constraint (14), we obtain

$$X_1^{*,y} = \xi + \left(\frac{\alpha\gamma_1}{yH(T)} \right)^{\frac{1}{1-\gamma_1}}.$$

If $X \leq \xi$, the $U_2(X)$ is convex, and the Weirestrass theorem implies that maximum $X_2^{*,y}$ must lie on the boundaries $X_2^{*,y} = 0$ or $X_2^{*,y} = \xi$.

In order to know whether $X_1^{*,y}$ or $X_2^{*,y}$ is the global maximum, we denote

$$f(H(T)) = U(X_1^{*,y}) - yH(T)X_1^{*,y} - [U(X_2^{*,y}) - yH(T)X_2^{*,y}],$$

if $f(H(T)) > 0$, then $X^{*,y}(T) = X_1^*$, otherwise $X^{*,y}(T) = X_2^*$.

Comparing $X_1^{*,y}$ with $X_2^{*,y} = \xi$, we find

$$\begin{aligned} f(H(T)) &= \alpha \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{\gamma_1}{1-\gamma_1}} - yH(T) \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{1}{1-\gamma_1}} \\ &= \left[\alpha - yH(T) \frac{\alpha \gamma_1}{yH(T)} \right] \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{\gamma_1}{1-\gamma_1}} \\ &= \alpha(1-\gamma_1) \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{\gamma_1}{1-\gamma_1}} \\ &> 0, \end{aligned}$$

so $X_2^{*,y} = \xi$ is never the optimal level of wealth.

Comparing $X_1^{*,y}$ with $X_2^{*,y} = 0$, we find

$$\begin{aligned} f(H(T)) &= \alpha \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{\gamma_1}{1-\gamma_1}} - yH(T)\xi - yH(T) \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{1}{1-\gamma_1}} + \beta \xi^{\gamma_2} \\ &= \frac{1-\gamma_1}{\gamma_1} yH(T) \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{1}{1-\gamma_1}} - yH(T)\xi + \beta \xi^{\gamma_2}, \end{aligned}$$

when $H(T) \leq \frac{\beta}{y} \xi^{\gamma_2-1}$, $-yH(T)\xi + \beta \xi^{\gamma_2} > 0$, so $f(H(T)) > 0$. Moreover,

$\lim_{H(T) \rightarrow \infty} f(H(T)) = -\infty$, $f'(H(T)) < 0$, hence $f(H(T))$ has an unique root

in the interval $\left(\frac{\beta}{y} \xi^{\gamma_2-1}, +\infty \right)$ which we denote the root of $f(H(T))$ by $\bar{H}$.

Summarizing the above analysis, we obtain

$$f(H(T)) \leq 0, H(T) \geq \bar{H},$$

$$f(H(T)) > 0, H(T) < \bar{H}.$$

Hence the global optimizer of problem (13) can be written as

$$X^{*,y}(T) = \begin{cases} \xi + \left(\frac{\alpha \gamma_1}{yH(T)} \right)^{\frac{1}{1-\gamma_1}}, & H(T) < \bar{H}, \\ 0, & H(T) \geq \bar{H}. \end{cases}$$

So far, we have got the optimal $X^{*,y}(T)$ with fixed y , and next we will begin to figure out the optimal y^* to solve the problem

$$\min_{y>0} L(X^{*,y}(T), y),$$

According to KKT condition as follows:

$$y^* E \left[H(T) X^{*,y^*}(T) - (\eta - \theta) \int_0^\infty z \nu(dz) \int_0^T H(s) ds \right] - y^* x_0 = 0. \quad (15)$$

When $H(T) < \bar{H}$, we can get the equation

$$y^* E \left[H(T) \left(\xi + \left(\frac{\alpha \gamma_1}{y^* H(T)} \right)^{\frac{1}{1-\gamma_1}} \right) - (\eta - \theta) \int_0^\infty z \nu(dz) \int_0^T H(s) ds \right] - y^* x_0 = 0,$$

so the optimal y^* satisfies

$$y^{\frac{1}{\gamma_1-1}} = \frac{x_0 - E \left[H(T) \xi \right] + (\eta - \theta) \int_0^\infty z \nu(dz) E \left[\int_0^T H(s) ds \right]}{E \left[(\alpha \gamma_1)^{\frac{1}{1-\gamma_1}} H(T)^{\frac{\gamma_1}{\gamma_1-1}} \right]}. \quad (16)$$

Substituting y^* into $X^{*,y^*}(T)$, we get the optimal $X^{*,y^*}(T)$ as follows:

$$X^{*,y^*}(T) = \begin{cases} \xi + \left(\frac{\alpha \gamma_1}{y^* H(T)} \right)^{\frac{1}{1-\gamma_1}}, & H(T) < \bar{H}, \\ 0, & H(T) \geq \bar{H}. \end{cases}$$

Let $X(T)$ represent another possible optimal solution satisfying the static budget equation,

$$\begin{aligned} & E \left[U \left(X^{*,y^*}(T) \right) \right] - E \left[U \left(X(T) \right) \right] \\ &= E \left[U \left(X^{*,y^*}(T) \right) \right] - E \left[U \left(X(T) \right) \right] - y^* X(0) \\ &\quad - y^* E \left[(\eta - \theta) \int_0^\infty z \nu(dz) \int_0^T H(s) ds \right] \\ &\quad + y^* X(0) + y^* E \left[(\eta - \theta) \int_0^\infty z \nu(dz) \int_0^T H(s) ds \right] \\ &\geq E \left[U \left(X^{*,y^*}(T) \right) \right] - E \left[U \left(X(T) \right) \right] \\ &\quad - y^* E \left[H(T) X^{*,y^*}(T) \right] + y^* E \left[H(T) X(T) \right] \\ &\geq 0. \end{aligned}$$

According to constraint (9) and (15), the first inequality follows from the fact that the static budget equation holds with equality for $X^{*,y^*}(T)$ and with inequality for $X(T)$, that is,

$$\begin{aligned} y^* X(0) + y^* E \left[(\eta - \theta) \int_0^\infty z \nu(dz) \int_0^T H(s) ds \right] &= y^* E \left[H(T) X^{*,y^*}(T) \right], \\ y^* X(0) + y^* E \left[(\eta - \theta) \int_0^\infty z \nu(dz) \int_0^T H(s) ds \right] &\geq y^* E \left[H(T) X(T) \right]. \end{aligned}$$

The second inequality holds because $X^{*,y^*}(T)$ is the optimal solution for problem (13). As such $X^{*,y^*}(T)$ is the optimal solution of the static problem. $\square$

From the Proposition 3.1, we find that the optimal terminal wealth for the

loss-averse insurer is discontinuous and achieves either $\xi + \left(\frac{\alpha \gamma_1}{y^* H(T)} \right)^{\frac{1}{1-\gamma_1}}$ or 0.

$\bar{H}$ means the breakpoint of the economic states. $H(T) < \bar{H}$ stands for a good economic states, at this time the insurer gains from participating in the financial

market, $\left(\frac{\alpha \gamma_1}{y^* H(T)} \right)^{\frac{1}{1-\gamma_1}}$. As economic conditions deteriorate, the terminal

wealth drops to 0. When $\xi = 0$, due to the $X(T)$ is no less than 0, the utility function (3) degenerates to the CRRA types $U(X(T)) = \alpha X(T)^{\gamma_1}$, in this case, $\bar{H} = +\infty$ and the optimal terminal wealth equals to $\left(\frac{\alpha\gamma_1}{y^*H(T)}\right)^{\frac{1}{1-\gamma_1}}$. Similar

results can be seen from Guan and Liang [16] and Chen *et al.* [17].

Remark 3.1. When $H(T) < \bar{H}$, substituting (16) into $X^{*,y}(T)$, we obtain

$$\begin{aligned} X^{*,y^*}(T) &= \xi + \left[x_0 - E(H(T)\xi) + (\eta - \theta) \int_0^\infty z\nu(dz) E\left(\int_0^T H(s)ds\right) \right] \\ &\quad \times \frac{H(T)^{\frac{1}{\gamma_1-1}}}{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]}. \end{aligned} \quad (17)$$

Next Lemma 3.1 and Lemma 3.2 will compute it.

If $X(t)$ is martingale, s.t.

$$E[X(t) | \mathcal{F}_s] = X(s), \quad s < t,$$

and Proposition 3.1 shows that $H(t)X(t) - (\eta - \theta) \int_0^\infty z\nu(dz) \int_0^t H(s)ds$ is a martingale, therefore

$$\begin{aligned} &E\left[H(T)X^{*,y^*}(T) - (\eta - \theta) \int_0^\infty z\nu(dz) \int_0^T H(s)ds \mid \mathcal{F}_t\right] \\ &= H(t)X^{*,y^*}(t) - (\eta - \theta) \int_0^\infty z\nu(dz) \int_0^t H(s)ds, \end{aligned}$$

and hence

$$\begin{aligned} H(t)X^{*,y^*}(t) &= E\left[H(T)X^{*,y^*}(T) \mid \mathcal{F}_t\right] \\ &\quad - (\eta - \theta) \int_0^\infty z\nu(dz) E\left[\left(\int_0^t H(s)ds + \int_t^T H(s)ds\right) \mid \mathcal{F}_t\right] \\ &\quad + (\eta - \theta) \int_0^\infty z\nu(dz) \int_0^t H(s)ds \\ &= E\left[H(T)X^{*,y^*}(T) \mid \mathcal{F}_t\right] - (\eta - \theta) \int_0^\infty z\nu(dz) E\left[\int_t^T H(s)ds \mid \mathcal{F}_t\right]. \end{aligned} \quad (18)$$

When $H(T) < \bar{H}$, substituting $X^{*,y^*}(T)$ in Equation (17) into $H(t)X^{*,y^*}(t)$, (18) can be rewritten as

$$\begin{aligned} &H(t)X^{*,y^*}(t) \\ &= \left[x_0 - E(H(T)\xi) + (\eta - \theta) \int_0^\infty z\nu(dz) E\left(\int_0^T H(s)ds\right) \right] \frac{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}} \mid \mathcal{F}_t\right]}{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]} \\ &\quad + E\left[\xi H(T) \mid \mathcal{F}_t\right] - (\eta - \theta) \int_0^\infty z\nu(dz) E\left[\int_t^T H(s)ds \mid \mathcal{F}_t\right]. \end{aligned} \quad (19)$$

When $H(T) \geq \bar{H}$, $X^{*,y}(T) = 0$, the $H(t)X^{*,y^*}(t)$ can be rewritten as

$$H(t)X^{*,y^*}(t) = -(\eta - \theta) \int_0^\infty z\nu(dz) E\left[\int_t^T H(s)ds \mid \mathcal{F}_t\right]. \quad (20)$$

Lemma 3.1. *Dueing to the martingale property and the conditional Fubini*

theorem, we obtain

$$\begin{aligned} E[\xi H(T) | \mathcal{F}_t] &= \xi H(t) e^{-r(T-t)}, \\ E\left[\int_t^T H(s) ds | \mathcal{F}_t\right] &= \frac{1}{r} H(t) (1 - e^{-r(T-t)}). \end{aligned}$$

Proof. $H(t)$ can be written as

$$H(t) = \exp\left(-\int_0^t r ds\right) K(t) = e^{-rt} K(t), \quad (21)$$

where $K(t)$ is defined by

$$K(t) = \exp\left\{-\frac{1}{2}\lambda_s^2 t - \lambda_s W(t) + \int_0^t \int_0^\infty \ln(1+\theta) N(dz, ds) - \int_0^t \int_0^\infty \theta v(dz) ds\right\},$$

using the Itô formula for Lévy-type stochastic integrals, we find that

$$dK(t) = K(t) \left(-\lambda_s dW(t) + \int_0^\infty \theta z \tilde{N}(dz, dt)\right),$$

therefore $K(t)$ is a martingale, and we obtain

$$E[H(T)] = E[e^{-rT} K(T)] = e^{-rT} E[K(T)] = e^{-rT} E[K(0)] = e^{-rT}.$$

Similarly, we have

$$\begin{aligned} E[\xi H(T) | \mathcal{F}_t] &= \xi E[e^{-rT} K(T) | \mathcal{F}_t] = \xi e^{-rT} E[K(T) | \mathcal{F}_t] \\ &= \xi e^{-rT} K(t) = \xi H(t) e^{-r(T-t)}. \end{aligned}$$

Using the conditional Fubini theorem, the order of integral and expectation can be exchanged, so we obtain

$$E\left(\int_0^T H(s) ds\right) = \int_0^T E[H(s)] ds = \int_0^T E[e^{-rs} K(s)] ds = \int_0^T e^{-rs} ds = \frac{1}{r} (1 - e^{-rT}),$$

and

$$\begin{aligned} E\left[\int_t^T H(s) ds | \mathcal{F}_t\right] &= \int_t^T E[H(s) | \mathcal{F}_t] ds = \int_t^T H(t) e^{-r(s-t)} ds \\ &= H(t) \int_t^T e^{-r(s-t)} ds = \frac{1}{r} H(t) (1 - e^{-r(T-t)}). \end{aligned}$$

□

Lemma 3.2.

$$\frac{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}} | \mathcal{F}_t\right]}{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]} = M(t),$$

where

$$\begin{aligned} M(t) &= \exp\left\{-\frac{\gamma_1}{\gamma_1-1} \int_0^t \lambda_s dW(s) - \frac{1}{2} \left(\frac{\gamma_1}{\gamma_1-1}\right)^2 \int_0^t \lambda_s^2 ds \right. \\ &\quad \left. + \frac{\gamma_1}{\gamma_1-1} \int_0^t \int_0^\infty \ln(1+\theta) N(dz, ds) + \int_0^t \int_0^\infty \left(1 - (1+\theta)^{\frac{\gamma_1}{\gamma_1-1}}\right) v(dz) ds\right\}. \end{aligned} \quad (22)$$

Proof. We substitute $H(T) = e^{-rT} K(T)$ into the $\frac{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}} | \mathcal{F}_t\right]}{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]}$, then we

obtain

$$\frac{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}} | \mathcal{F}_t\right]}{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]} = \frac{E\left[e^{-\frac{\gamma_1}{\gamma_1-1}rT} K(T)^{\frac{\gamma_1}{\gamma_1-1}} | \mathcal{F}_t\right]}{E\left[e^{-\frac{\gamma_1}{\gamma_1-1}rT} K(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]},$$

Using the Itô formula for lévy-type stochastic integrals, we know that $K(t)^{\frac{\gamma_1}{\gamma_1-1}}$ is not a martingale, so we introduce an exponential martingale

$$\begin{aligned} M(t) = \exp & \left\{ -\frac{\gamma_1}{\gamma_1-1} \lambda_s W(t) - \frac{1}{2} \left(\frac{\gamma_1}{\gamma_1-1} \right)^2 \lambda_s^2 t \right. \\ & \left. + \frac{\gamma_1}{\gamma_1-1} \int_0^t \int_0^\infty \ln(1+\theta) N(dz, ds) + \int_0^t \int_0^\infty \left(1 - (1+\theta)^{\frac{\gamma_1}{\gamma_1-1}} \right) \nu(dz) ds \right\}, \end{aligned} \quad (23)$$

and the differential form of $M(t)$ as follows:

$$\frac{dM(t)}{M(t)} = -\frac{\gamma_1}{\gamma_1-1} \lambda_s dW(t) + \int_0^\infty \left(1 - (1+\theta)^{\frac{\gamma_1}{\gamma_1-1}} \right) \tilde{N}(dz, ds), \quad (24)$$

then denote

$$\begin{aligned} N(t) = \exp & \left\{ -\frac{\gamma_1}{\gamma_1-1} rt + \frac{1}{2} \frac{\gamma_1}{(\gamma_1-1)^2} \lambda_s^2 t \right. \\ & \left. + \int_0^t \int_0^\infty \left[-\frac{\gamma_1}{\gamma_1-1} \theta - 1 + (1+\theta)^{\frac{\gamma_1}{\gamma_1-1}} \right] \nu(dz) ds \right\}, \end{aligned}$$

so $H(t)^{\frac{\gamma_1}{\gamma_1-1}}$ can be written as

$$H(t)^{\frac{\gamma_1}{\gamma_1-1}} = M(t) N(t).$$

Therefore,

$$\begin{aligned} \frac{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}} | \mathcal{F}_t\right]}{E\left[H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right]} &= \frac{E\left[M(T)N(T) | \mathcal{F}_t\right]}{E\left[M(T)N(T)\right]} = \frac{N(T)E\left[M(T) | \mathcal{F}_t\right]}{N(T)E\left[M(T)\right]} \\ &= \frac{M(t)}{M(0)} = M(t). \end{aligned}$$

□

Theorem 3.2. 1) When $H(T) < \bar{H}$, the optimal portfolio $\pi^*(t)$ and $l^*(t)$ are

$$\begin{aligned}\pi^*(t) &= \frac{\mu_s - r}{\sigma_s^2} \frac{1}{1 - \gamma_1} \left[X^{*,y^*}(t) - \xi e^{-r(T-t)} + \frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z \nu(dz) \right], \\ l^*(t) &= \frac{\theta - 1 + (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}}}{\theta + 1} \left[X^{*,y^*}(t) - \xi e^{-r(T-t)} + \frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z \nu(dz) \right] \wedge z.\end{aligned}\quad (25)$$

The optimal wealth at time t is given by

$$\begin{aligned}X^{*,y^*}(t) &= \xi e^{-r(T-t)} - \frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z \nu(dz) \\ &\quad + \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \right] \exp \left\{ rt + \frac{1}{1 - \gamma_1} \lambda_s W(t) \right. \\ &\quad + \frac{1}{2} \frac{1 - 2\gamma_1}{(\gamma_1 - 1)^2} \lambda_s^2 t - \frac{1}{1 - \gamma_1} \int_0^t \int_0^\infty \ln(1 + \theta) N(dz, ds) \\ &\quad \left. + \int_0^t \int_0^\infty \left(\theta + 1 - (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}} \right) \nu(dz) ds \right\}.\end{aligned}\quad (26)$$

2) When $H(T) \geq \bar{H}$, the optimal portfolio $\pi^*(t)$ and $l^*(t)$ are given, respectively, by

$$\begin{aligned}\pi^*(t) &= 0, \\ l^*(t) &= 0.\end{aligned}\quad (27)$$

The optimal wealth at time t is given by

$$X^{*,y^*}(t) = -\frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z \nu(dz). \quad (28)$$

Proof. 1) According to Lemma 3.1 and Lemma 3.2 in the case of $H(T) < \bar{H}$, $H(t)X^{*,y^*}(t)$ in the (19) can be written as

$$\begin{aligned}&H(t)X^{*,y^*}(t) \\ &= \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} \int_0^\infty z \nu(dz) (1 - e^{-rT}) \right] M(t) + \xi H(t) e^{-r(T-t)} \\ &\quad - \frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) H(t) \int_0^\infty z \nu(dz),\end{aligned}\quad (29)$$

so the optimal wealth at time t is given by

$$\begin{aligned}X^{*,y^*}(t) &= \xi e^{-r(T-t)} - \frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z \nu(dz) \\ &\quad + \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \right] \frac{M(t)}{H(t)} \\ &= \xi e^{-r(T-t)} - \frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z \nu(dz) + \left[x_0 - \xi e^{-rT} \right. \\ &\quad + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \left. \right] \exp \left\{ rt + \frac{1}{1 - \gamma_1} \lambda_s W(t) + \frac{1}{2} \frac{1 - 2\gamma_1}{(\gamma_1 - 1)^2} \lambda_s^2 t \right. \\ &\quad \left. - \frac{1}{1 - \gamma_1} \int_0^t \int_0^\infty \ln(1 + \theta) N(dz, ds) + \int_0^t \int_0^\infty \left(\theta + 1 - (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}} \right) \nu(dz) ds \right\}.\end{aligned}$$

Taking differential on both sides of (29)

$$\begin{aligned}
& d\left[H(t)X^{*,y^*}(t)\right] \\
&= \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r}(1 - e^{-rT}) \int_0^\infty z\nu(dz)\right] dM(t) \\
&\quad + M(t) d\left(x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r}(1 - e^{-rT}) \int_0^\infty z\nu(dz)\right) \\
&\quad + \left(\xi e^{-r(T-t)} - \frac{\eta - \theta}{r}(1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz)\right) dH(t) \\
&\quad + H(t) d\left(\xi e^{-r(T-t)} - \frac{\eta - \theta}{r}(1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz)\right).
\end{aligned}$$

Note that the coefficient of $M(t)$ in (29) can be replaced by

$$\begin{aligned}
& x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r}(1 - e^{-rT}) \int_0^\infty z\nu(dz) \\
&= \left[X^{*,y^*}(t) - \xi e^{-r(T-t)} + \frac{\eta - \theta}{r}(1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz)\right] \frac{H(t)}{M(t)}.
\end{aligned} \tag{30}$$

According to (7) (24) and (30),

$$\begin{aligned}
& d\left[H(t)X^{*,y^*}(t)\right] \\
&= (\cdot)dt + H(t)\lambda_s \left[\frac{\gamma_1}{1 - \gamma_1} X^{*,y^*}(t) \right. \\
&\quad \left. - \frac{1}{1 - \gamma_1} \left(\xi e^{-r(T-t)} - \frac{\eta - \theta}{r}(1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz)\right)\right] dW(t) \\
&\quad + H(t) \left[X^{*,y^*}(t) \int_0^\infty \left(1 - (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}}\right) \right. \\
&\quad \left. + \left(\xi e^{-r(T-t)} - \frac{\eta - \theta}{r} \int_0^\infty z\nu(dz)\right) \left(\int_0^\infty \left(\theta - 1 + (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}}\right)\right) \right] \tilde{N}(dz, dt).
\end{aligned} \tag{31}$$

We are only interested in diffusion part, so Comparing (31) with (8), we obtain

$$\begin{aligned}
\pi^*(t) &= \frac{\mu_s - r}{\sigma_s^2} \frac{1}{1 - \gamma_1} \left[X^{*,y^*}(t) - \xi e^{-r(T-t)} + \frac{\eta - \theta}{r}(1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz)\right], \\
l^*(t) &= \frac{\theta - 1 + (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}}}{\theta + 1} \left[X^{*,y^*}(t) - \xi e^{-r(T-t)} + \frac{\eta - \theta}{r}(1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz)\right] \wedge z.
\end{aligned}$$

2) Similarly, when $H(t) \geq \bar{H}$, the $H(t)X^{*,y^*}$ in the (20) can be written as

$$H(t)X^{*,y^*}(t) = -\frac{\eta - \theta}{r} \int_0^\infty z\nu(dz) H(t) (1 - e^{-r(T-t)}), \tag{32}$$

so the optimal wealth at time t is given by

$$X^{*,y^*}(t) = -\frac{\eta - \theta}{r} (1 - e^{-r(T-t)}) \int_0^\infty z\nu(dz).$$

Taking differential on both sides of Equation (32)

$$d\left[H(t)X^{*,y^*}(t)\right] = (\cdot)dt + X^{*,y^*}(t)H(t) \left(-\lambda_s dW(t) + \int_0^\infty \theta \tilde{N}(dz, dt)\right),$$

and comparing it with (8), we obtain

$$\pi^*(t) = 0,$$

$$l^*(t) = 0.$$

□

Remark 3.2. If we put the (26) into equation (25), we find that

$$\begin{aligned} \pi^*(t) = & \frac{\mu_s - r}{\sigma_s^2} \frac{1}{1 - \gamma_1} \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \right] \\ & \times \exp \left\{ rt + \frac{1}{1 - \gamma_1} \lambda_s W(t) + \frac{1}{2} \frac{1 - 2\gamma_1}{(\gamma_1 - 1)^2} \lambda_s^2 t \right. \\ & \left. - \frac{1}{1 - \gamma_1} \int_0^t \int_0^\infty \ln(1 + \theta) N(dz, ds) + \int_0^t \int_0^\infty \left(\theta + 1 - (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}} \right) \nu(dz) ds \right\}. \end{aligned}$$

Similarly,

$$\begin{aligned} l^*(t) = & \frac{\theta - 1 + (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}}}{\theta + 1} \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \right] \\ & \times \exp \left\{ rt + \frac{1}{1 - \gamma_1} \lambda_s W(t) + \frac{1}{2} \frac{1 - 2\gamma_1}{(\gamma_1 - 1)^2} \lambda_s^2 t \right. \\ & \left. - \frac{1}{1 - \gamma_1} \int_0^t \int_0^\infty \ln(1 + \theta) N(dz, ds) + \int_0^t \int_0^\infty \left(\theta + 1 - (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}} \right) \nu(dz) ds \right\}. \end{aligned}$$

As the insurance has more initial wealth, as measured by x_0 , the π^* and l^* also increase linearly. If the appreciation rate μ_s of risky asset increases, the amount invested in the risky asset obviously increases. Furthermore, as the insurance market becomes more volatile, as measured by σ_s , the amount invested in the risky asset decreases nonlinearly. Also, it makes sense that μ_s and σ_s have no effect on the excess-of-loss reinsurance.

Remark 3.3. Theorem 3.2 shows that, $\pi^*(t) = 0$ and $l^*(t) = 0$ in the case of $H(T) \geq \bar{H}$. That is to say, when the market deteriorate, the insurer will stop investing in the risky asset and purchasing reinvestment strategy.

Remark 3.4. Optimal y^* in the Equation (16) can be rewritten as

$$y^* = \frac{\left(x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \right)^{\gamma_1 - 1}}{(\alpha \gamma_1)^{-1} N(T)^{\gamma_1 - 1}}. \quad (33)$$

Therefore, the optimal terminal wealth in Theorem 3.1 can be rewritten as

$$X^{*, y^*}(T) = \begin{cases} \xi + \left[x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} (1 - e^{-rT}) \int_0^\infty z \nu(dz) \right] \frac{H(T)^{\frac{1}{\gamma_1 - 1}}}{N(T)}, & H(T) < \bar{H}, \\ 0, & H(T) \geq \bar{H}, \end{cases} \quad (34)$$

where

$$N(T) = \exp \left\{ -\frac{\gamma_1}{\gamma_1 - 1} rT + \frac{1}{2} \frac{\gamma_1}{(\gamma_1 - 1)^2} \lambda_s^2 T + \left(-\frac{\gamma_1}{\gamma_1 - 1} \theta - 1 + (\theta + 1)^{\frac{\gamma_1}{\gamma_1 - 1}} \right) T \int_0^\infty \nu(dz) \right\},$$

$$\frac{H(T)^{\frac{1}{\gamma_1-1}}}{N(T)} = \exp \left\{ rT + \frac{1-2\gamma_1}{(1-\gamma_1)^2} \lambda_s T + \frac{1}{1-\gamma_1} \lambda_s W(T) \right. \\ \left. - \frac{1}{1-\gamma_1} \int_0^T \int_0^\infty \ln(1+\theta) N(dz, ds) + \left(1+\theta - (1+\theta)^{\frac{\gamma_1}{\gamma_1-1}} \right) T \int_0^\infty \nu(dz) \right\}.$$

4. Numerical Illustration

In this section, we present several numerical examples to illustrate our results in the previous section. Thus throughout this section, unless otherwise stated, the values parameters are taken as:

$$\alpha = 3; \beta = 4; \xi = 4; \gamma_1 = 0.2; \gamma_2 = 0.15; r = 0.05; \lambda = 6 \\ \eta = 0.5; \theta = 0.6; \mu_s = 0.1; \sigma_s = 0.3; x_0 = 4; T = 4.$$

In addition, we assume that the claim size Z_i follows the exponential distribution and the claim size density function is $f(z) = 6e^{-6z}$, hence $\mu_\infty = \frac{1}{6}$ and according to the remark 2.1, $\int_0^\infty zv(dz) = \lambda\mu_\infty = 1$. Subsequently, we analyze the effects of parameters on the optimal terminal wealth and the strategy. For convenience but without loss of generality, we focus on the case at time $t = 0$.

In this paper, Proposition 3.1 and **Figure 2** show that the optimal terminal wealth with S-shaped utility lies above the reference point ξ , which is similar to the smooth CRRA utility in good states of the market with low quasi-pricing kernel $H(T)$, while the wealth under loss aversion drops to 0 gradually with high quasi-pricing kernel $H(T)$ in very bad states. In particular, $\bar{H} = 1.31$ in **Figure 2** stands for the breakpoint of good states and bad states.

Note that the breakpoint $\bar{H}$ is of importance in Theorem 3.1 and Theorem 3.2, and hence we next discuss the sensitivity of the strategy with $\bar{H}$. We know

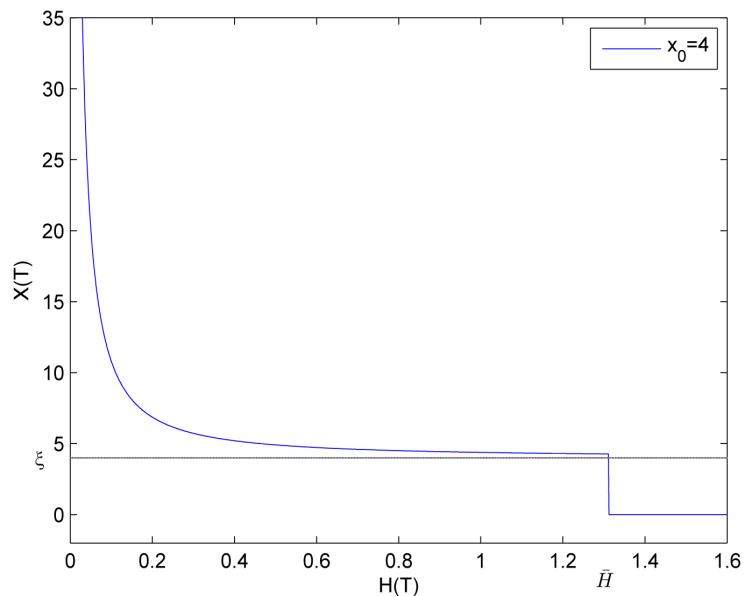


Figure 2. Optimal terminal wealth $X(T)$ with loss aversion.

y^* is the optimal lagrange multiplier and $\bar{H}$ is the root of $f(H(T))$. There is a negative relationship between y^* and $\bar{H}$.

$$y^* = \frac{\left(x_0 - \xi e^{-rT} + \frac{\eta - \theta}{r} \int_0^\infty z v(dz) (1 - e^{-rT})\right)^{\gamma_1 - 1}}{(\alpha \gamma_1)^{-1} N(T)^{\gamma_1 - 1}}, \quad (35)$$

$$f(H(T)) = \frac{1 - \gamma_1}{\gamma_1} y^* H(T) \left(\frac{\alpha \gamma_1}{y^* H(T)} \right)^{\frac{1}{1 - \gamma_1}} - y^* H(T) \xi + \beta \xi^{\gamma_2}, \quad (36)$$

Figure 3 shows the effects of the insurer's initial wealth x_0 on the breakpoint $\bar{H}$ and the optimal terminal wealth $X(T)$. Since $0 < \gamma_1 < 1$, from the expression of y^* , a larger x_0 leads to a decrease of the lagrange multiplier. Then we find from the **Figure 3(a)** that $\bar{H}$ increases with x_0 , which means an expansion of the insurance region. **Figure 3(b)** shows the optimal terminal wealth $X(T)$ is an increasing function of x_0 , which is the same as shown in the Theorem 3.1 and the expression in (34).

Figure 4 illustrates the effect of interest rate r on the optimal portfolio weight in risky asset. We find that the optimal investment weight in risky asset is a decreasing function of r , which is reasonable for the insurer to decrease the weight invested in the risky asset as the risk-free asset becomes more attractive.

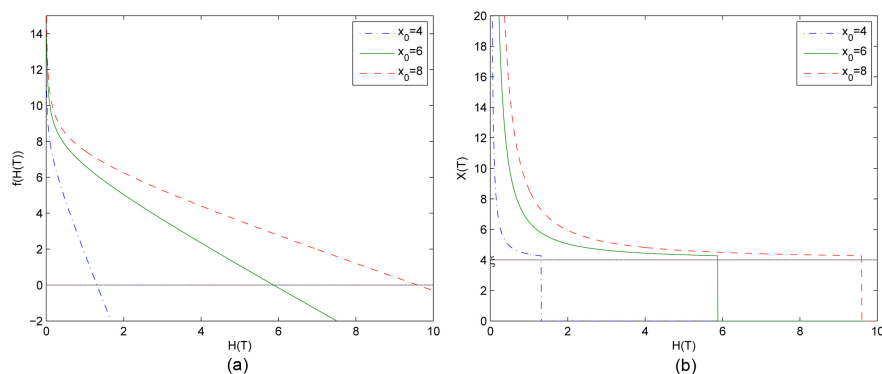


Figure 3. Effects of x_0 on $\bar{H}$ and $X(T)$.

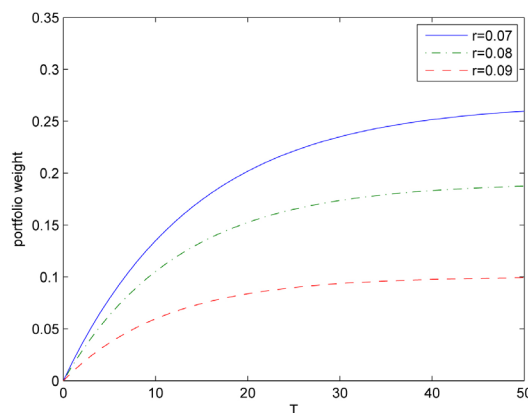


Figure 4. Effect of r on the investment weight.

Moreover, **Figure 4** also shows the optimal investment weight in risky asset is increasing function of the T , which agrees with the popular advice, in other words, to invest less in risky asset as one gets older and as the terminal time approaches.

Figure 5 and **Figure 6** describe the effects of parameters α and β of the S-shaped utility function on the breakpoint $\bar{H}$ and the optimal terminal wealth $X(T)$. As is shown in **Figure 5(a)**, when α is smaller, $\bar{H}$, the root the function $f(H(T))$ is larger. With α increasing, $\bar{H}$ decreases and the insurance region will be shortcut. On the contrary, when β is small, the zero $\bar{H}$ is small, and $\bar{H}$ increases along with the α . The effects of parameters α and β are opposite. The reason is that investors are risk-averse towards gains while they are risk-seeking towards losses.

From **Figure 7**, we find that γ_1 exerts positive effect on $X(T)$ and a negative effect on the optimal investment weight $\frac{\pi(t)}{X(t)}$. We know the

coefficient of relative risk aversion of the utility $u(x)$ equals $-\frac{u''(x)}{u'(x)}$, which

means that $1-\gamma_1$ stands for the coefficient of relative risk aversion. In other words, $1-\gamma_1$ decreases with the increasing of γ_1 , and the insurer becomes more gain-aversion. Therefore, increasing γ_1 leads to a increase in the optimal

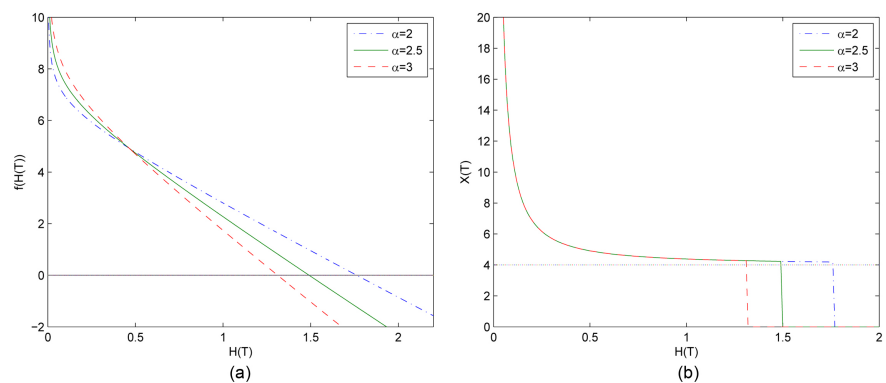


Figure 5. Effects of α on $\bar{H}$ and $X(T)$.

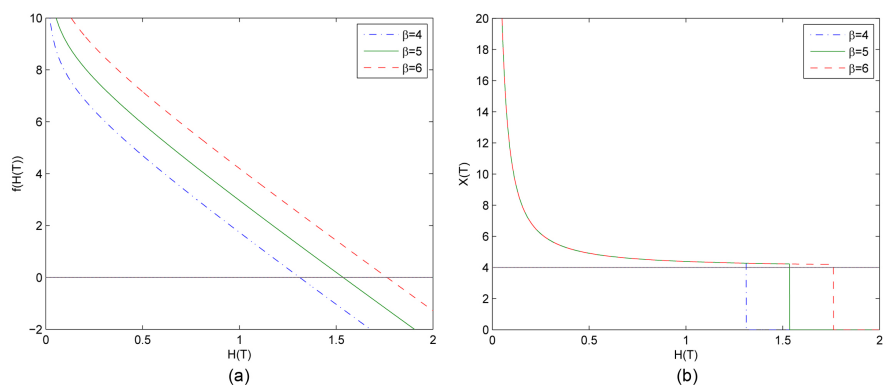


Figure 6. Effects of β on $\bar{H}$ and $X(T)$.

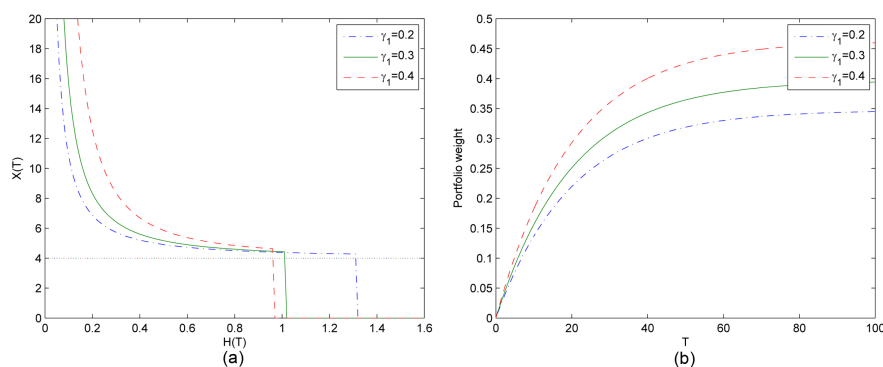


Figure 7. Effects of γ_1 and T on $X(T)$ and portfolio weight.

terminal wealth $X(T)$, a decrease in the breakpoint $\bar{H}$, as is shown in **Figure 7(a)**, and an increase in the proportion of wealth invested in the risky asset, as shown in **Figure 7(b)**. Similarly, **Figure 7(b)** also shows the initial portfolio weight of risky asset is an increasing function of the T .

5. Conclusions

In this paper, we consider the optimal investment and reinsurance strategy for an insurer with loss aversion. The insurer aims to maximize the expected utility of terminal wealth and the wealth is allowed to invest in a risk-free asset and a risky asset. Furthermore, the insurer can purchase excess-of-loss reinsurance. Since the S-shaped utility is convex-concave, the stochastic programming method is not suitable, and we obtain a close-form solution for the optimal strategy by using the martingale method. We find that the optimal terminal wealth is a piecewise function. In good states of the market, the optimal wealth is of the same form as the smooth CRRA utility function case; on the contrary, the optimal wealth approaches 0 in bad states of the market. Similarly, the optimal investment and reinsurance strategy are also divided into two cases respectively. When the market deteriorates, the insurer will stop investing in the risky asset and purchasing the reinvestment strategy. Finally, we present some numerical examples to show the effects of model parameters on the optimal terminal wealth and the optimal strategy.

Based on our current work, various directions may be followed in the future research. 1) the price process of the risky asset now is described by the GBM, further we can add the diffusion term to the model, or try to use the CEV model or Heston model; 2) notice that the interest rate used in this paper is a fixed constant, so we can introduce the stochastic interest rate process, such as the Vasicek model or Ornstein-Uhlenbeck model; 3) derivatives, such as options, can be added to the paper and purchased by the insurer, thus we can research the effect of derivatives on investment strategies and control risks; 4) the reference point in the S-shaped utility function can be dynamic by introducing inflation, which will change with time and correlate with inflation factors. All the future research directions will make the problem more comprehensive and complex,

but also more relevant to reality in insurers daily business.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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