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N-Expansive Property for Flows

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Abstract

In this paper, we discuss the dynamics of *n*-expansive homeomorphisms with the shadowing property defined on compact metric spaces in continuous case. For every $n \in \mathbb{N}$, we exhibit an *n*-expansive homeomorphism but not $(n-1)$-expansive. Furthermore, that flow has the shadowing property and admits an infinite number of chain-recurrent classes.

Keywords

Expansive, Flow, *N*-Expansive, Shadowing

1. Introduction and Preliminaries

The classical terms, expansive flows on a metric space are presented by Bowen and Walters [1] which generalized the similar notion by Anosov [2]. Besides, Walters [3] investigated continuous transformations of metric spaces with discrete centralizers and unstable centralizers and proved that expansive homeomorphisms have unstable centralizers; other result was studied in [4]. In discrete case, this concept originally introduced for bijective maps by Utz [5] has been generalized to positively expansiveness in which positive orbits are considered instead [6]. Further generalizations are the pointwise expansiveness (with the above radius depending on the point [7]), the entropy-expansiveness [8], the continuum-wise expansiveness [9], the measure-expansiveness and their corresponding positive counterparts. However, as far as we know, no one has considered the generalization in which at most *n* companion orbits are allowed for a certain prefixed positive integer *n*. For simplicity we call these systems *n*-expansive (or positively *n*-expansive if positive orbits are considered instead). A generalization of the expansiveness property that has been given attention recently is the *n*-expansive property (see [10] [11] [12] [13] [14]).

In this paper, we introduce a notion of *n*-expansivity for flows which is generalization of expansivity, and show that there is an *n*-expansive flow but not $(n-1)$-expansive flow. Moreover, that flow is shadowable and has infinite number of chain-recurrent classes.

Let (X,d) be a metric space. A *flow* on *X* is a map $\phi: X\times\mathbb{R}\to X$ satisfying $\phi(x,0)=x$ and $\phi(\phi(x,s),t)=\phi(x,s+t)$ for $x\in X$ and $t,s\in\mathbb{R}$. For convenience, we will denote

$$\phi(x,s)=\phi_s(x) \text{ and } \phi_{(a,b)}(x)=\{\phi_t(x): t\in(a,b)\}.$$

The set $\phi_{\mathbb{R}}(x)$ is called the orbit of ϕ through $x\in X$ and will be denoted by $\mathrm{Orb}_\phi(x)$. We have the following several basis concepts (see [1] [15] [16]).

Definition 1.1. *Let* ϕ *be a flow in a metric space* (X,d). *We say that* ϕ *is n-expansive (*$n\in\mathbb{N}$*) if there exists* $c>0$ *such that for every* $x\in X$ *the set*

$$\Gamma(x,c):=\{y\in X; d(\phi_t(x),\phi_t(y))\le c, \forall t\in\mathbb{R}\},$$

contains at most *n* different points of *X*.

We say that ϕ is finite expansive if there exists $c>0$ such that for every $x\in X$ the set $\Gamma(x,c)$ is finite.

Definition 1.2. *Let* $x\in X$. *We say that x is a period point if there exists* $T>0$ *such that* $\phi_{t+T}(x)=\phi_t(x),\forall t\in\mathbb{R}$. *Denote that* $\pi(x)$ *is the period of x, which is the smallest non-negative number satisfying this equation.*

Definition 1.3. *Give* $\delta,T\ge 0$. *We say that a sequence of pairs* $(x_i,t_i)_{i\in\mathbb{Z}}\subset X\times\mathbb{R}$ *is a* (δ,T)*-pseudo orbit of* ϕ *if* $t_i\ge T$ *and* $d(\phi_{t_i}(x_i),x_{i+1})\le\delta,\forall i\in\mathbb{Z}$.

We define

$$s_i=\begin{cases}\sum_{j=0}^{i-1}t_j, & i>0,\\ 0, & i=0,\\ -\sum_{j=i}^{-1}t_j, & i<0,\end{cases}$$

and $x_0\star t=\phi_{t-s_i}(x_i)$ whenever $s_i\le t<s_{i+1}$.

Definition 1.4. *We say that* ϕ *is shadowing property if for each* $\epsilon>0$ *there is* $\delta>0$ *such that for any* $(\delta,1)$*-pseudo orbit* $(x_i,t_i)_{i\in\mathbb{Z}}$, *there exists* $x\in X$ *and an orientation preserving homeomorphism* $h:\mathbb{R}\to\mathbb{R}$ *such that* $h(0)=0$ *and* $d(x_0\star t,\phi_{h(t)}(x))\le\epsilon$.

Denote by *Rep* the set of orientation preserving homeomorphism $h:\mathbb{R}\to\mathbb{R}$ such that $h(0)=0$.

Definition 1.5. *Give two points p and q in X. We say p and q are* (δ,T)*-related if there are two* (δ,T)*-chains* $(x_i,t_i)_{i=0}^m$ *and* $(y_i,s_i)_{i=0}^n$ *such that* $p=x_0=y_n$ *and* $q=y_0=x_m$. *We say that p and q are related* $(p\sim q)$ *if they are* (δ,T)*-related for every* $\delta,T>0$. *The chain-recurrent class of a point* $p\in X$ *is the set of all points* $q\in X$ *such that* $p\sim q$.

Theorem 1.1. *For every $n \in \mathbb{N}$, there is an n-expansive flow, define in a compact metric space, that is not $(n-1)$-expansive, has the shadowing property and admits an infinite number of chain-recurrent classes.*

2. Proof of the Main Theorem

Consider a flow ϕ defined in a compact metric space (M, d_0), and ϕ has 1-expansive, and has the shadowing property. Further, suppose it has an infinite number of period points $\{p_k\}_{k\in\mathbb{N}}$, which we can suppose belong to different orbits. Let E be an infinite set, such that there exists a bijection $r:\mathbb{R}\to E$. Let

$$Q = \bigcup_{k\in\mathbf{N}} \{1,\cdots,n-1\}\times\{k\}\times\left[0,\pi(p_k)\right),$$

and note that there exists a bijection $s:Q\to\mathbb{R}$. Consider the bijection $q:Q\to E$ defined by

$$q(i,k,j) = r\circ s(i,k,j).$$

Let $X = M\cup E$. Thus, any point $x\in E$ has the form $x = q(i,k,j)$ for some $(i,k,j)\in Q$. Define a function $d: X\times X\to\mathbb{R}^+$ by

$$d(x,y) = \begin{cases} 0, & x = y, \\ d_0(x,y), & x,y\in M, \\ \dfrac{1}{k}+d_0\left(y,\phi_j(p_k)\right), & x = q(i,k,j),\ y\in M, \\ \dfrac{1}{k}+d_0\left(x,\phi_j(p_k)\right), & x\in M,\ y = q(i,k,j), \\ \dfrac{1}{k}, & x = q(i,k,j),\ y = q(l,k,j),\ i\neq l, \\ \dfrac{1}{k}+\dfrac{1}{m}+d_0\left(\phi_t(p_k),\phi_r(p_m)\right), & x = q(i,k,j),\ y = q(l,m,r),\ k\neq m \text{ or } j\neq r. \end{cases}$$

Now we prove that function d is a metric in X. Indeed, we see that $d(x,y)=0$ iff $x=y$, and that $d(x,y)=d(y,x)$ for any pair $(x,y)\in X\times X$. We shall prove that the triangle inequality $d(x,z)\le d(x,y)+d(y,z)$ for any triple $(x,y,z)\in X\times X\times X$. When $(x,y,z)\in M\times M\times M$ we have that $d_{|M\times M} = d_0$, and d_0 is a metric in M. When $(x,y,z)\in M\times M\times E$ then $z = q(i,k,j)$ and

$$d(x,z) = \frac{1}{k}+d_0\left(x,\phi_j(p_k)\right)\le d_0(x,y)+\frac{1}{k}+d_0\left(y,\phi_j(p_k)\right) = d(x,y)+d(y,z).$$

Therefore, when $(x,y,z)\in E\times M\times M$, changing the role of x and z in the previous case, we obtain this result. When $(x,y,z)\in M\times E\times M$, we have $y = q(i,k,j)$ and

$$d(x,z) = d_0(x,z)\le\frac{2}{k}+d_0\left(x,\phi_j(p_k)\right)+d_0\left(z,\phi_j(p_k)\right) = d_0(x,y)+d_0(y,z).$$

When $(x,y,z)\in M\times E\times E$, we have $y = q(i,k,j)$ and $z = (l,m,r)$. If

$k \neq m$ or $j \neq r$ then

$$\begin{aligned} d(x,z) &= \frac{1}{m} + d_0(x, \phi_r(p_m)) \\ &< \frac{2}{k} + \frac{1}{m} + d_0(x, \phi_j(p_k)) + d_0(\phi_j(p_k), \phi_r(p_m)) \\ &= d(x,y) + d(y,z). \end{aligned}$$

If $k = m$, $j = r$ and $i \neq l$ then

$$d(x,z) = \frac{1}{m} + d_0(x, \phi_r(p_m)) < \frac{1}{k} + \frac{1}{m} + d_0(x, \phi_j(p_k)) = d(x,y) + d(y,z).$$

So if $(x,y,z) \in E \times E \times M$, change the role of x and z in previous case, and we get the result. If $(x,y,z) \in E \times M \times E$ then $x = q(i,k,j)$ and $z = q(l,m,r)$. Hence,

$$d(x,y) + d(y,z) = \frac{1}{k} + \frac{1}{m} + d_0(y, \phi_j(p_k)) + d_0(y, \phi_r(p_m))$$

and

$$d(x,z) = \begin{cases} \frac{1}{k} + \frac{1}{m} + d_0(\phi_j(p_k), \phi_r(p_m)) & \text{if } k \neq m \text{ or } j \neq r, \\ \frac{1}{k} & \text{if } k = m,\ j = r \text{ and } i \neq l. \end{cases}$$

Thus, we always get the result $d(x,z) < d(x,y) + d(y,z)$ for both of 2 cases. When $(x,y,z) \in E \times E \times E$, we let

$$x = q(i_1, k_1, j_1), y = q(i_2, k_2, j_2), z = q(i_3, k_3, j_3).$$

Case 1. If $k_1 = k_3$ and $j_1 = j_3$ we have $d(x,z) = \frac{1}{k_1}$, and

$$\begin{aligned} & d(x,y) + d(y,z) \\ &= \begin{cases} \frac{2}{k_1}, & k_1 = k_2 = k_3 \text{ and } j_1 = j_2 = j_3, \\ \frac{2}{k_1} + \frac{2}{k_2} + d_0(\phi_{j_1}(k_1), \phi_{j_2}(k_2)) + d_0(\phi_{j_2}(k_2), \phi_{j_3}(k_3)), & k_1 = k_3 \neq k_2 \text{ or } j_1 = j_3 \neq j_2. \end{cases} \end{aligned}$$

It means that $d(x,z) < d(x,y) + d(y,z)$ for both of 2 cases.

Case 2. If $k_1 \neq k_3$ or $j_1 \neq j_3$, we have

$$d(x,z) = \frac{1}{k_1} + \frac{1}{k_3} + d_0(\phi_{j_1}(k_1), \phi_{j_3}(k_3)),$$

and

$$\begin{aligned} & d(x,y) + d(y,z) \\ &= \begin{cases} \frac{2}{k_1} + \frac{1}{k_3} + d_0(\phi_{j_2}(k_2), \phi_{j_3}(k_3)), & k_1 = k_2 \text{ and } j_1 = j_2, \\ \frac{1}{k_1} + \frac{2}{k_3} + d_0(\phi_{j_1}(k_1), \phi_{j_2}(k_2)), & k_2 = k_3 \text{ and } j_2 = j_3, \\ \frac{1}{k_1} + \frac{2}{k_2} + \frac{1}{k_3} + d_0(\phi_{j_1}(k_1), \phi_{j_2}(k_2)) + d_0(\phi_{j_2}(k_2), \phi_{j_3}(k_3)), & k_1 \neq k_2 \neq k_3 \text{ or } j_1 \neq j_2 \neq j_3. \end{cases} \end{aligned}$$

Hence, $d(x,z) < d(x,y) + d(y,z)$.

It implies d is a metric in X.

Next, we prove that (X,d) is a compact metric space. Let any sequences $(x_n)_{n\in\mathbb{N}} \in X$. We prove that this sequence has a convergent subsequence. If $(x_n)_{n\in\mathbb{N}}$ has infinite elements in M, then the compactness of M and the fact $d_{|M\times M} = d_0$, so $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. We consider $(x_n)_{n\in\mathbb{N}}$ has finite elements in M; therefore, it has infinite elements in E. We can assume that $(x_n)_{n\in\mathbb{N}} \subset E$ then $x_n = q(i_n, k_n, j_n)$. If there is $N \in \mathbb{N}$ such that $k_n < N, \forall n \in \mathbb{N}$ then the set $\{x_n; n\in\mathbb{N}\}$ is finite, so at least one point of $(x_n)_{n\in\mathbb{N}}$ appears infinite times, forming a convergent subsequence. Now suppose $(k_n)_{n\in\mathbb{N}}$ is unbounded, therefore, $\lim_{n\to\infty} k_n = \infty$. We choose $y_n = \phi_{j_n}(p_{k_n})$, so $(y_n)_{n\in\mathbb{N}} \subset M$ and $d(x_n, y_n) = \frac{1}{k_n}, \forall n \in \mathbb{N}$. Since $(y_n)_{n\in\mathbb{N}}$ is a subset of compact set M, $(y_n)_{n\in\mathbb{N}}$ has a subsequence $(y_{n_l})_{l\in\mathbb{N}}$ converging to $y \in M$. Thus, we have

$$d(x_{n_l}, y) < d(x_{n_l}, y_{n_l}) + d(y_{n_l}, y) = \frac{1}{k_{n_l}} + d(y_{n_l}, y) \to 0 \text{ when } l \to \infty.$$

It implies that $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_l})_{l\in\mathbb{N}}$ which converges to y. Thus, (X,d) is a compact metric space.

For all $t \in \mathbb{R}$, we define a map ψ_t by

$$\psi_t(x) = \begin{cases} \phi_t(x) & \text{if } x \in M, \\ q(i,k,(j+t)\bmod \pi(p_k)) & \text{if } x = q(i,k,j). \end{cases}$$

We can see that j, t, $j+t$ cannot be in $\mathbb{N}$, but we can define a real number: $t \bmod \pi(p_k) := r$, when

$$t = m\pi(p_k) + r,\ m \in \mathbb{Z},\ 0 \le r < \pi(p_k).$$

By definition of flow, it's easy to see that ψ is a flow of X. Indeed, we can prove that $\psi_{t+s} = \psi_t \circ \psi_s, \forall t,s \in \mathbb{R}$. If $x \in M$, we get

$$\psi_{t+s}(x) = \phi_{t+s}(x) = \phi_t \circ \phi_s(x) = \psi_t \circ \psi_s(x),\ \forall t,s \in \mathbb{R}.$$

If $x = q(i,k,j)$, we have

$$\psi_{t+s}(x) = q(i,k,(j+t+s)\bmod \pi(p_k)) = \psi_t \circ \psi_s(x).$$

Therefore, ψ is the flow with the previous properties.

In order to prove that ψ is n-expansive, first we see that ϕ is 1-expansive; so there is $a > 0$ such that if $d(\phi_t(x), \phi_t(y)) \le a, \forall t \in \mathbb{N}$, then $x = y$. Suppose that $\{x_1, \cdots, x_{n+1}\}$ are $n+1$ different points of X satisfying

$$d(\psi_t(x_i), \psi_t(x_j)) \le a,\ \forall t \in \mathbb{R},\ \forall (i,j) \in \{1,\cdots,n+1\} \times \{1,\cdots,n+1\}.$$

Hence, at most one of these points belong to M. Consequently, at least n of them belong to E. Without loss of generality, we get $x_m = q(i_m, k_m, j_m), m \in \{1,\cdots,n\}$. Because $i_m \in \{1,\cdots,n-1\}$ and we have n number i_m; thus, by Pigeonhole principle, at least two of these points are of the form $q(i,k,j)$ and $q(i,m,r)$. We prove that $k \neq m$. Indeed, if $k = m$, we have 2 points are $q(i,k,j)$ and $q(i,k,r)$ with $j \neq r$ (because all of $n+1$ points are different). For each $s \in \mathbb{R}$ we have

$$\begin{aligned} & d\left(\phi_s\left(\phi_j\left(p_k\right)\right)\right), d\left(\phi_s\left(\phi_r\left(p_k\right)\right)\right) \\ & = d\left(\psi_s\left(q\left(i,k,j\right), \psi_s\left(q\left(i,k,r\right)\right)\right)\right) - \frac{2}{k} \\ & < d\left(\psi_s\left(q\left(i,k,j\right)\right), \psi_s\left(q\left(i,k,r\right)\right)\right) < a. \end{aligned}$$

This implies that $\phi_j\left(p_k\right) = \phi_r\left(p_k\right)$ (by the Proposition of 1-expansive of ϕ), which implies that $j = r$ and we obtain a contradiction. Therefore, $k \neq m$.

Now we implies that: for every $s \in \mathbb{R}$ we have:

$$\begin{aligned} & d\left(\phi_s\left(\phi_j\left(p_k\right)\right)\right), d\left(\phi_s\left(\phi_r\left(p_m\right)\right)\right) \\ & = d\left(\psi_s\left(q\left(i,k,j\right)\right), \psi_s\left(q\left(i,m,r\right)\right)\right) - \frac{1}{k} - \frac{1}{m} \\ & < d\left(\psi_s\left(q\left(i,k,j\right)\right), \psi_s\left(q\left(i,m,r\right)\right)\right) < a. \end{aligned}$$

So similarly, we have $\phi_j\left(p_k\right) = \phi_r\left(p_m\right)$; hence, $p_m = p_k$, which is contradiction with the fact that $k \neq m$. Thus, we cannot choose $n+1$ points satisfy this proposition; it means ψ is *n*-expansive in *X*.

Next, we prove that ψ is not $(n-1)$-expansive. For any $a > 0$, we choose $k \in \mathbb{N}$ such that $\frac{1}{k} < a$, so we have

$d\left(\phi_j\left(p_k\right), q\left(i,k,j\right)\right) = \frac{1}{k} < a, \forall j \in \mathbb{R}, \forall i \in \{1, \cdots, n-1\}$. So $\Gamma\left(p_k, a\right)$ contain at least n points $\{p_k, q(1,k,0), \cdots, q(n-1,k,0)\}$ and that ψ is not $(n-1)$-expansive, because there is not $a > 0$ satisfies this define about $(n-1)$-expansive.

Now we prove that ψ has the shadowing property. Since ϕ has the shadowing property, for each $\epsilon > 0$, we can consider $\delta_\phi > 0$, so for any $(\delta_\phi, 1)$-pseudo-orbit in *M* we have the $\frac{\epsilon}{2}$-shadowing. Now consider $(x_n, t_n)_{n \in \mathbb{Z}}$ has the $(\delta, 1)$-pseudo-orbit by ψ in *X*. We assume that $\delta < \frac{\delta_\phi}{3} < \frac{\epsilon}{3}$. So we have $d\left(\psi_{t_n}\left(x_n\right), x_{n+1}\right) < \delta$. Let *N* is a smallest integer number such that $\frac{1}{N} < \delta$, and we consider (x_n, x_{n+1}) in 3 cases.

Case 1. If $(x_n, x_{n+1}) \in E \times M$, we have $x_n = q(i,k,j)$ and

$d\left(\psi_{t_n}\left(x_n\right), x_{n+1}\right) = \frac{1}{k} + d_0\left(x_{n+1}, \phi_{j+t_n}\left(p_k\right)\right)$, so $\frac{1}{k} < \delta$; hence, $k \geq N$.

Case 2. If $(x_n, x_{n+1}) \in M \times E$, we obtain $x_{n+1} = q(i,k,j)$ and

$d\left(\psi_{t_n}\left(x_n\right), x_{n+1}\right) = \frac{1}{k} + d_0\left(\phi_j\left(p_k\right), \phi_j\left(x_n\right)\right)$, so $\frac{1}{k} < \delta$; hence, $k \geq N$.

Case 3. If $(x_n, x_{n+1}) \in E \times E$, we have $x_n = q(i,k,j)$ and $x_{n+1} = q(l,m,r)$. So $\psi_{t_n}\left(x_n\right) = q\left(i,k,j+t_n\right)$. Thus, if we want $d\left(\psi_{t_n}\left(x_n\right), x_{n+1}\right) < \delta$, we have either if $k \geq N$, so $m \geq N$ (by similarly) or if $k < N$, we have $x_{n+1} = \psi_{t_n}\left(x_n\right)$, such that $x_{n+1} = q\left(i,k,j+t_n\right)$. When $(x_n, t_n)_{n \in \mathbb{Z}}$ is one of orbit $\{q(l,k,j_n)\}_{n \in \mathbb{Z}}$, and $j_{n+1} = j_n + t_n, \forall n \in \mathbb{Z}$. So one obtain $s_n = j_n - j_0$, thus,

$$d\left(\psi_{t-s_n}(x_n), \psi_t(x_0)\right) = d\left(q(l,k,t-s_n+j_n), q(l,k,t+j_0)\right) = 0,\ s_n \le t < s_{n+1}.$$

Therefore, the shadowing property is proved.

When $x_i = q(l,k,j)$, then $k > N$. Define a sequence $(y_n, t_n)_{n\in\mathbb{Z}} \subset M$ by

$$y_n = \begin{cases} x_n & \text{if } x_n \in M, \\ \phi_j(p_k) & \text{if } x_n = q(l,k,j). \end{cases}$$

Then $(y_n, t_n)_{n\in\mathbb{Z}}$ is δ_ϕ-pseudo-orbit in M since

$$\begin{aligned} d\left(\phi_{t_n}(y_n), y_{n+1}\right) &= d\left(\psi_{t_n}(y_n), y_{n+1}\right) \\ &\le d\left(\psi_{t_n}(y_n), \psi_{t_n}(x_n)\right) + d\left(\psi_{t_n}(x_n), x_{n+1}\right) + d\left(x_{n+1}, y_{n+1}\right) \\ &< \frac{1}{N} + \delta + \frac{1}{N} < \delta_\phi. \end{aligned}$$

Hence, there exists $y \in M$ and a function $h \in Rep$ such that

$$d\left(\phi_{t-s_n}(y_n), \phi_{h(t)}(y)\right) < \frac{\epsilon}{2},\ \forall s_n \le t < s_{n+1}.$$

So

$$\begin{aligned} d\left(\phi_{t-s_n}(x_n), \phi_{h(t)}(y)\right) &< d\left(\phi_{t-s_n}(x_n), \phi_{t-s_n}(y_n)\right) + d\left(\phi_{t-s_n}(y_n), \phi_{h(t)}(y)\right) \\ &< \frac{1}{N} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore, $(x_n, t_n)_{n\in\mathbb{Z}}$ is ϵ-shadowing. Hence, ψ has the shadowing property.

Finally, we have ψ admits an infinite number of chain-recurrent classes. Indeed, if we have $q(i,k,l) \in E$ then

$$d\left(q(i,k,j), x\right) \ge \frac{1}{k},\ \forall x \in X \setminus \{q(i,k,j)\}.$$

So if $0 < \epsilon < \frac{1}{k}$ then the orbit of $q(i,k,j)$ cannot be connected by ϵ-pseudo orbits with any other point of X. This proves that the chain-recurrent classes of $q(i,k,j)$ contains only its orbit. Therefore different periodic orbits in E belong to different chain-recurrent classes and we conclude the proof.

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Open Questions

How are the properties of the local stable (unstable) sets of n-expansive flows?

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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