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The Signless Laplacian Spectral Radius of Some Special Bipartite Graphs

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Abstract

This paper mainly researches on the signless laplacian spectral radius of bipartite graphs $D_r(m_1,m_2;n_1,n_2)$. We consider how the signless laplacian spectral radius of $D_r(m_1,m_2;n_1,n_2)$ changes under some special cases. As application, we give two upper bounds on the signless laplacian spectral radius of $D_r(m_1,m_2;n_1,n_2)$, and determine the graphs that obtain the upper bounds.

Keywords

The Signless Laplacian Spectral Radius, The Largest Eigenvalue, Bipartite Graph

1. Introduction

Let $G=(V(G),E(G))$ be a connected graph of order $n\geq 1$ with vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$ and edge set $E(G)$ is considered in this paper. In spectral graph theory, one usually uses the spectrum of related matrices to characterize the structure of graphs. The most studied matrix associated with G appears to be the adjacency matrices $A(G)=(a_{ij})$ with $a_{ij}=1$ when there's an edge between v_i and v_j, otherwise $a_{ij}=0$. Some other well studied matrices are the Laplacian matrix and the signless Laplacian matrix of *G*. The former is defined by $L(G)=D(G)-A(G)$ where $D(G)=diag(d(v_1),\cdots,d(v_n))$ is the degree diagonal matrix, whereas the latter is defined as $Q(G)=D(G)-A(G)$. For polynomials of $f(x)$ with only real roots, let $\tau(f(x))$ be the largest root of $f(x)$; the maximum eigenvalue of the (unsigned) Laplace of graph G is denoted as $\tau(G)$, δ and Δ respectively represent the minimum and maximum degrees of the graph.

The actical [1] studied the bipartite graph $D(m_1,m_2;n_1,n_2)$ with the fixed order of n and the size of m, and the graph with the largest Laplace spectrum radius in the graph class was determined, as well as the upper bound of the Laplace spectrum radius of $D(m_1, m_2; n_1, n_2)$. The actical [2] studies the bipartite graph of the fixed order of n and the size of $n+k(k>0)$, and describes the structure of the bipartite graph with maximum adjacency spectrum radius. Reference [3] is to determine the structure of the bipartite graph of the fixed order of n and the size of m after removing the given k edges. In Reference [4], by studying the signless Laplacian spectrum, the structure of the maximum spectrum of bipartite graphs with fixed order and size is determined, and the upper and lower bounds of the spectrum are given again.

Inspired by the above results, this paper studies the bipartite graph $G = D_r(m_1, m_2; n_1, n_2)$. We fix the order and the size of the bipartite graphs G, and observe what influence may have on the signless Laplacian radius of G after transforming the neighborhood of some vertexes.

Before giving the main conclusion of this paper, we first introduce the bipartite graph $D_r(m_1, m_2; n_1, n_2)$, and then give the definitions of equitable division and quotient matrix that need to be used in the later proof:

Definition 1.1. Let G be a connected bipartite graph with two vertex sets of U and V, each vertex set has the following partition, $U = U_1 \cup U_2, V = V_1 \cup V_2$, and every vertex in U_1 is connected to all vertexes in V_1 and V_2, every vertex in U_2 is connected to all vertexes in V_2, every vertex in V_1 is connected to all vertexes in U_1 and U_2, each vertex in V_2 is connected to all vertexes in U_1, showing in **Figure 1**. If the number of vertexes in U_1, U_2, V_1, V_2 is m_1, m_2, n_1, n_2 respectively, and the induction sub-graphs of U_1, U_2, V_1, V_2 are all r-regular, then denoting $G = D_r(m_1, m_2; n_1, n_2)$. For convenience, the order and the size of G are n and m, that is

$$n = m_1 + m_2 + n_1 + n_2, m = m_1(n_1 + n_2) + m_2 n_1 \tag{1}$$

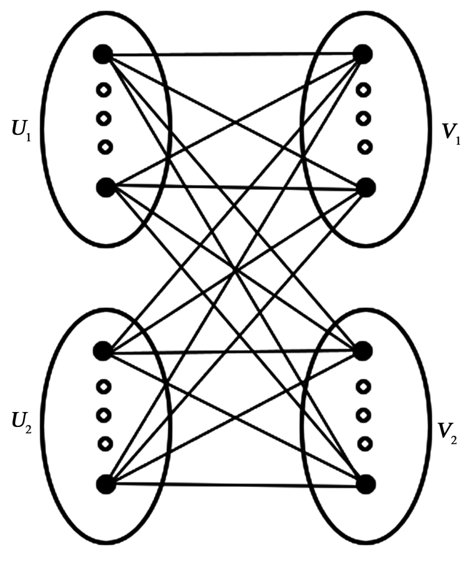

Figure 1. $D_1(m_1, m_2; n_1, n_2)$.

Definition 1.2. Let G be a connected graph, the vertex set $V(G)$ of G is divided into $V(G)=V_1\cup V_2\cup\cdots\cup V_K$. If every vertex in V_i has the same number of adjacent vertex in V_j, for any $i,j\in\{1,2,\cdots,k\}$, such partitions $V_1\cup V_2\cup\cdots\cup V_K$ are called equitable division of G.

Definition 1.3. Let A as real symmetric matrix, and its row and column are divided equally. The matrix formed by elements that is the average row sum of each sub-blocks, according to the position of its sub-blocks is called the quotient matrix of A.

Lemma 1. [5] The spectral radius of G must be the eigenvalue of any quotient matrix of G.

Lemma 2. [5] For any graph G, let $\tau(G)$ be the maximum (signless) Laplace eigenvalue of G, then $\tau(G)\le\max\{d_u+d_v,u,v\in E(G)\}$.

Lemma 3. Let $D_r(m_1,m_2;n_1,n_2)$ be a connected bipartite graph and defined in definition 1.1, then U_1,U_2,V_1,V_2 is an equitable division.

Proof: Because every vertex in U_1 is connected to all vertexes in U_2,V_1 and V_2, so every vertex of U_1 has the same number of adjacent vertexes in U_2,V_1,V_2. Similarly, every vertex in U_2 has the same number of adjacent vertexes in U_1,V_1,V_2, every vertex of V_1 has the same number of adjacent vertexes in U_1,U_2,V_2, every vertex of V_2 has the same number of adjacent vertexes with U_1,U_2,V_1. So $U_1\cup U_2\cup V_1\cup V_2$ is an equitable divide.

2. Regular

In this section, we mainly discuss the graph $D_0(m_1,m_2;n_1,n_2)$, that is, the induction sub-graphs of U_1,U_2,V_1,V_2 are all independent sets and are all 0-regular graphs. For such figure $D_0(m_1,m_2;n_1,n_2)$, we observe the change of the maximum eigenvalue of the graph by taking neighborhood transformation, and then determine the structure of the graph when the graph achieve the maximum spectral radius.

Lemma 2.1. Let $G=D_0(m_1,m_2;n_1,n_2), H=D_0(m_1+a,m_2;n_1-a,n_2)$, when $a-m_1+n_1<0$, $\tau(G)<\tau(H)$.

If $a-m_1+n_1>0$, $\tau(G)>\tau(H)$, else $\tau(G)=\tau(H)$.

Proof: Since U_1,U_2,V_1,V_2 is an equitable division of G, Q_1 and Q_2 are a quotient matrix of $Q(G)$ and $Q(H)$,

$$Q_1=\begin{pmatrix} n_1+n_2 & 0 & n_1 & n_2\\ 0 & n_1 & n_1 & 0\\ m_1 & m_2 & m_1+m_2 & 0\\ m_1 & 0 & 0 & m_1\end{pmatrix} \tag{2}$$

$$Q_2=\begin{pmatrix} n_1+n_2+a & 0 & n_1+a & n_2\\ 0 & n_1+a & n_1+a & 0\\ m_1-a & m_2 & m_1+m_2-a & 0\\ m_1-a & 0 & 0 & m_1-a\end{pmatrix} \tag{3}$$

The characteristic polynomials of Q_1 and Q_2 are obtained by calculation as follows:

$$
\begin{aligned}
f_1(x) &= x^4 + (-2m_1 - m_1 - 2n_1 - n_2)x^3 \\
&\quad + \left(m_1m_2 + 3m_1n_1 + m_1n_2 + m_2n_1 + m_2n_2 + n_1n_2 + m_1^2 + n_1^2\right)x^2 \\
&\quad - \left(m_1n_1^2 + m_1^2n_1 + m_1m_2n_1 + m_1n_1n_2\right)x
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
f_2(x) &= x^4 + (-2m_1 - m_1 - 2n_1 - n_2)x^3 + (am_1 - an_1 + m_1n_2 + 3m_1n_2 \\
&\quad + m_1n_2 + m_1n_1 + m_1n_2 + n_1n_2 - a^2 + m_1^2 + n_1^2)x^2 + (a^2m_1 - am_1^2 \\
&\quad + a^2m_2 + an_1^2 + a^2n_1 + a^2n_2 - m_1n_1^2 - m_1^2n_1 - am_1m_2 - am_1n_2 \\
&\quad + am_2n_1 + an_1n_2 - m_1m_2n_1 - m_1n_1n_2)x
\end{aligned} \tag{5}
$$

$$
f_3(x) = f_1(x) - f_2(x) = ax(a - m_1 + n_1)(x - m_1 - m_2 - n_1 - n_2) \tag{6}
$$

The largest root of $f_1(x), f_2(x)$ is denoted as $\tau(f_1(x)), \tau(f_2(x))$. By Lemma 3, it's easily to know $\tau(f_1(x)) < d_i + d_j < m_1 + m_2 + n_1 + n_2$, $\tau(f_2(x)) < d_i + d_j < m_1 + m_2 + n_1 + n_2$, so $x - m_1 - m_2 - n_1 - n_2 < 0$, then

When $a - m_1 + n_1 < 0$,

$$
f_3(x) = f_1(x) - f_2(x) = ax(a - m_1 + n_1)(x - m_1 - m_2 - n_1 - n_2) = -f_2(\tau(f_1(x)))
$$

so $f_2(\tau(f_1(x))) < 0$, then $\tau(f_1) < \tau(f_2), \tau(G) < \tau(H)$.

When $a - m_1 + n_1 > 0$,

$$
f_3(x) = f_1(x) - f_2(x) = ax(a - m_1 + n_1)(x - m_1 - m_2 - n_1 - n_2) = f_1(\tau(f_2(x))) < 0
$$

so $\tau(f_1(x)) > \tau(f_2(x)), \tau(G) > \tau(H)$.

When $a - m_1 + n_1 = 0$,

$$
f_3(x) = f_1(x) - f_2(x) = ax(a - m_1 + n_1)(x - m_1 - m_2 - n_1 - n_2) = 0
$$

so $\tau(f_1(x)) = \tau(f_2(x)), \tau(G) = \tau(H)$.

3. Complete Graph

The induction sub-graphs of U_i, V_i $(i = 1,2; j = 1,2)$ studied in the second section are all independent sets, namely 0-regular graphs. Based on the second section, this section continues to study the situation U_1, U_2, V_1, V_2 that are all complete graphs. For convenience, this paper write this graphas $D(m_1, m_2; n_1, n_2)$.

Lemma 3.1. Let $G = D(1,1; n_1, 1), H = D(a+1, 1; n_1 - a, 1)$, $\tau(G) > \tau(H)$ will be constant when $n > a + 2$ and $n \geq 8$.

Proof: Since U_1, U_2, V_1, V_2 is an equitable division of *G*, Q_1 and Q_2 are a quotient matrix of *Q*(*G*) and *Q*(*H*)

$$
Q_1 = \begin{pmatrix} n_1 + 1 & 0 & n_1 & 1 \\ 0 & n_1 & n_1 & 0 \\ 1 & 1 & 2n_1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{7}
$$

$$
Q_2 = \begin{pmatrix} a + n_1 + 1 & 0 & n_1 - a & 1 \\ 0 & n_1 - a & n_1 - a & 0 \\ a + 1 & 1 & 2n_1 - a & 0 \\ a + 1 & 0 & 0 & a + 1 \end{pmatrix} \tag{8}
$$

The characteristic polynomials of Q_1 and Q_2 are obtained by calculation as follows:

$$\begin{aligned} f_1(x) &= x^4 - (4n_1+2)x^3 + (5n_1^2+5n_1)x^2 \\ &\quad + (-2n_1^3 - 5n_1^2 + 3n_1)x + 2n_1^3 - 2n_1^2 \end{aligned} \tag{9}$$

$$\begin{aligned} f_2(x) &= x^4 + (a-4n_1-2)x^3 + (5n_1 - 2a - 3an_1 + 5n_1^2)x^2 \\ &\quad + (-3a^2 + 2an_1^2 + 7an_1 - 3a - 2n_1^3 - 5n_1^2 + 3n_1)x \\ &\quad - a^3 - a^2 - 4an_1^2 + 3an_1 + 2n_1^3 - 2n_1^2 \end{aligned} \tag{10}$$

$$\begin{aligned} f_3(x) &= f_1(x) - f_2(x) \\ &= -ax^3 + (2a+3an_1)x^2 + (3a^2 - 2an_1^2 - 7an_1 + 3a)x \\ &\quad + a^3 - 3a^2n_1 + a^2 + 4an_1^2 - 3an_1 \end{aligned} \tag{11}$$

First of all need to prove $\tau(f_3) < 2n_1$, because

$$f_3(x) = a\left[-x^3 + (2+3n_1)x^2 + (3a - 2n_1^2 - 7n_1 + 3)x + a^2 - 3an_1 + a + 4n_1^2 - 3n_1\right]$$

Set

$$g_1(x) = -x^3 + (2+3n_1)x^2 + (3a - 2n_1^2 - 7n_1 + 3)x + a^2 - 3an_1 + a + 4n_1^2 - 3n_1 \tag{12}$$

There is a common factor constant a in $f_3(x)$, and a does not affect the root of $f_3(x)$. Therefore, for the convenience of discussion, we study the polynomial $g_1(x)$, $g_1(x)$ after elimination of a is equal to the root of $f_3(x)$, because $a^2 - 3an_1 + a + 4n_1^2 - 3n_1$ can be viewed as a unary quadratic equation about a, what's more $\Delta = b^2 - 4ac = -7n_1^2 + 1 + 6n_1 < 0$, so $a^2 - 3an_1 + a + 4n_1^2 - 3n_1 > 0$ is constant. Let

$$g_2(x) = -x^3 + (2+3n_1)x^2 + (3a - 2n_1^2 - 7n_1 + 3)x \tag{13}$$

the root of $g_2(x)$ are

$$x_1 = 0, x_2 = \frac{3n_1 + 2 - \sqrt{\Delta}}{2}, x_3 = \frac{3n_1 + 2 + \sqrt{\Delta}}{2}, \text{ and } \Delta = \sqrt{(n_1^2 - 8) + 12a - 48}$$

Because $n_1 - 8 < \Delta = \sqrt{(n_1^2 - 8) + 12a - 48} < \sqrt{n_1^2 - 8n_1 + 16} = n_1 - 2$, So the range of the largest root of $g_2(x)$ is $(2n_1 - 3, 2n_1)$. Moreover $a^2 - 3an_1 + a + 4n_1^2 - 3n_1 > 0$, so we assume the largest root of $g_1(x)$ is $x_3 + k, (k > 0)$.

Next proof $\tau(g_1(x)) = x_3 + k < 2n_1$. We know

$$g_2(x)' = -3x^2 + (4 + 6n_1)x + 3a - 2n_1^2 - 7n_1 + 3$$

and

$$\Delta = 12n_1^2 - 36n_1 + 52 + 36a > 0$$

by calculation the root of $g_2'(x)$ is $\dfrac{6n_1 + 4 \pm \sqrt{12n_1^2 - 36n_1 + 52 + 36a}}{6} < 2n_1$, so $g_1(2n)$ is monotonically decreasing in

$$\left(\frac{6n_1+4+\sqrt{12n_1^2-36n_1+52+36a}}{6},2n_1\right)$$

and since $g_1(2n)<0$ when $n\geq a+2$, so $\tau(g_1(x))=x_3+k<2n_1$, then $\tau(f_3)=x_3+k<2n_1$ proofed.

Finally proof $\tau(f_1(x))>2n_1$, $\tau(f_2(x))>2n_1$, by calculation

$$f_1(x)=-4n_1^3+4n_1^2<0$$

and

$$f_2(x)=-4n_1^3+(2a+4)n_1^2-(3a^2-3a)n-a^3-a<0$$

when $x=2n_1$, so $\tau(f_1(x))>2n_1$, $\tau(f_2(x))>2n_1$.

Because $f_3(x)=f_1(x)-f_2(x)$, and $\tau(f_2(x))$ is the largest root of the characteristic polynomial $f_2(x)$. There have

$$f_3(\tau(f_2(x)))=f_1(\tau(f_2(x)))-f_2(\tau(f_2(x)))=f_1(\tau(f_2(x)))<0$$

so $\tau(f_1(x))>\tau(f_2(x))$.

Lemma 3.2. Let $G=D(1,1;n,n)$, $H=D(a+1,a+1;n-a,n-a)$, then $\tau(G)<\tau(H)$ is constant.

Proof: Since U_1,U_2,V_1,V_2 is an equitable division of *G*, Q_1 and Q_2 are a quotient matrix of *Q*(*G*) and *Q*(*H*)

$$Q_1=\begin{pmatrix}2n & 0 & n & n\\ 0 & n & n & 0\\ 1 & 1 & 2n & 0\\ 1 & 0 & 0 & 2n-1\end{pmatrix} \tag{14}$$

$$Q_2=\begin{pmatrix}a+2n & 0 & n-a & n\\ 0 & n-a & n-a & 0\\ a+1 & 1 & 2n-a & 0\\ a+1 & 0 & 0 & 2n+a-1\end{pmatrix} \tag{15}$$

The characteristic polynomials $g_1(x)$, $g_2(x)$ of Q_1 and Q_2 are obtained by calculation as follows:

$$\begin{aligned}g_1(x)=&\,x^4+(1-7n)x^3+(18n^2-8n)x^2\\&+(-20n^3+18n^2-2n)x+8n^4-12n^3+4n^2\end{aligned} \tag{16}$$

$$\begin{aligned}g_2(x)=&\,x^4+(1-7n)x^3+(-a^2-3an+3a+18n^2-8n)x^2\\&+(2a^2n-2a^2+2a-20n^3+18n^2-2n)x\\&+12an^2-4an-8an^3+8n^4-12n^3+4n^2\end{aligned} \tag{17}$$

let

$$h(x)=g_1(x)-g_2(x),$$

$$\begin{aligned}h(x)=&\,(3a-3an+a^2)x^2+(2a^2n-2a^2+10an^2-12an+2a)x\\&-8an^3+12an^2-4an\end{aligned} \tag{18}$$

because $3a-3an+a^2>0$, we obtain that $h(x)$ is an open up quadratic function, and $\Delta=4a^2(n-1)(a^2n-a^2+2an^2-8an+2a+n^3+n^2-n-1)>0$, because of $4a^2(n-1)>0$, we let $\Delta_1=a^2n-a^2+2an^2-8an+2a+n^3+n^2-n-1$, and get the derivative for *a* and *n*, $\Delta_1'(a)=(2n-2)a-8n+2n^2+2>0$, $\Delta_1'(n)=a^2+4an-8a+2n+3n^2-1$, so $\Delta>0$. Let the two roots be x_1 and x_2 respectively, and $x_1<x_2$. By the root formula $x=\frac{-b\pm\sqrt{\Delta}}{2a}$, $\Delta<2a^2(n-1)$, $x_2<\frac{-b+2a^2(n-1)}{2a}$ both can be calculated. x_2 is monotonically increasing with respect to a, so $x_2<2n-1$. When $a=1$, then

$$x_2=\frac{5(n^2-n)+\sqrt{n(n-1)(n^2+3n-8)}}{3n-2}$$

the four roots of $g_1(x)$ are $2n\pm\sqrt{2n}$, $2n$, $n-1$ and $x_2>2n-\sqrt{2n}$. When $x>2n-1$, then $g_1(x)>g_2(x)$. when $x<2n-1$, then $g_1(x)$, $g_2(x)$ have two intersections, and $g_1(x)$ haven't gotten to the maximum yet, so $\tau(G)<\tau(H)$ is constant.

4. Conclusion

It can be seen from the above conclusions, the structure of the graph $G=D_0(m_1,m_2;n_1,n_2)$ is $D_0(m_1+n_1-1,m_2;1,n_2)$ when the signless laplacian spectral radius of G has reached maximum. Similarly, the structure of the graph $G=D(1,1;n_1,1)$ is $D(1,1;n_1,1)$, when the signless laplacian spectral radius of G has reached maximum, and the structure of the graph $G=D(1,1;n,n)$ is $D(n,n;1,1)$, when the signless laplacian spectral radius of G has reached maximum.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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