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Is $-A^{-1}$ an Infinitesimal Generator?

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Abstract

There are some researchers considering the problem whether A^{-1} is the generator of a bounded C_0-semigroup if A generates a bounded C_0-semigroup. Actually, it is a very basic and important problem. In this paper, we discuss whether $-A^{-1}$ is the generator of a bounded α-times resolvent family if $-A$ generates a bounded α-times resolvent family.

Keywords

α-Times Resolvent Family, Analytic α-Times Resolvent Family, Fractional Power of Generator

1. Introduction

In paper [1], the author studies the problem whether A^{-1} is the generator of a bounded C_0-semigroup if A generates a bounded C_0-semigroup. We know that α-times resolvent operator family is generalization of C_0-semigroup and C_0-semigroup is 1-times resolvent operator family. So, in this paper, we will show that when the operator $-A$ generates a bounded α-times resolvent operator family, under certain condition, $-A^{-1}$ is also the generator of a bounded α-times resolvent operator family. The representation of such bounded α-times resolvent operator family will be obtained, too. Furthermore, we will consider the problem whether $-A^{-b}$ owns this property.

Let us first recall the definitions of α-times resolvent operator family. Let A be a closed densely defined linear operator on a Banach space X and $\alpha \in (0,2]$.

$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+1)}$ is a Mittag-Leffler function.

Definition 1.1 [2] *A family* $S_\alpha(t) \subset B(X)$ *is called an* α*-times resolvent operator family for A if the following conditions are satisfied*:

1) $S_\alpha(t)$ is strongly continuous for $t \ge 0$ and $S_\alpha(0) = I$;

2) $S_\alpha(t)D(A)\subset D(A)$ and $AS_\alpha(t)x=S_\alpha(t)Ax$ for $x\in D(A)$ and $t\ge 0$;

3) For $x\in D(A)$, $S_\alpha(t)x$ satisfies

$$S_\alpha(t)x = x+\int_0^t g_\alpha(t-s)S_\alpha(s)Ax\mathrm{d}s,\ t\ge 0,$$

where $g_\alpha(t):=\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t>0$.

If $\|S_\alpha(t)\|\le M_A\mathrm{e}^{\omega_A t}$ where $M_A\ge 1,\omega_A\ge 0$, we write as $A\in C^\alpha(M_A,\omega_A)$ (or shortly $A\in C^\alpha$). Then we give the definitions of analytic α-times resolvent operator family.

Definition 1.2 [2] *An α-times resolvent family $S_\alpha(\cdot,A)$ is called analytic if $S_\alpha(\cdot,A)$ admits an analytic extension to a sector $\Sigma_{\theta_0}\setminus\{0\}$ for some $\theta_0\in(0,\pi/2]$, where $\Sigma_{\theta_0}:=\{\lambda\in\mathbb{C}:|\arg\lambda|<\theta_0\}$. An analytic solution operator is said to be of analyticity type (θ_0,ω_0) if for each $\theta<\theta_0$ and $\omega>\omega_0$, there is $M=M(\theta,\omega)$ such that $\|S_\alpha(z,A)\|\le M\mathrm{e}^{\omega Re z}, z\in\Sigma_\theta$.*

Then we give a Lemma which will be used later.

Lemma 1.1 [2] *$0\le\alpha\le 2$. Then $A\in C^\alpha(M_A,\omega_A)$ if and only if $(\omega_A^\alpha,\infty)\subset\rho(A)$ and there is a strongly continuous operator-valued function $S(t)$ satisfying $\|S(t)\|\le M_A\mathrm{e}^{\omega_A t}$, $t\ge 0$, and such that*

$$\lambda^{\alpha-1}R(\lambda^\alpha,A)x=\int_0^\infty \mathrm{e}^{-\lambda t}S(t)x\mathrm{d}t,\ \lambda>\omega_A,\ x\in X.$$

2. Main Theorem and Conclusion

Theorem 2.1. *On a Hilbert space H, the following statements are equivalent:*

(1) $A\in C^\alpha(1,0)$, $\alpha\in(0,2]$;

(2) A is a closed, densely defined operator, $(0,\infty)\subset\rho(A)$, and for $\forall\lambda>0$, $\|\lambda^\alpha R(\lambda^\alpha,A)\|\le 1$;

(3) A is a closed, densely defined operator, for $\forall x\in D(A)$, $Re(Ax,x)\le 0$ and $\lambda^\alpha I-A$ is invertible for some $\lambda>0$.

Proof. (2) $\Rightarrow$ (3) For $\forall\lambda>0$, $\|\lambda^\alpha R(\lambda^\alpha,A)\|\le 1$, then we have for $\forall x\in D(A)$,

$$\|(\lambda^\alpha-A)x\|\ge\lambda^\alpha\|x\|, \tag{1}$$

hence we know $\lambda^\alpha-A$ is invertible from the proposition 1.5 of chapter 3 in book [3]. While, from equation (1), we can also have for $\forall x\in D(A)$, $(\lambda^\alpha x-Ax,x)\ge\lambda^\alpha(x,x)$, then $Re(Ax,x)\le 0$.

(3) $\Rightarrow$ (2) Since $\forall x\in D(A)$, $Re(Ax,x)\le 0$, then for $\forall\lambda>0$, $(\lambda^\alpha x-Ax,x)\ge\lambda^\alpha(x,x)$. $\lambda^\alpha I-A$ is invertible for some $\lambda>0$ imply that $\lambda^\alpha I-A$ is invertible for any $\lambda>0$. Together with A is closed and densely defined, we have $(x,x)\ge(\lambda^\alpha R(\lambda^\alpha,A)x,x)$, hence $(0,\infty)\subset\rho(A)$ and $\|\lambda^\alpha R(\lambda^\alpha,A)\|\le 1$.

(1) $\Rightarrow$ (2) From lemma 1.3 of [4], we know that A is a closed, densely defined operator. And we can get the other conclusion from theorem 2.8 of [2].

(2) $\Rightarrow$ (1). Firstly, set $A_n := n^{\alpha} AR(n^{\alpha}, A) = n^{2\alpha} R(n^{\alpha}, A) - n^{\alpha} I$. For every $n \in N$, A_n is a bounded operator and can commute with one another. It follows from Theorem 2.5 of [2] that A_n generates an α-times resolvent family $S_{n,\alpha}(t) := E_{\alpha}(A_n t^{\alpha})$ which is also uniformly continuous and exponential bounded.

For $\forall n \in N, \forall x \in D(A)$, $Re(A_n x, x) = Re(n^{\alpha} AR(n^{\alpha}, A)x, x)$. There exists a $y \in D(A)$, such that $R(n^{\alpha}, A)x = y$, that is $x = (n^{\alpha} - A)y$. Then

$$\begin{aligned} Re(A_n x, x) &= Re(n^{\alpha} Ay, (n^{\alpha} - A)y) \\ &= Re(n^{\alpha} Ay, n^{\alpha} y) - Re(n^{\alpha} Ay, Ay) \\ &= n^{2\alpha} Re(Ay, y) - n^{\alpha} Re(Ay, Ay) \end{aligned}$$

Since $Re(Ay, y) \le 0$ and $Re(Ay, Ay) \ge 0$, then we have that $Re(A_n x, x) \le 0$. It means that for $\forall \lambda > 0$, $\left\| \lambda^{\alpha} R(\lambda^{\alpha}, A_n) \right\| \le 1$. Consequently, $A_n \in \mathcal{C}^{\alpha}(1,0)$, and $\left\| S_{n,\alpha}(t) \right\| \le 1$.

From Lemma II, 3.4(ii) of [5], we have that A_n converges to A pointwise on $D(A)$. If we can get the following properties, we will have $A \in \mathcal{C}^{\alpha}(1,0)$.

(a) $S_{\alpha}(t)x := \lim_{n\to\infty} S_{n,\alpha}(t)x$ (*) exists for $\forall x \in H$;

(b) $S_{\alpha}(t)$ is an α-times resolvent family which is generated by A;

(c) $M = 1, \omega = 0$.

(a) For $S_{n,\alpha}(t)$ is bounded, we can only need to prove (*) on $D(A)$. For $\forall x \in D(A)$,

$$S_{n,\alpha}(t)x = x + \int_0^t P_{n,\alpha}(\tau) Ax \mathrm{d}\tau,$$

where $P_{n,\alpha}(t) = \int_0^t g_{\alpha-1}(t-s) S_{n,\alpha}(s) \mathrm{d}s$ ((2.52) and (2.53) of [2]). Together with $\left\| S_{n,\alpha}(t) \right\| \le 1$, we can get that $\left\| P_{n,\alpha}(t) \right\| \le g_{\alpha}(t)$. Thus for $\forall m, n \in N$,

$$\left\| S_{n,\alpha}(t)x - S_{m,\alpha}(t)x \right\| \le \int_0^t g_{\alpha}(\tau) \mathrm{d}\tau \left\| A_n x - A_m x \right\| = g_{\alpha+1}(t) \left\| A_n x - A_m x \right\|$$

By Lemma II, 3.4(ii) of [5], $\{A_n x\}_{n\in N}$ is a Cauchy sequence for each $x \in D(A)$. Therefore $\{S_{n,\alpha}(t)x\}_{n\in N}$ converges uniformly on each interval $[0, t_0]$.

(b) I. For $\forall x \in D(A)$, $S_{\alpha}(t)$ is the uniformly continuous functions and so is continuous itself. For each $n \in N$, $S_{n,\alpha}(t)$ is uniformly bounded on every interval $[0, t_0]$ and $S_{n,\alpha}(0) = I$, then so is $S_{\alpha}(t)$. By Lemma I, 5.2 of [5], $S_{\alpha}(t)$ is strongly continuous and $S_{\alpha}(0) = I$.

II. For $\forall n \in N$, $\forall x \in D(A)$, $S_{n,\alpha}(t)x \in D(A)$ and $S_{n,\alpha}(t)x \to S_{\alpha}(t)x$. Together with that A is an closed operator, we have that $S_{\alpha}(t)x \in D(A)$. That is $S_{\alpha}(t)D(A) \subset D(A)$.

We have $A_n S_{n,\alpha}(t)x = S_{n,\alpha} A_n x$, A_n and $S_{n,\alpha}(t)$ converge to A and $S_{\alpha}(t)$ pointwise, respectively. So, we have $AS_{\alpha}(t)x = S_{\alpha} Ax$.

III. We know that

$$S_{n,\alpha}(t)x = x + \int_0^t g_\alpha(t-s)S_{n,\alpha}(s)A_n x\mathrm{d}s.$$

And for $\forall x \in D(A)$, $g_\alpha(t)S_{n,\alpha}(t)A_n x$ converges uniformly on the interval $[0,t]$, then

$$\begin{aligned} S_\alpha(t)x &= \lim_{n\to\infty} S_{n,\alpha}(t)x = x + \lim_{n\to\infty}\int_0^t g_\alpha(t-s)S_{n,\alpha}(s)A_n x\mathrm{d}s \\ &= x + \int_0^t g_\alpha(t-s)\lim_{n\to\infty} S_{n,\alpha}(s)A_n x\mathrm{d}s \\ &= x + \int_0^t g_\alpha(t-s)S_\alpha(s)Ax\mathrm{d}s. \end{aligned}$$

For all the above, we can obtain that $S_\alpha(t)$ is an α-times resolvent family which is generated by A.

(c) For each $n \in N$, $\|S_{n,\alpha}(t)\| \le 1$ and $S_{n,\alpha}(t)$ converges to $S_\alpha(t)$ pointwise, so $\|S_\alpha(t)\| \le 1$, too. That is $M = 1, \omega = 0$.

To sum up the above (a), (b) and (c), we can conclude that $A \in \mathcal{C}^\alpha(1,0)$.

Theorem 2.2. $A \in \mathcal{C}^\alpha(1,0)$, $\alpha \in (0,1]$ $\Leftrightarrow$ *A is a closed, densely defined operator,* $(0,\infty) \subset \rho(A)$, *and for* $\forall \lambda > 0$, $\|\lambda^\alpha R(\lambda^\alpha, A)\| \le 1$.

Proof. In the proof of the previous theorem, we have only used the properties of Hilbert space in the acquisition of $\|S_{n,\alpha}(t)\| \le 1$ and we can get $\|S_{n,\alpha}(t)\| \le 1$ without the properties of Hilbert space.

In fact, on a Banach space, for each $n \in N$, A_n can generate a C_0-semigroup $T_n(t)$. Moreover,

$$\|T_n(t)\| = \left\|\mathrm{e}^{A_n t}\right\| = \left\|\mathrm{e}^{n^{2\alpha}R(n^\alpha, A)t - n^\alpha It}\right\| \le 1.$$

From the subordination principle, we have $S_{n,\alpha}(t) = \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})T_n(s)\mathrm{d}s$, where $\Phi_\gamma(z) := \sum_0^\infty \frac{(-z)^n}{n!\Gamma(-\gamma n - \gamma + 1)} = \frac{1}{2\pi i}\int_\Gamma \mu^{\gamma-1}\exp(\mu - z\mu^\gamma)\mathrm{d}\mu, 0 < \gamma < 1$. So

$$\|S_{n,\alpha}(t)\| \le \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})\|T_n(s)\|\mathrm{d}s = \int_0^\infty \Phi_\alpha(st^{-\alpha})\mathrm{d}(st^{-\alpha}) = 1.$$

We can obtain that this theorem is tenable from the proof of the previous theorem.

Theorem 2.3. *On a Hilbert space H, if* $-A \in \mathcal{C}^\alpha(1,0)$, $\alpha \in (0,2]$ *and* $-A^{-1}$ *exists as a closed, densely defined operator, then* $-A^{-1} \in \mathcal{C}^\alpha(1,0)$.

Proof. From the above Theorem 2.1, we have $(0,\infty) \subset \rho(-A)$, and for $\forall x \in D(-A)$, $Re(-Ax, x) \le 0$. And $-A^{-1}$ exists as a closed, densely defined operator, so it is easy to show that $\lambda^\alpha I + A^{-1}$ is bounded invertible for some $\lambda > 0$, from (8) of [1]. Further more, for $\forall y \in D(-A^{-1})$, then $y \in R(-A)$, thus there exists an $x \in D(-A)$, such that $-Ax = y$. Then $Re(-A^{-1}y, y) = Re(x, -Ax) \le 0$. By Theorem 2.1, we can obtain that $-A^{-1} \in \mathcal{C}^\alpha$.

Theorem 2.4. *If* $-A \in \mathcal{C}^\alpha(0)$, $\alpha \in (0,2]$, $S_\alpha(t)$ *is the α-times resolvent family generated by it and* $\|S_\alpha(t)\| = O\left(t^{-\frac{1}{4}-\varepsilon}\right), t \to \infty, \forall \varepsilon > 0$. *And if* $-A^{-1}$ *exists as a closed, densely defined operator, then* $-A^{-1}$ *generates an α-times resolvent family* $S_{\alpha_-}(t)$, *which is given by*

$$S_{\alpha_-}(t)x = x - \int_0^\infty t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x\mathrm{d}\tau, \forall x \in X,$$

where J_1 is the first order Bessel function [6]. Moreover, there exists an $M > 0$, such that $\left\|S_{\alpha_-}(t)\right\| \le 1 + Mt^{\frac{1}{4}}, \forall t \ge 0$.

Proof. Since $-A \in C^\alpha(0)$, then $\{\lambda^\alpha : Re\lambda > 0\} \subset \rho(-A)$. Together with the assumption that $-A^{-1}$ is a closed, densely defined operator, we have that $\{\lambda^\alpha : Re\lambda > 0\} \subset \rho(-A^{-1})$. Because of the property of Bessel function $J_1(t) \approx \frac{1}{\sqrt{\pi}} t^{-\frac{1}{4}} \cos\left(2\sqrt{t} - \frac{3}{4}\pi\right)$ and for large t, $\left\|S_\alpha(t)\right\| = O\left(t^{-\frac{1}{4}-\varepsilon}\right), t \to \infty$, then

$$\begin{aligned}
&\int_0^\infty \left\| t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x \right\| \mathrm{d}\tau \\
&= \int_0^1 \left\| t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x \right\| \mathrm{d}\tau + \int_1^\infty \left\| t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x \right\| \mathrm{d}\tau \\
&\le \int_0^1 t\frac{1}{\sqrt{t\tau}} \frac{M'}{(t\tau)^{\frac{1}{4}}} M_A \|x\| \mathrm{d}\tau + \int_1^\infty t\frac{1}{\sqrt{t\tau}} \frac{M'}{(t\tau)^{\frac{1}{4}}} \frac{M''}{\tau^{\frac{1}{4}+\varepsilon}} \|x\| \mathrm{d}\tau \\
&= M'M_A t^{\frac{1}{4}} \|x\| \int_0^1 \tau^{-\frac{3}{4}} \mathrm{d}\tau + M'M'' t^{\frac{1}{4}} \|x\| \int_1^\infty \tau^{-(1+\varepsilon)} \mathrm{d}\tau \\
&= 4M'M_A t^{\frac{1}{4}} \|x\| + \frac{M'M''}{\varepsilon} t^{\frac{1}{4}} \|x\| := Mt^{\frac{1}{4}} \|x\|
\end{aligned}$$

Thus, the integral is well defined. Set $S(t)x := x - \int_0^\infty t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x\mathrm{d}\tau, \forall x \in X$. Obviously, $S(t)$ is strongly continuous and $S(0) = I$. From the above discussion, we can get that $\left\|S(t)\right\| \le 1 + Mt^{\frac{1}{4}}$. For $\forall \lambda, Re\lambda > 0$,

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\lambda t} S(t)x\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-\lambda t} \left[x - \int_0^\infty t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x\mathrm{d}\tau \right] \mathrm{d}t \\
&= \lambda^{\alpha-1} R\left(\lambda^\alpha, -A^{-1}\right)x.
\end{aligned}$$

Consequently, we can obtain a conclusion that $-A^{-1}$ generates an α-times resolvent family $S_{\alpha_-}(t)$ from Lemma 2.1 and

$$S_{\alpha_-}(t)x = x - \int_0^\infty t\frac{1}{\sqrt{t\tau}} J_1\left(2\sqrt{t\tau}\right) S_\alpha(\tau)x\mathrm{d}\tau,\ \left\|S_{\alpha_-}(t)\right\| \le 1 + Mt^{\frac{1}{4}}, \forall t \ge 0.$$

Theorem 2.5. *A satisfies the assumption of Theorem 2.4, for $\forall b, 0 < b < 1$, $-A^{-b}$ generates a bounded analytic α-times resolvent family $H_{b,\alpha_-}(t)$. If $\frac{1}{4\alpha} \le b < 1$, then*

$$H_{b,\alpha_-}(t)x = \int_0^\infty f_{\alpha,\alpha}^b(t,s) S_{\alpha_-}(s)x\mathrm{d}s, \forall t > 0,$$

where

$$f_{\alpha,\alpha}^b(t,s) = \frac{1}{2\pi i} \int_\Gamma E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} \mathrm{e}^{-(-\mu)^{\frac{1}{\alpha}}s} \mathrm{d}\mu,$$

and

$$\Gamma = \Gamma_1 \cup \Gamma_1' \cup \Gamma_2 \cup \Gamma_2' \cup \Gamma_3 \cup \Gamma_3'$$

is oriented counterclockwise, where

$$\Gamma_1 = \left\{\rho e^{i\omega} : \rho \ge t^{-\alpha/b}\right\}, \quad \Gamma_1' = \left\{\rho e^{-i\omega} : \rho \ge t^{-\alpha/b}\right\},$$

$$\Gamma_2 = \left\{t^{-\alpha/b} e^{i\theta} : \omega \le \theta \le \pi\right\}, \Gamma_2' = \left\{t^{-\alpha/b} e^{-i\theta} : \omega \le \theta \le \pi\right\},$$

$$\Gamma_3 = \left\{\rho e^{i\pi} : 0 \le \rho \le t^{-\alpha/b}\right\}, \Gamma_3' = \left\{\rho e^{-i\pi} : 0 \le \rho \le t^{-\alpha/b}\right\},$$

and $\omega \in \left(\pi - \frac{\alpha}{2}\pi, \frac{1}{b}\left(\pi - \frac{\alpha}{2}\pi\right)\right)$.

Proof. $-A \in \mathcal{C}^\alpha(0)$, so $A \in Sect\left(\pi - \frac{\alpha}{2}\pi\right)$ and $A^{-1} \in Sect\left(\pi - \frac{\alpha}{2}\pi\right)$. For $\forall b, 0 < b < 1$, $A^{-b} \in Sect(\varphi), \varphi < \pi - \frac{\alpha}{2}\pi$. It follows from the Remark 2.8(a) of [7] that $-A^{-b}$ generates a bounded analytic α-times resolvent family $H_{b,\alpha_-}(t)$. If $\frac{1}{4\alpha} \le b < 1$, we set

$$S(t)x = \int_0^\infty \frac{1}{2\pi i} \int_\Gamma E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{\frac{1}{\alpha}} s} d\mu S_{\alpha_-}(s) x ds.$$

Since $\left\|S_{\alpha_-}(t)\right\| \le 1 + Mt^{\frac{1}{4}}, \forall t \ge 0$, then

$$\begin{aligned}
\|S(t)\| &= \left\|\int_0^\infty \frac{1}{2\pi i} \int_\Gamma E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{\frac{1}{\alpha}} s} d\mu S_{\alpha_-}(s) ds\right\| \\
&\le \int_0^\infty \left|\frac{1}{2\pi i} \int_\Gamma E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{\frac{1}{\alpha}} s} d\mu\right| ds \\
&\quad + M\int_0^\infty \left|\frac{1}{2\pi i} \int_\Gamma E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{\frac{1}{\alpha}} s} d\mu\right| s^{\frac{1}{4}} ds \\
&:= I + II.
\end{aligned}$$

From [8], we have that there exists an M_1, such that $I \le M_1$. Next we estimate II, it follows from (2.4) of [8] that

$$\begin{aligned}
&\int_0^\infty \left|\frac{1}{2\pi i} \int_{\Gamma_1} E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{\frac{1}{\alpha}} s} d\mu\right| s^{\frac{1}{4}} ds \\
&\le \frac{1}{2\pi} \int_0^\infty \int_{t^{-\frac{\alpha}{b}}}^\infty \left|E_\alpha\left(-\rho^b t^\alpha e^{ib\omega}\right)\right| \rho^{\frac{1}{\alpha}-1} e^{-\rho^{\frac{1}{\alpha}} \cos\left(\frac{\pi-\omega}{\alpha}\right) s} d\rho s^{\frac{1}{4}} ds \\
&= \frac{\alpha}{2\pi} \int_0^\infty \int_{t^{-\frac{1}{b}}}^\infty \left|E_\alpha\left(-\sigma^{b\alpha} t^\alpha e^{ib\omega}\right)\right| e^{-\sigma \cos\left(\frac{\pi-\omega}{\alpha}\right) s} d\sigma s^{\frac{1}{4}} ds \\
&= \frac{\alpha}{2\pi} \int_{t^{-\frac{1}{b}}}^\infty \frac{\Gamma\left(\frac{5}{4}\right)}{\left[\sigma \cos\left(\frac{\pi-\omega}{\alpha}\right)\right]^{\frac{5}{4}}} \left|E_\alpha\left(-\sigma^{b\alpha} t^\alpha e^{ib\omega}\right)\right| d\sigma
\end{aligned}$$

$$\leq \frac{\alpha\Gamma\left(\frac{5}{4}\right)}{2\pi\cos^{\frac{5}{4}}\left(\frac{\pi-\omega}{\alpha}\right)}\int_{t^{-\frac{1}{b}}}^{\infty}\frac{C}{\sigma^{\frac{5}{4}}\left(1+\sigma^{b\alpha}t^{\alpha}\right)}\mathrm{d}\sigma$$
$$\leq \frac{\alpha\Gamma\left(\frac{5}{4}\right)}{2\pi\cos^{\frac{5}{4}}\left(\frac{\pi-\omega}{\alpha}\right)}\int_{t^{-\frac{1}{b}}}^{\infty}\frac{C}{\sigma^{b\alpha+\frac{5}{4}}}\mathrm{d}\sigma\frac{1}{t^{\alpha}}$$
$$= \frac{2\alpha\Gamma\left(\frac{5}{4}\right)}{\pi\cos^{\frac{5}{4}}\left(\frac{\pi-\omega}{\alpha}\right)(4b\alpha+1)}t^{\frac{1}{4b}}$$

The same estimate holds for the integral on Γ_1'.

$$\int_0^{\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_2}E_{\alpha}\left(-\mu^b t^{\alpha}\right)(-\mu)^{\frac{1}{\alpha}-1}\mathrm{e}^{-(-\mu)^{\frac{1}{\alpha}}s}\mathrm{d}\mu\right|s^{\frac{1}{4}}\mathrm{d}s$$
$$\leq \frac{1}{2\pi}\int_0^{\infty}\int_{\omega}^{\pi}\left|E_{\alpha}\left(-\mathrm{e}^{ib\theta}\right)\right|t^{-\frac{1}{b}}\mathrm{e}^{-t^{-\frac{1}{b}}\cos\left(\frac{\pi-\theta}{\alpha}\right)s}\mathrm{d}\theta s^{\frac{1}{4}}\mathrm{d}s$$
$$= \frac{\Gamma\left(\frac{5}{4}\right)t^{\frac{1}{4b}}}{2\pi}\int_{\omega}^{\pi}\frac{\left|E_{\alpha}\left(-\mathrm{e}^{ib\theta}\right)\right|}{\cos^{\frac{5}{4}}\left(\frac{\pi-\theta}{\alpha}\right)}\mathrm{d}\theta \leq \frac{\Gamma\left(\frac{5}{4}\right)E_{\alpha}(1)}{2\cos^{\frac{5}{4}}\left(\frac{\pi-\omega}{\alpha}\right)}t^{\frac{1}{4b}}$$

The same estimate holds for the integral on Γ_2'.

$$\int_0^{\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_3\cup\Gamma_3'}E_{\alpha}\left(-\mu^b t^{\alpha}\right)(-\mu)^{\frac{1}{\alpha}-1}\mathrm{e}^{-(-\mu)^{\frac{1}{\alpha}}s}\mathrm{d}\mu\right|s^{\frac{1}{4}}\mathrm{d}s$$
$$\leq \frac{1}{2\pi}\int_0^{\infty}\int_0^{t^{-\frac{\alpha}{b}}}\left|E_{\alpha}\left(-\rho^b t^{\alpha}\mathrm{e}^{-ib\pi}\right)-E_{\alpha}\left(-\rho^b t^{\alpha}\mathrm{e}^{ib\pi}\right)\right|\rho^{\frac{1}{\alpha}-1}\mathrm{e}^{-\rho^{\frac{1}{\alpha}}s}\mathrm{d}\rho s^{\frac{1}{4}}\mathrm{d}s$$
$$= \frac{\alpha}{2\pi}\int_0^{t^{-\frac{1}{b}}}\frac{\Gamma\left(\frac{5}{4}\right)}{\sigma^{\frac{5}{4}}}\left|E_{\alpha}\left(-\sigma^{b\alpha}t^{\alpha}\mathrm{e}^{-ib\pi}\right)-E_{\alpha}\left(-\sigma^{b\alpha}t^{\alpha}\mathrm{e}^{ib\pi}\right)\right|\mathrm{d}\sigma$$
$$\leq \frac{\alpha\Gamma\left(\frac{5}{4}\right)}{\pi}\int_0^{t^{-\frac{1}{b}}}\sigma^{-\frac{5}{4}}\sum_{k=1}^{\infty}\frac{\left(\sigma^{b\alpha}t^{\alpha}\right)^k}{\Gamma(k\alpha+1)}\mathrm{d}\sigma = \frac{\alpha\Gamma\left(\frac{5}{4}\right)}{\pi}\sum_{k=1}^{\infty}\frac{1}{\Gamma(k\alpha+1)\left(kb\alpha-\frac{1}{4}\right)}t^{\frac{1}{4b}}$$

To sum up, we can conclude that there exists an M_2, such that $II \leq M_2 t^{\frac{1}{4b}}$. So $\|S(t)\| \leq M_1 + M_2 t^{\frac{1}{4b}}$. Then we should show that $S(t)$ is strongly continuous at 0. It following from the dominated convergence Theorem and Fubini Theorem that

$$\lim_{t\to 0}S(t)x = \lim_{t\to 0}\int_0^{\infty}\frac{1}{2\pi i}\int_{\Gamma}E_{\alpha}\left(-\mu^b t^{\alpha}\right)(-\mu)^{\frac{1}{\alpha}-1}\mathrm{e}^{-(-\mu)^{\frac{1}{\alpha}}s}\mathrm{d}\mu S_{\alpha_-}(s)x\mathrm{d}s$$
$$= \int_0^{\infty}\frac{1}{2\pi i}\int_{\Gamma}(-\mu)^{\frac{1}{\alpha}-1}\mathrm{e}^{-(-\mu)^{\frac{1}{\alpha}}s}\mathrm{d}\mu S_{\alpha_-}(s)x\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{\Gamma}\left(A^{-1}-\mu\right)^{-1}x\mathrm{d}\mu = x$$

For $\forall \lambda, Re\lambda > 0$, it follows from Fubini Theorem that

$$\int_0^\infty e^{-\lambda t} S(t) x \mathrm{d}t = \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{1}{2\pi i} \int_\Gamma E_\alpha\left(-\mu^b t^\alpha\right)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{\frac{1}{\alpha}} s} \mathrm{d}\mu S_{\alpha_-}(s) x \mathrm{d}s \mathrm{d}t$$

$$= \frac{1}{2\pi i} \int_\Gamma (-\mu)^{\frac{1}{\alpha}-1} \int_0^\infty E_\alpha\left(-\mu^b t^\alpha\right) \mathrm{d}t \int_0^\infty e^{-(-\mu)^{\frac{1}{\alpha}} s} S_{\alpha_-}(s) x \mathrm{d}s \mathrm{d}\mu$$

$$= \frac{1}{2\pi i} \int_\Gamma \lambda^{\alpha-1}\left(\lambda^\alpha + \mu^b\right)^{-1}\left(A^{-1} - \mu\right)^{-1} x \mathrm{d}\mu = \lambda^{\alpha-1} R\left(\lambda^\alpha, -A^{-b}\right) x$$

From all the above, we can obtain a conclusion that if $\frac{1}{4\alpha} \le b < 1$, $-A^{-b}$ generates a bounded analytic α-times resolvent family $H_{b,\alpha_-}(t)$,

$$H_{b,\alpha_-}(t) x = \int_0^\infty f_{\alpha,\alpha}^b(t,s) S_{\alpha_-}(s) x \mathrm{d}s, \forall t > 0.$$

3. Conclusion

In this paper, we considered when the operator $-A$ generates a bounded α-times resolvent operator family, under certain condition, $-A^{-1}$ as well as $-A^{-b}$ is also the generator of a bounded α-times resolvent operator family. Through the study of the problem whether A^{-1} is the generator of a bounded α-times resolvent operator family if A generates a bounded α-times resolvent operator family, we can know the generator A more clearly. Furthermore, this work can improve the study of the inverse problem.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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