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The Traveling Wave Solutions of Space-Time Fractional Differential Equation Using Fractional Riccati Expansion Method

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Abstract

In this paper, we firstly give a counterexample to indicate that the chain rule $D_x^\alpha u = \sigma \frac{\mathrm{d}u}{\mathrm{d}\xi} D_x^\alpha \xi$ is lack of accuracy. After that, we put forward the fractional Riccati expansion method. No need to use the chain rule, we apply this method to fractional KdV-type and fractional Telegraph equations and obtain the tangent and cotangent functions solutions of these fractional equations for the first time.

Keywords

Conformable Fractional Derivative, The Chain Rule, Exact Solution, Fractional Riccati Expansion Method

1. Introduction

As we know that the global quasi-operator fractional-order derivative owns the properties of depending on history, and posses more advantages than the local operator integral-order in describing the memory and hereditary characteristic of different substances, the fractional-order derivative is usually used also by simulating the dynamic behavior of soft material, which is a kind of material between solid and fluid. Recently, the study on properties of fractional derivatives and fractional-order equations increasingly causes attention to many authors. Tarasov [1] [2] [3] investigated properties of the chain rule and Leibniz rule for fractional derivatives. Manuel and Tenreiro [4] analyzed the definitions of the Grnwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives.

Authors [5] [6] studied the exact solution of fractional equations by using

Jumarie's modified Riemann-Liouville derivative

$$D_x^{\alpha}u(x)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x(x-\tau)^{-\alpha}\left[u(\tau)-u(0)\right]\mathrm{d}\tau,(0<\alpha\le 1) \tag{1}$$

and the following two basic formulae:

$$D_x^{\alpha}\left[f(x)g(x)\right]=g(x)D_x^{\alpha}f(x)+f(x)D_x^{\alpha}g(x), \tag{2}$$

$$D_x^{\alpha}f(g(x))=f_g'\left[g(x)\right]D_x^{\alpha}g(x)=D_g^{\alpha}f(g(x))\left[g'(x)\right]^{\alpha}. \tag{3}$$

Liu [7] gave the two counterexamples and stated the opinion that Jumarie's two above formulae (2) and (3) are incorrect. Author's [8] developed the fractional complex transform

$$u(x,t)=U(\xi),\quad \xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\beta}}{\Gamma(1+\beta)},(0<\alpha,\beta\le 1) \tag{4}$$

and the following chain rule for fractional derivative

$$D_x^{\alpha}u=\sigma_1\frac{\mathrm{d}U}{\mathrm{d}\xi}D_x^{\alpha}\xi,\quad D_t^{\beta}u=\sigma_2\frac{\mathrm{d}U}{\mathrm{d}\xi}D_t^{\beta}\xi, \tag{5}$$

where σ_1,σ_2 are constants.

The above formulae (4) and (5) have the advantage of converting a fractional differential equation with Jumarie's modified of Riemann-Liouville derivative into its ordinary differential equations. Recently, many authors [9] [10] studied the exact traveling wave solutions of space-time fractional equations by using of formulae (4) and (5)and *G'/G*-expansion method, improved F-expansion, first integral method etc.

In here, we must point out that the constants σ_1,σ_2 are lack of accuracy, if the fractional transformation $U(\xi)$ contains only one term, formula (5) is correct. But $U(\xi)$ contains two terms and above, formula (5) is incorrect. We give the counterexample to show that σ of formula (5) does not exist and therefore the corresponding results reported in many literatures are not correct. Inspired by this, we present the new fractional Riccati expansion method. By this method, we firstly transform fractional partial differential equations into fractional ordinary equations with the same order by using traveling wave transformation. Then we can give the exact solutions of fractional ordinary equations using the solutions of fractional Riccati equation. In this process, the chain rule formulae (4) and (5) do not need to be used.

The content is as follows: in Section 2, one counterexample is given to show that the formula (5) is not true. The properties and definition of conformable fractional derivative are listed, steps of the fractional Riccati expansion method are presented. In Section 3, the proposed method is applied to solve space-time fractional differential KdV-type and Telegraph equations, exact solitary wave solutions can be obtained. Furthermore, the method can be used to obtain exact solutions of many other fractional equations without formulae (4) and (5). The brief conclusions are arranged in Section 4.

2. Counterexample and the Fractional Riccati Expansion Method

2.1. Counterexample

In some literatures, authors [9] [10] reduced the fractional differential equations to the ordinary differential equations by using formulae (4) and (5), with the help of solving ordinary differential equations, the traveling wave solutions of fractional differential equations can be obtained. Inhere we must point out formula (5) is incorrect. Certainly, results obtained by using (5) and (4) are lack of accuracy. In the following content, we give one counterexample to show that formula (5) is not true.

Counterexample Takes $f(g(x)) = g^{\frac{1}{2}} + g^{\frac{1}{4}}, g(x) = x^2, \alpha = \frac{1}{2}$, the left side of the first formula to expression (5) is denoted by

$$\begin{aligned} D_x^{\frac{1}{2}} f(x) &= D_x^{\frac{1}{2}}\left[x + x^{\frac{1}{2}}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\tau)^{-\frac{1}{2}}\left[\tau + \tau^{\frac{1}{2}}\right]\mathrm{d}\tau \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\tau)^{-\frac{1}{2}}\tau\,\mathrm{d}\tau + \frac{1}{\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\tau)^{-\frac{1}{2}}\tau^{\frac{1}{2}}\mathrm{d}\tau \\ &= 2\sqrt{\frac{t}{\pi}} + \frac{\sqrt{\pi}}{2}. \end{aligned} \tag{6}$$

But from

$$\frac{\mathrm{d}f}{\mathrm{d}g} = \frac{1}{2}g^{-\frac{1}{2}} + \frac{1}{4}g^{-\frac{3}{4}} \tag{7}$$

and

$$D_x^{\frac{1}{2}} g(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-\tau)^{-\frac{1}{2}}\tau^2\,\mathrm{d}\tau = \frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}}, \tag{8}$$

we know that the right side of first formula to expression (5) equals to

$$\frac{\mathrm{d}f}{\mathrm{d}g}\cdot D_x^{\frac{1}{2}} g(x) = \left[\frac{1}{2}g^{-\frac{1}{2}} + \frac{1}{4}g^{-\frac{3}{4}}\right]\cdot\frac{8}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} = \frac{4}{3}\sqrt{\frac{t}{\pi}} + \frac{4}{3\pi}\frac{\sqrt{\pi}}{2}. \tag{9}$$

Comparing (6) with (9), we find that the “ σ “ of formula (5) doesn’t exist. That is to say, we can not give a constant “ σ “ such that

$$D_x^{\frac{1}{2}} f(x) = \sigma\frac{\mathrm{d}f}{\mathrm{d}g}\cdot D_x^{\frac{1}{2}} g(x) \tag{10}$$

holds. It is obviously that formula (5) is right for the compound functions containing one term, such as $f(g(x)) = g^{\frac{1}{2}}, g(x) = x^2$.

2.2. Definition and Properties of Fractional Derivative

The fractional derivative is described in the sense of the following “conformable fractional derivative" defined by Khalil [11] as

Definition 2.1. For a given function $f:[0,\infty)\to\Re$, the "conformable fractional derivative" of *f* with α order is denoted by

$$T_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \tag{11}$$

for all $t>0,\alpha\in(0,1]$.

We known that Riemann-Liouville derivative and Caputo derivative have the following setbacks:

(1) they do not satisfy $D_a^\alpha(1)=0$,

(2) they do not satisfy formulae (2) and (3).

However, The conformable fractional derivative makes up these setbacks and it satisfies the known formulae (2), (3) and Rolle's Theorem. For $\alpha\in(0,1],b,c\in\Re$, the conformable fractional derivative of some functions are listed as the following:

(1) $T_\alpha\left(\mathrm{e}^{cx}\right)=cx^{1-\alpha}\mathrm{e}^{cx}$;

(2) $T_\alpha\left(-x^{-\alpha}\right)=\dfrac{\alpha}{x^{2\alpha}}$;

$T_\alpha(\sinh(bx))=bx^{1-\alpha}\cosh(bx)$, $T_\alpha(\cosh(bx))=bx^{1-\alpha}\sinh(bx)$,

(3) $T_\alpha(\operatorname{sech}(bx))=-bx^{1-\alpha}\operatorname{sech}(bx)\tanh(bx)$,

$T_\alpha(\tanh(bx))=bx^{1-\alpha}\operatorname{sech}^2(bx)$, $T_\alpha(\coth(bx))=-bx^{1-\alpha}\operatorname{csch}^2(bx)$.

The above properties play a very important role in the fractional Riccati expansion method. Khalil [11] gave proofs of (1) and (2). After simple calculating, We know that (3) are true by (1), (2) and the definitions of hyperbolic functions.

Proposition 2.2. For $\alpha\in(0,1]$, f,g be conformable fractional derivative, $f(g)$ is differential at point *g*, *g* is continuous at point *x*, then the chain rule

$$T_\alpha f(g)(x)=\frac{\mathrm{d}f}{\mathrm{d}g}\cdot T_\alpha g(x) \tag{12}$$

holds, where "$\dfrac{\mathrm{d}}{\mathrm{d}g}$" denote the derivative of integer-order with respect to *g*.

Proof By $f(g)(x)$ is differential at point x and the definition (2.1) of conformable fractional derivative, then

$$\begin{aligned}T_\alpha f(g)(x)&=\lim_{\varepsilon\to 0}\frac{f(g)\left(x+\varepsilon x^{1-\alpha}\right)-f(g)(x)}{\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{f(g)\left(x+\varepsilon x^{1-\alpha}\right)-f(g)(x)}{g\left(x+\varepsilon x^{1-\alpha}\right)-g(x)}\cdot\frac{g\left(x+\varepsilon x^{1-\alpha}\right)-g(x)}{\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{f(g)\left(x+\varepsilon x^{1-\alpha}\right)-f(g)(x)}{g\left(x+\varepsilon x^{1-\alpha}\right)-g(x)}\cdot\lim_{\varepsilon\to 0}\frac{g\left(x+\varepsilon x^{1-\alpha}\right)-g(x)}{\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{\Delta f}{\Delta g}\cdot\lim_{\varepsilon\to 0}\frac{g\left(x+\varepsilon x^{1-\alpha}\right)-g(x)}{\varepsilon}=\frac{\mathrm{d}f}{\mathrm{d}g}\cdot T_\alpha g(x).\end{aligned}$$

In the above process, the fourth equality holds because of continuity of function $g(x)$. Then the chain rule (12) holds.

2.3. Method of Fractional Riccati Expansion

For a given fractional differential equation, we write in two variables x and t as

$$P\left(u, T_\beta u(x), T_\beta u(t), T_{2\beta} u(x), T_{2\beta} u(t), \cdots\right)=0, \tag{13}$$

Inhere, $T_\beta u(x), T_\beta u(t)$ are comfortable derivatives of $u=u(x,t)$ with respect to x and t respectively, u is an unknown function, P is a polynomial.

Step 1: By using the traveling wave transformation

$$u(x,t)=u(\xi), \quad \xi=x+ct, \tag{14}$$

where c is constants to be determined. Fractional differential Equation (13) is reduced to the following nonlinear fractional ordinary differential equation of u with the same order:

$$P_1\left(u, T_\beta u(\xi), c^\beta T_\beta u(\xi), T_{2\beta} u(\xi), c^{2\beta} T_{2\beta} u(\xi), \cdots\right)=0, \tag{15}$$

Step 2: Suppose that $u(\xi)$ solution of Equation (15) can be expressed by the following form:

$$u(\xi)=\sum_{i=0}^{n} a_i F^i(\xi), \quad a_n \neq 0, \tag{16}$$

Inhere, $a_i\,(i=0,1,\cdots,n)$ are undetermined constants, n is a positive integer to be determined by balancing the nonlinear term and the linear term of the highest order in Equation (15), $F(\xi)$ satisfys the following fractional Riccati equation

$$T_\alpha F(x)=m+F^2, \quad (m<0, 0<\alpha \leq 1), \tag{17}$$

where m is parameter. By using the definition (2.1), proposition (2.2) of comfortable fractional derivative and the derivatives (3) of some functions, we can obtain the following solutions of Equation (17).

Theorem 2.3. For given $m<0, 0<\alpha \leq 1$, Equation (17) exist the solutions

$$F(x)=-\sqrt{-m}\tanh\left(\frac{2-\alpha}{\alpha}\sqrt{-m}\,x^{\frac{2-\alpha}{\alpha}}\right), \tag{18}$$

and

$$F(x)=-\sqrt{-m}\coth\left(\frac{2-\alpha}{\alpha}\sqrt{-m}\,x^{\frac{2-\alpha}{\alpha}}\right), \tag{19}$$

Proof: By using the proposition (2.2) of comfortable fractional derivative and the derivative (3) of tanh function,the left side of expression (17)

$$\begin{aligned} T_\alpha F(x) &= -\sqrt{-m}\,\frac{2-\alpha}{\alpha}\sqrt{-m}\left(x^{\frac{\alpha}{2-\alpha}}\right)^{1-\alpha} \operatorname{sech}^2\left(\frac{2-\alpha}{\alpha}\sqrt{-m}\,x^{\frac{\alpha}{2-\alpha}}\right)\frac{\alpha}{2-\alpha}x^{\frac{\alpha}{2-\alpha}-\alpha} \\ &= m\operatorname{sech}^2\left(\frac{2-\alpha}{\alpha}\sqrt{-m}\,x^{\frac{\alpha}{2-\alpha}}\right). \end{aligned} \tag{20}$$

After simple calculation, we know that the right side of expression (17)

$$m+F^2=m+(-m)\tanh^2\left(\frac{2-\alpha}{\alpha}\sqrt{-m}x^{\frac{\alpha}{2-\alpha}}\right)$$
$$=m\operatorname{sech}^2\left(\frac{2-\alpha}{\alpha}\sqrt{-m}x^{\frac{\alpha}{2-\alpha}}\right). \tag{21}$$

By (20) and (21), then (18) is the solutions of Equation (17). Similarly, we can obtain that the proof of (19).

Step 3: Determining integer n by the principle of homogeneous balance, and Substituting (16) and (17) into Equation (15) and collecting the terms with same order of $F(\xi)$, then setting each coefficient of $F(\xi)$ to zero. So we get a system of algebraic equations of $a_0,a_1,\cdots,a_n$ and c. By solving this algebraic equations, yields $a_0,a_1,\cdots,a_n$ and c can be expressed by parameters a,b,m and δ. With the help of the solutions of Equation (17) and expression (18), the solutions of Equation (16) can be arrived. So we have the traveling wave solutions of Equation (13).

3. Application

In this section, we apply the fractional Riccati expansion method to KdV-type equations and Telegrph equations. The fractional derivative is comforable fractional derivative with order α with respect to t, marked by notation $\frac{D^\alpha}{Dt^\alpha}$.

3.1. Solutions of Space-Time Fractional KdV-Type Equation

The KdV equation is a very important shallow water wave equation derived by Korteweg and de Vries in 1895. The modified KdV equation arises in many fields, such as fluid physics, solid-state physics, phasma physics, and have been studied by many authors [12] [13] [14]. The space-time fractional modified KdV equation is

$$\frac{D^\alpha u}{Dt^\alpha}+au\frac{D^\alpha u}{Dx^\alpha}+bu^2\frac{D^\alpha u}{Dx^\alpha}+\delta\frac{D^{3\alpha}u}{Dx^{3\alpha}}=0, \tag{22}$$

inhere a,b and δ are constants.

Burgers equation is a nonlinear partial differential equation simulating the propagation and reflection of shock waves and can be applied in many fields, such as fluid mechanics, nonlinear acoustics, and gas dynamics. The space-time fractional modified KdV-Burgers equation is

$$\frac{D^\alpha u}{Dt^\alpha}+au\frac{D^\alpha u}{Dx^\alpha}+bu^2\frac{D^\alpha u}{Dx^\alpha}+r\frac{D^{2\alpha}u}{Dx^{2\alpha}}+\delta\frac{D^{3\alpha}u}{Dx^{3\alpha}}=0, \tag{23}$$

Many authors [15] [16] [17] have investigated the existence and stability of KdV-Burgers equation. Jose [18] discussed numerically the stability of the shock solution of KdV-Burgers equation,when diffusion dominates dispersion,the steady-state solution is a monotonic shock, when dispersion dominates diffusion, the steady-state solution is a shock which is oscillatory upstream and monotonic downstream. Pego [19] studied the stability of traveling wave solutions of a generalization of the Kdv-Burgers.

To solve the Equation (23), substituting the traveling wave transformation (14) to Equation (23), then it can written as the following fractional ordinary equation

$$c\frac{D^{\alpha}u}{D\xi^{\alpha}}+au\frac{D^{\alpha}u}{D\xi^{\alpha}}+bu^{2}\frac{D^{\alpha}u}{D\xi^{\alpha}}+r\frac{D^{2\alpha}u}{D\xi^{2\alpha}}+\delta\frac{D^{3\alpha}u}{D\xi^{3\alpha}}=0, \tag{24}$$

Suppose that $u(\xi)$ can be expressed by the following form

$$u(\xi)=\sum_{i=0}^{n}a_{i}F^{i}. \tag{25}$$

By using the homogeneous balance principle and balancing $\frac{D^{3\alpha}u}{D\xi^{3\alpha}}$ with $u^{2}\frac{D^{\alpha}u}{D\xi^{\alpha}}$, we known that $n=1$. Therefore, expression (25) can be expressed as

$$u(\xi)=a_{0}+a_{1}F(\xi), \tag{26}$$

where $F(\xi)$ are solutions of fractional Riccati Equation (17), coefficients a_0,a_1 will be determined. Substituting (26) and (17) into (24), we obtain that

$$ca_{1}+aa_{0}a_{1}+ba_{1}a_{0}^{2}+2m\delta a_{1}+aa_{1}^{2}F+2ba_{1}^{2}a_{0}F+2ra_{1}F+ba_{1}^{3}F^{2}+6\delta a_{1}F^{2}=0. \tag{27}$$

In Equation (27), setting the coefficients of $F^{i}(i=0,1,2)$ to zero, yields algebraic equations of parameters a_0,a_1 and c

$$\begin{cases}ca_{1}+aa_{0}a_{1}+ba_{1}a_{0}^{2}+2m\delta a_{1}=0\\ aa_{1}^{2}+2ba_{1}^{2}a_{0}+2ra_{1}=0\\ ba_{1}^{3}+6\delta a_{1}=0\end{cases} \tag{28}$$

By solving algebraic system (28), solutions are denoted by:

$$a_{0}=-\frac{a}{2b}\mp\frac{r}{\sqrt{-6b\delta}},\quad a_{1}=\pm\sqrt{-\frac{6\delta}{b}},\quad c=\frac{a^{2}}{4b}-2m\delta+\frac{r^{2}}{6\delta}, \tag{29}$$

where $b\delta<0$. By (14), (18), (19), (26) and (29), we have the following theorem.

Theorem 3.1. For $b\delta<0,m<0,0<\alpha\le 1$, the following solitary wave solutions of Equation (23) are

$$u_{1}(x,t)=-\frac{a}{2b}\mp\frac{r}{\sqrt{-6b\delta}}\mp\sqrt{\frac{6m\delta}{b}}\tanh\left[\frac{2-\alpha}{\alpha}\sqrt{-m}\,(x+ct)^{\frac{\alpha}{2-\alpha}}\right] \tag{30}$$

and

$$u_{2}(x,t)=-\frac{a}{2b}\mp\frac{r}{\sqrt{-6b\delta}}\mp\sqrt{\frac{6m\delta}{b}}\coth\left[\frac{2-\alpha}{\alpha}\sqrt{-m}\,(x+ct)^{\frac{\alpha}{2-\alpha}}\right]. \tag{31}$$

Remark 3.2. When $\alpha=1$, solution (30) become the following form

$$u_{3}(x,t)=-\frac{a}{2b}\mp\frac{r}{\sqrt{-6b\delta}}\mp\sqrt{\frac{6m\delta}{b}}\mp\sqrt{\frac{6m\delta}{b}}\tanh\left[\sqrt{-m}\,(x+ct)\right]. \tag{32}$$

Expression (32) is the result (4.7) given in [15] in case $p=1$.

Setting $r=0$ in the expression (30) and (31), yields that:

Theorem 3.3. Suppose that $b\delta<0, m<0, 0<\alpha\le 1$, Equation (22) have the following solitary wave solutions

$$u_4(x,t)=-\frac{a}{2b}\mp\sqrt{\frac{6m\delta}{b}}\tanh\left[\frac{2-\alpha}{\alpha}\sqrt{-m}\,(x+ct)^{\frac{\alpha}{2-\alpha}}\right] \tag{33}$$

and

$$u_5(x,t)=-\frac{a}{2b}\mp\sqrt{\frac{6m\delta}{b}}\coth\left[\frac{2-\alpha}{\alpha}\sqrt{-m}\,(x+ct)^{\frac{\alpha}{2-\alpha}}\right]. \tag{34}$$

Remark 3.4. (1) When $\alpha=1$, solution (33) become the following form

$$u_6(x,t)=u_1(\xi)=-\frac{a}{2b}\mp\sqrt{\frac{6m\delta}{b}}\tanh\left[\sqrt{-m}\,(x+ct)\right]. \tag{35}$$

Expression (35) is the result (4.9) given in [15] in case $p=1$.

(2) Setting a to zero in the expressions (33) and (34), we can obtain that the results (14) and (15) derived by Abdel-Salam [20].

3.2. Solutions of Space-Time Fractional Telegraph Equation

The space-time fractional Telegraph equation

$$\frac{D^{2\alpha}u}{Dt^{2\alpha}}-\frac{D^{2\alpha}u}{Dx^{2\alpha}}+\frac{D^{\alpha}u}{Dt^{\alpha}}+\gamma u+\beta u^3=0, \tag{36}$$

where γ,β are constants. Telegraph equation is the important model in the description of the transmission of energetic particle distributions [21] [22] and it has been applied in a wide range, such as astrophysical and plasma physics [23]. Tawfik [24] obtained analytical solutions of the space-time fractional Telegraph equation in the case of $\gamma=\beta=0$. The traveling wave solution of Telegraph Equation (36) with $\alpha=1$ was derived by Wang and Li [25]. In virtue of exp-function method, many traveling wave solutions of Equation (36) was studied by Guner and Bekir [10].

Substituting (14) into (36), Telegraph Equation (36) can be deduced by the following form

$$c^2\frac{D^{2\alpha}u}{D\xi^{2\alpha}}-\frac{D^{2\alpha}u}{D\xi^{2\alpha}}+c\frac{D^{\alpha}u}{D\xi^{\alpha}}+\gamma u+\beta u^3=0 \tag{37}$$

By using the principle of homogeneous balance and (16), supposing that the solutions of Equation (37) is

$$u(\xi)=a_0+a_1F(\xi), \tag{38}$$

where $F(\xi)$ are solutions of fractional Riccati Equation (18), a_0,a_1 are undetermined coefficients. Substituting (38) and (17) into (37), after simply calculating and setting the coefficients of F^0,F,F^2,F^3 into zero. The algebraic equations with respect to a_0,a_1 and c can be described in the following form

$$\begin{cases} cma_1 + \gamma a_0 + \beta a_0^3 = 0 \\ 2m\left(c^2 - 1\right) + \gamma + 3\beta a_0^2 = 0 \\ c + 3\beta a_0 a_1 = 0 \\ 2\left(c^2 - 1\right) + \beta a_1^2 = 0 \end{cases} \tag{39}$$

From the second and fourth equation of (39), we can obtain respectively,

$$a_0^2 = \frac{-\gamma - 2m\left(c^2 - 1\right)}{3\beta}, \quad a_1^2 = \frac{-2\left(c^2 - 1\right)}{\beta}, \tag{40}$$

By the third equation of (39), we have

$$c = -3\beta a_0 a_1. \tag{41}$$

Substituting (41) into the first equation of (39), we yields that

$$-3m\beta a_1^2 + \gamma + \beta a_0^2 = 0. \tag{42}$$

By means of (40) and (42), for $m < 0, \beta > 0, \gamma \le 8m$, if γ, m satisfying $9\gamma^2 - 2\gamma + 16m = 0$, the solutions can be given in the following form

$$a_0^2 = -\frac{\gamma}{4\beta}, \quad a_1^2 = \frac{\gamma}{4m\beta}, \quad c^2 = 1 - \frac{\gamma}{8m}. \tag{43}$$

By mean of (14), (18), (19), (38) and (43), we can obtain the following solutions of Equation (36):

Theorem 3.5. For $m < 0, \beta > 0, \gamma \le 8m$, if γ, m satisfying $9\gamma^2 - 2\gamma + 16m = 0$, then we have the following solitary wave solutions of Equation (36), that is

$$u_7(x,t) = \pm\sqrt{-\frac{\gamma}{4\beta}} \pm \sqrt{\frac{\gamma}{4m\beta}} \tanh\left[\frac{2-\alpha}{\alpha}\sqrt{-m}\left(x \pm \sqrt{1 - \frac{\gamma}{8m}}t\right)^{\frac{\alpha}{2-\alpha}}\right] \tag{44}$$

and

$$u_8(x,t) = \pm\sqrt{-\frac{\gamma}{4\beta}} \pm \sqrt{\frac{\gamma}{4m\beta}} \coth\left[\frac{2-\alpha}{\alpha}\sqrt{-m}\left(x \pm \sqrt{1 - \frac{\gamma}{8m}}t\right)^{\frac{\alpha}{2-\alpha}}\right]. \tag{45}$$

Remark 3.6. (1) Solutions (44) and (45) are the new solitary wave solutions not be found by Guner and Bekir [10].

(2) when $\alpha = 1$, (44) and (45) are the new solitary wave solutions of Telegraph equation not be obtained by wang [25].

4. Conclusion

In this paper, we point out firstly that the chain rule of fractional derivative is incorrect by giving a counterexample. Especially, the results reported in literatures by using chain rule for fractional derivative are lack of accuracy. After that, we put forward the Riccati expansion method and solve fractional KdV-type equation and Telegraph equation by applying this method. Some solitary wave solutions can be obtained for the first time. In this process, the chain rule is not required, method raised in here is simple and can be used to get solitary wave solutions of more fractional differential equations.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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