

The Proof of the Generalized Piemann's Hypothesis

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Abstract

The article presents the proof of the validity of the generalized Riemann hypothesis on the basis of adjustment and correction of the proof of the Riemann's hypothesis in the work [1], obtained by a finite exponential functional series and finite exponential functional progression.

Keywords

Riemann's Hypothesis, Natural Series, Function of Möbius, Mertens Function, Finite Exponential Functional Series, Finite Exponential Functional Progression

1. Introduction

In this paper we give the proof of the generalized Riemann's hypothesis on the basis of adjustments and corrections to the proof of the Riemann's hypothesis of the zeta function, which was undertaken in [1], as well as values of specified limits of the coefficient c . The paper also provides a refutation of the hypothesis of Mertens.

The formulation of the problem (Generalized Riemann's hypothesis). All non-trivial zeros of Dirichle's function $L(s, \chi)$ have a real part that is equal to

$$\sigma = \frac{1}{2}.$$

1). For this we first will prove the Riemann's hypothesis for the zeta function $\zeta(n)$.

1.1). **The solution.** For the confirmation of the Riemann's hypothesis we will give the definitions and prove the following theorem.

Definition 1. The expression

$$\sum_{k=1}^{\lfloor \sqrt{n} \rfloor} u^{\frac{1}{k}}(x) = u(x) + u^{\frac{1}{2}}(x) + u^{\frac{1}{3}}(x) + \dots + u^{\frac{1}{\lfloor \sqrt{n} \rfloor}}(x) \quad (1)$$

is called finite exponential functional series with respect to the variable exponent $\frac{1}{k}$, where $k = \{1, 2, 3, \dots, [\sqrt{n}]\}$.

Definition 2. The progression of the type

$$a(x), a(x) \cdot q(x), a(x) \cdot q^2(x), \dots, a(x) \cdot q^{\sqrt{x}}(x) \tag{2}$$

is called finite exponential functional progression, if their first member $a(n)$ is a function of x or is equal to 1 and the denominator $q(n)$ is a function of the variable $x^{\frac{1}{\sqrt{x}}}$.

Theorem 1. If the set of natural numbers $N_n^+ = \{1, 2, \dots, k, \dots, n\}$ is the union of subsets $M_1, M_2, M_3, M_4, M_5, M_6$ and these subsets are disjoint and have appropriately for m, r, s, t, v, u the elements, the number of elements of the set $N_n^+ = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 \cup M_6$ equal to

$$n = m + r + s + t + v + u.$$

Proof. The theorem is proved similarly to the theorem 7.11 ([2], p. 50).

Theorem 2. $M(n) = \sum_{k=1}^{\infty} \mu(k)$ of a series $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ equals.

$$|M(n)| < 1.5\sqrt{n}$$

Proof. Let the number $N = N(n)$ be the quantity of elements of the set of natural numbers $N_n^+ = \{1, 2, 3, \dots, n\}$. The positive integers N_n^+ consist: of 1; primes, the quantity of which is denoted by $\Pi = \Pi(n)$; of natural numbers, which are divided on p^m with quotient from 1 with $m \geq 2$, the number of which is denoted by $K^m = K^m(n)$; the number of positive integers, which are decomposed into a product of a pair number of primes, we denote by $T^n = T^n(n)$, and the amount of numbers, which can be converted into product of unpaired number of primes, will be denoted by $T^u = T^u(n)$. Then the amount of natural numbers N , of the set of natural numbers N_n^+ , according to the theorem 1, equals

$$N_n^+ = 1 + \Pi + T^n + T^u + K^2 + K \tag{3}$$

The number of the natural series N_n^+ approximately can be expressed as a finite exponential function series

$$f_1(n) = \sum_{k=2}^{[n]} n^{\frac{1}{k}} = \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} + \dots + \sqrt[n]{n} \tag{4}$$

We will write the number of natural numbers $K(n)$, approximately as finite exponential function series consisting of the first $(\sqrt{n}-1)$ members of the series (4), and we denote it as $f_2(n)$, then

$$f_2(n) = \sum_{k=2}^{[\sqrt{n}]} n^{\frac{1}{k}} = \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} + \dots + \sqrt[\sqrt{n}]{n}. \tag{5}$$

In the sum of the series (4) each term of the series is taken, as the amount of natural numbers. For example, $\sqrt[k]{n} = \lceil \sqrt[k]{n} \rceil$; $\lceil \sqrt{100} \rceil = \{1; 2; 3; 4; 5; 6; 7; 8; 9; 10\}$; $\lceil \sqrt[3]{100} \rceil = \{1; 2; 3; 4\}$ and so on.

A series $(\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} + \dots + \sqrt[n]{n})$ is greater than the series of the natural number of positive integers $f_1(n) > N_n^+$.

Definition 3. The natural numbers that overlap, are called the finite exponential function series of (5), in which they occur more than once. Let that $N = n$ and $\frac{1}{\sqrt{n}} = \frac{1}{\lceil \sqrt{n} \rceil}$. The function $f_2(n) > K(n)$, because in the function except the numbers $K(n)$ are included and the numbers which overlap.

Definition 4. Two infinitely great $f(n)$ and $\varphi(n)$, which are not equal to each other $f(n) \neq \varphi(n)$ are called equivalent if $\lim_{n \rightarrow \infty} \frac{f(n)}{\varphi(n)} = 1$ when $\varphi(n) \neq 0$.

We assume that the set of natural numbers $M_1 = (1, 2, 3, \dots, n-1, n)$ with algebra $A_1 = \langle N, +, \cdot, -, 0, 1 \rangle$ is vector space P_1 . In this space we set the standard $\|x\|_n = \max_n |n| = n$, and the set of numbers $M_2 = (\sqrt[n]{n}, \sqrt[n-1]{n}, \dots, \sqrt{n})$ with algebra $A_2 = \langle \sqrt[n]{n}, +, \cdot, -, 0, 1 \rangle$ will be assumed as vector space P_2 with standard

$$\|x\|_{\sqrt{n}} = \max_{\sqrt{n}} |\sqrt{n}| = \sqrt{n}.$$

Then, the denominator of an exponential function of finite progression, which operates in the space P_1 will take as $q = n^{\frac{1}{n}}$, and the denominator of an exponential function of finite progression, which operates in the space P_2 , as $q = n^{\frac{1}{\sqrt{n}}}$.

The finite exponential functional series (4) is approximable by the sum of the finite exponential functional progression

$$\varphi_1\left(n^{\frac{1}{n}}\right) = 1 + n^{\frac{1}{n}} + n^{\frac{2}{n}} + n^{\frac{3}{n}} + \dots + n^{\frac{1}{2}} \tag{6}$$

Proposition 1. The finite exponential functional series (4) and the sum of the finite exponential functional progression (6) are equivalent.

Proof. To prove the equivalence of the finite exponential functional series $f_1(n)$ with the finite functional progression $\varphi_1\left(n^{\frac{1}{n}}\right)$ the sum of the functional series is written in the form of

$$f_1(n) = (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n}) < \left(n^{\frac{1}{\sqrt{n}}} (n - \sqrt{n} + 1) + (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n}) \right),$$

Then let us write that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f_1(n)}{\varphi_1\left(\frac{1}{n^n}\right)} &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n})}{\left(1 + n^{\frac{1}{n}} + n^{\frac{2}{n}} + n^{\frac{3}{n}} + \dots + n^{\frac{1}{2}}\right)} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n} \cdot \left(1 + n^{\frac{1}{3} \cdot \frac{1}{2}} + n^{\frac{1}{4} \cdot \frac{1}{2}} + \dots + n^{\frac{1}{n} \cdot \frac{1}{2}}\right)}{\sqrt{n} \cdot \left(n^{-\frac{1}{2}} + n^{\frac{1}{n} \cdot \frac{1}{2}} + \dots + 1\right)} = 1 \end{aligned}$$

Therefore, in accordance with the definition 4, the functional series and functional progression will be equivalent.

Proposition 1 is proved.

Proposition 2. The finite exponential functional series (4) and the sum of the finite exponential functional progression (6) $2 \cdot S_1(N)$ are equivalent.

Proof. The sum of the finite exponential functional progression $\varphi_1\left(\frac{1}{n^n}\right)$ can be calculated by the formula

$$2 \cdot S_1(N) = 2 \cdot \frac{\sqrt{n} - 1}{n^{\frac{1}{n}} - 1} > 2 \cdot \frac{\sqrt{n} - 1}{\frac{2}{\sqrt{n}}} = (n - \sqrt{n}) \quad (7)$$

and then the limit $\lim_{n \rightarrow \infty} \frac{f_1(n)}{\varphi_1\left(\frac{1}{n^n}\right)}$ will be equal to:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f_1(n)}{\varphi_1\left(\frac{1}{n^n}\right)} &= \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\sqrt{n}}}(n - \sqrt{n} + 1) + (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n})}{n - \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{\left(n^{\frac{1}{\sqrt{n}}}n - n^{\frac{1}{\sqrt{n}}}\sqrt{n} + n^{\frac{1}{\sqrt{n}}}1\right) + (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n})}{n - \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n + \sqrt[3]{n} + \sqrt[4]{n} + \dots + \sqrt[n]{n} + 1}{n - \sqrt{n}} = 1, \end{aligned}$$

where $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ and $0 < \sqrt[n]{n} - 1 < \frac{2}{\sqrt{n}}$ ([3], p. 67).

Therefore, in accordance with the definition 4, the functional series (4) and the finite sum $2 \cdot S_1$ of the functional progression (6) at $\left(\sqrt[n]{n} - 1\right) \approx \frac{2}{\sqrt{n}}$, when $n \rightarrow \infty$, will be equivalent.

Proposition 2 is proved.

From the expression (4) and (7) one can see that $\varphi_1\left(\frac{1}{n^n}\right)$ is within limit of function

$$f_1(n) \approx \varphi_1\left(n^{\frac{1}{n}}\right) \geq 2 \cdot S_1(n) > n - \sqrt{n}.$$

The sum of the finite exponential functional progression (6) with $q = n^{\frac{1}{n}}$ equals $2 \cdot S_1(N) \approx (n - \sqrt{n})$. When $(n \rightarrow \infty), S_1(N) \rightarrow \infty$. We will compare the function $2 \cdot S_1(n)$ with the function $f(n) = N_n^+$. We find

$$k_1 = \lim_{n \rightarrow \infty} \frac{f(n)}{S_1(n)} = \lim_{n \rightarrow \infty} \frac{N}{2 \cdot \frac{\sqrt{n}-1}{\sqrt[n]{n}-1}} < \lim_{n \rightarrow \infty} \frac{N}{2 \cdot \frac{\sqrt{n}-1}{2}} = \lim_{n \rightarrow \infty} \frac{N}{n - \sqrt{n}} = 1.$$

Therefore, $k_1 < 1$.

Lemma 1. The number of natural numbers that overlap is less than $1,5\sqrt{n}$.

Proof. To prove this proposition let us denote through K^n is the numbers that occur more than once in the finite exponential functional series (4) when $n \rightarrow \infty$, and use the exponential functional series $f_2(n)$.

The series (5) is taken in this form to be because it includes all numbers that overlap. This follows from the expression $2^{\sqrt{n}} > N, \sqrt{n} \cdot \ln 2 > \ln N$. Two is taken because it is the smallest prime number that can not be decomposed into prime factors.

The finite exponential functional series (5) will be replaced by the sum of finite exponential functional progression

$$\varphi_2\left(n^{\frac{1}{\sqrt{n}}}\right) = \left(1 + \sum_{k=1}^{\frac{\sqrt{n}}{2}} n^{\frac{k}{\sqrt{n}}}\right). \tag{8}$$

Proposition 3. The finite exponential functional series (5) and the finite exponential functional progression (8) are equivalent.

Proof. The functional series (4) can be written as

$$f_1(n) = \sum_{k=\sqrt{n}}^n n^{\frac{1}{k}} + \left(n^{\frac{1}{2}} + n^{\frac{1}{3}} + \dots + n^{\frac{1}{\sqrt{n}}}\right),$$

and the functional progression (6) is as

$$\varphi_1\left(n^{\frac{1}{n}}\right) = \left(1 + n^{\frac{1}{n}} + n^{\frac{2}{n}} + \dots + n^{\frac{n-2\sqrt{n}}{2n}}\right) + \left(1 + \sum_{k=1}^{\frac{\sqrt{n}}{2}} n^{\frac{k}{\sqrt{n}}}\right).$$

Discard the first members of the series and progression, we find that

$$f_2(n) = n^{\frac{1}{2}} + n^{\frac{1}{3}} + \dots + n^{\frac{1}{\sqrt{n}}} \text{ or } \varphi_2\left(n^{\frac{1}{\sqrt{n}}}\right) = \left(1 + \sum_{k=1}^{\frac{\sqrt{n}}{2}} n^{\frac{k}{\sqrt{n}}}\right).$$

We show that

$$f_3(n) = \sum_{k=\sqrt{n}}^n n^{\frac{1}{k}} \text{ and } \varphi_3\left(n^{\frac{1}{n}}\right) = \left(1 + n^{\frac{1}{n}} + n^{\frac{2}{n}} + n^{\frac{3}{n}} + \dots + n^{\frac{n-2\sqrt{n}}{2n}}\right)$$

are equivalent:

$$k_2 = \lim_{n \rightarrow \infty} \frac{f_3(n)}{\varphi_3\left(n^{\frac{1}{n}}\right)} = \lim_{n \rightarrow \infty} \frac{\sum_{k=\sqrt{n}}^n n^{\frac{1}{k}}}{\left(1 + n^{\frac{1}{n}} + n^{\frac{2}{n}} + \dots + n^{\frac{n-2\sqrt{n}}{2n}}\right)} = \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{\sqrt{n}}}(n - \sqrt{n} + 1)}{2\left(\frac{n - \sqrt{n}}{2}\right)} = 1.$$

It follows that the finite exponential functional series $f_2(n)$ and the finite exponential functional progression $\varphi_2\left(n^{\frac{1}{n}}\right)$ are equivalent.

Proposition 3 is proved.

To prove the theorem, we introduce the functions series

$$f_4\left(n^{\frac{1}{k_1}}\right) = \sum_{k=2k_1}^{\lfloor \sqrt{n} \rfloor} n^{\frac{1}{k}} = n^{\frac{1}{4}} + n^{\frac{1}{6}} + \dots + n^{\frac{1}{\sqrt{n}}}, \quad (9)$$

where $k_1 = \{2, 3, \dots, \lfloor \sqrt{n} \rfloor / 2\}$. And functional progression

$$\varphi_4\left(n^{\frac{2}{\sqrt{n}}}\right) = \left(1 + n^{\frac{2}{\sqrt{n}}} + n^{\frac{4}{\sqrt{n}}} + \dots + n^{\frac{\sqrt{n}-8}{4\sqrt{n}}} + n^{\frac{1}{4}}\right). \quad (10)$$

If we express the series (5), as a series $\lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor + \lfloor \sqrt[4]{n} \rfloor + \dots + \lfloor \sqrt[n]{n} \rfloor$, then the series (8) is taken $\lfloor \sqrt{n} \rfloor + \lfloor \sqrt[4]{n} \rfloor + \lfloor \sqrt[6]{n} \rfloor + \dots + \lfloor \sqrt[2k]{n} \rfloor$ is such form so that each element of the series (8) overlaps the each element of the series (5) with unpaired exponents of the root. And then we can write that

$$\lfloor \sqrt{n} \rfloor + \lfloor \sqrt[4]{n} \rfloor + \lfloor \sqrt[6]{n} \rfloor + \dots + \lfloor \sqrt[2k]{n} \rfloor > \lfloor \sqrt[3]{n} \rfloor + \lfloor \sqrt[5]{n} \rfloor + \dots + \lfloor \sqrt[2k+1]{n} \rfloor. \quad (11)$$

Hence the amount of numbers that cover more numbers that overlap.

Proposition 4. The finite exponential functional series (9) and the finite exponential progression (10) are equivalent.

Proof. To prove the equivalence of the finite exponential functional series with the finite exponential functional progression in the form of the relation

$$\begin{aligned} k_3 &= \lim_{n \rightarrow \infty} \frac{f_4(n)}{\varphi_4(n)} = \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{4}} + n^{\frac{1}{6}} + \dots + n^{\frac{1}{\sqrt{n}}}}{\left(1 + n^{\frac{2}{\sqrt{n}}} + n^{\frac{4}{\sqrt{n}}} + \dots + n^{\frac{\sqrt{n}-8}{4\sqrt{n}}} + n^{\frac{1}{4}}\right)} \\ &= \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{4}} \left(1 + n^{\frac{1}{6} - \frac{1}{4}} + \dots + n^{\frac{1}{\sqrt{n}} - \frac{1}{4}}\right)}{n^{\frac{1}{4}} \cdot \left(n^{\frac{1}{4}} + n^{\frac{2}{\sqrt{n}} - \frac{1}{4}} + \dots + 1\right)} = 1 \end{aligned}$$

Therefore, in accordance with the definition 4, the functional series (9) and a

functional progression (10) are equivalent.

Proposition 4 is proved.

Proposition 5. The finite exponential functional series (9) and the finite sum $4 \cdot S_4(n)$ of the exponential function progression (10) are equivalent when

$$\sqrt[n]{n} - 1 \approx \frac{2}{\sqrt[n]{n}} \quad ([3], \text{p. } 67).$$

Proof. The sum of functional series $f_4(n)$ is more than $\left(\frac{\sqrt[n]{n}}{2} - 1\right)$, and the sum of functional progression $\varphi_4\left(n^{\frac{2}{\sqrt[n]{n}}}\right)$ is considered, as the sum of the functional progression with $q = n^{\frac{2}{\sqrt[n]{n}}}$. Then we find that $4 \cdot S_4(n) = 4 \cdot \frac{\sqrt[n]{n} - 1}{\sqrt[n]{n^2} - 1}$. In order to calculate the functional $\left(n^{\frac{2}{\sqrt[n]{n}}} - 1\right)$, let us set $t^2 = n$ or $t = \sqrt{n}$, and then we obtain

$$\left(n^{\frac{2}{\sqrt[n]{n}}} - 1\right) = \left(t^{\frac{4}{t}} - 1\right) = \left(t^{\frac{2}{t}} - 1\right) \cdot \left(t^{\frac{2}{t}} + 1\right) = \left(t^{\frac{1}{t}} - 1\right) \cdot \left(t^{\frac{1}{t}} + 1\right) \cdot \left(t^{\frac{2}{t}} + 1\right).$$

Since $\lim_{t \rightarrow \infty} \sqrt[t]{t} = 1$ then $\left(n^{\frac{2}{\sqrt[n]{n}}} - 1\right) < \frac{8}{\sqrt[n]{n}}$. And then we will have

$$4 \cdot S_4(n) \approx 4 \cdot \frac{\sqrt[n]{n} - 1}{\frac{8}{\sqrt[n]{n}}} = \frac{\sqrt[n]{n} - \sqrt[n]{n}}{2}, \quad (12)$$

Using the definition 4 we will have

$$k_3 = \lim_{n \rightarrow \infty} \frac{f_4(n)}{\varphi_4\left(n^{\frac{2}{\sqrt[n]{n}}}\right)} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt[n]{n} - 1}{2}}{\frac{\sqrt[n]{n} - \sqrt[n]{n}}{2}} = 1.$$

Therefore, a function of series (9) and the sum $4 \cdot S_4(n)$ of functional progression (10) are equivalent.

Proposition 5 is proved.

From the expressions (10) and (12) it is clear that $\varphi_4(n)$ is within

$$f_4\left(n^{\frac{2}{\sqrt[n]{n}}}\right) \approx \varphi_4(n) \geq 4 \cdot S_4(n) > \frac{\sqrt[n]{n} - \sqrt[n]{n}}{2}.$$

Then we compare function $\sqrt[n]{n}$ with the function $f_4(n)$ when $n \rightarrow \infty$ and we obtain

$$k_4 = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{4 \cdot \frac{\sqrt[n]{n} - 1}{\sqrt[n]{n^2} - 1}} < \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{4 \cdot \frac{\sqrt[n]{n} - 1}{\frac{8}{\sqrt[n]{n}}}} = \lim_{n \rightarrow \infty} \frac{8 \cdot \sqrt[n]{n}}{4 \cdot (\sqrt[n]{n} - \sqrt[n]{n})} = 2.$$

Hence, we have that $k_4 < 2$, or $\sqrt[n]{n} - 2 \cdot f_4\left(n^{\frac{2}{\sqrt[n]{n}}}\right) < 0$.

Therefore, $\frac{\sqrt{n}}{2} < f_4(n)$. Let us take into account the value of the finite exponential functional series $f_4(n)$, and write that

$$\frac{\sqrt{n}}{2} < K^n(n) < \sqrt{n} + \frac{\sqrt{n}}{2} = \frac{3}{2}\sqrt{n}.$$

Lemma 1 is proved.

Then we can write that

$$N(n) - c \cdot f_1(n) < 0, \quad (13)$$

Using the inequality $f_1(n) > 2 \cdot K(n)$ we will write that

$$f_1(n) \approx K(n) + (n - \sqrt{n} + 1) + K^n(n) \quad (14)$$

If we substitute value of the function $f_2(n)$ (14) into (13), we obtain

$$N(n) - c \cdot (K(n) + (n - \sqrt{n} + 1) + K^n(n)) < 0.$$

Using Lemma 1, we obtain

$$N(n) - c \cdot \left(K(n) + (n - \sqrt{n} + 1) + \frac{3}{2}\sqrt{n} \right) < 0.$$

Hence; we find that

$$N(n) - c \cdot (K(n) + (n - \sqrt{n} + 1)) < c \cdot \frac{3}{2}\sqrt{n}.$$

The value N from (3) is substituted instead N , we obtain

$$1 + \Pi + T^n + T^n + K^k + K(n) < c \cdot (1.5\sqrt{n} + K(n) + (n - \sqrt{n} + 1)),$$

or

$$1 + \Pi + T^n + T^n + K^k < c \cdot (1.5\sqrt{n} + (1 + \Pi + T^n + T^n + K^k - \sqrt{n} + 1)). \quad (15)$$

Then we can write that appropriately of the properties of the function of Mobius- $\mu(n) = 1$, when $n = 1$; $\mu(n) = (-1)^k$, where k is the amount of prime factors of the numbers $n = p_1 \cdot p_2 \cdot \dots \cdot p_k$ and $\mu(n) = 0$ when n is multiple p^m for $m \geq 2$,

$$1 + T^n - (\Pi + T^n) < c \cdot ((1 + T^n) - (\Pi + T^n) - \sqrt{n} + 1 + 1.5\sqrt{n}). \quad (16)$$

We write that

$$M(n) = 1 + T^n - (\Pi + T^n).$$

Then the expression (16) takes the form

$$M(n) < c \cdot ((M(n) - \sqrt{n} + 1) + 1.5\sqrt{n}).$$

Therefore

$$|M(n)| < \frac{c}{c-1} \cdot (0.5\sqrt{n} + 1). \quad (17)$$

The theorem is proved.

For the Mertens function we can find a more precise estimate.

Lemma 2. The accurate assessment. $M(n) = \sum_{k=1}^{\infty} \mu(k)$ in a series

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$
 will be equal

$$M(n) < \frac{c}{c-1} (1.25 \cdot \sqrt{n} + \sqrt[3]{n} - 1.25 \cdot \sqrt[4]{n} - \sqrt{n} + 1).$$

Proof. In order to find a more accurate estimate than $M(n) < \frac{c}{c-1} (0.5\sqrt{n} + 1)$, let us find the sum of the finite exponential functional series (5)

$$f_2(n) = \sum_{k=2}^{\lfloor \sqrt{n} \rfloor} n^{\frac{1}{k}} = \left(\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} + \dots + \sqrt{\lfloor \sqrt{n} \rfloor} \right).$$
 For that we use the functional

progression (6) $\varphi_2\left(n^{\frac{1}{\sqrt{n}}}\right) = \left(1 + \sum_{k=1}^{\frac{\sqrt{n}}{2}} n^{\frac{k}{\sqrt{n}}} \right)$, then we obtain that

$$f_2(n) = \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} + \dots + \sqrt{\lfloor \sqrt{n} \rfloor} = 2.5\sqrt{n} + 2\sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n}.$$

We find from the expression (10) that the quantity of numbers that overlap is less than $\frac{f_2(n)}{2}$ because $\frac{f_2(n)}{2} > \lfloor \sqrt[3]{n} \rfloor + \lfloor \sqrt[5]{n} \rfloor + \dots + \lfloor \sqrt[2k+1]{n} \rfloor$. Using this

method, we define what $M(n) < \frac{c}{c-1} (0.5 \cdot \sqrt{n} + 1)$ and the expression

$$\frac{f_2(n)}{2} > \lfloor \sqrt[3]{n} \rfloor + \lfloor \sqrt[5]{n} \rfloor + \dots + \lfloor \sqrt[2k+1]{n} \rfloor$$
 we obtain that

$$M(n) < \frac{c}{c-1} (1.25 \cdot \sqrt{n} + \sqrt[3]{n} - 1.25 \cdot \sqrt[4]{n} - \sqrt{n} + 1).$$

Hence, we find that the upper limit of the value functions $\lim_{n \rightarrow \infty} \frac{M(n)}{\sqrt{n}}$ will be the value

$$\limsup_{n \rightarrow \infty} \frac{M(n)}{\sqrt{n}} < \frac{c}{c-1} \cdot 0.25,$$

and the lower limit is

$$\liminf_{n \rightarrow \infty} \frac{M(n)}{\sqrt{n}} > -\frac{c}{c-1} \cdot 0.25.$$

Therefore, the evaluation $M(n) < \frac{c}{c-1} (0.25\sqrt{n} + \sqrt[3]{n} - 1.25\sqrt[4]{n} + 1)$ is a more accurate estimate than $M(n) < \frac{c}{c-1} (0.5 \cdot \sqrt{n} + 1)$ when $n \rightarrow \infty$.

Lemma 2 is proved.

The theorem 2 proves that the upper limit value of the function $\lim_{n \rightarrow \infty} \frac{M(n)}{\sqrt{n}}$ equals

$$\limsup_{n \rightarrow \infty} \frac{M(n)}{\sqrt{n}} < 0.5 \cdot \frac{c}{c-1},$$

and the lower limit is

$$\liminf_{n \rightarrow \infty} \frac{M(n)}{\sqrt{n}} > -0.5 \cdot \frac{c}{c-1},$$

Proposition 6. $\frac{c}{c-1} \cdot (0.5\sqrt{n} + 1) \ll n^{\frac{1}{2} + \varepsilon}$ when $n \rightarrow \infty$.

Proof. According to the theorem 54 ([4], p. 114) we have that $M(n) = O\left(n^{\frac{1}{2} + \varepsilon}\right)$. The value $M(n) = O\left(\frac{c}{c-1} \cdot (0.5\sqrt{n} + 1)\right)$ is compared with $M(n) = O\left(n^{\frac{1}{2} + \varepsilon}\right)$,

we will write that $\left(\frac{c}{c-1} \cdot (0.5\sqrt{n} + 1)\right) = n^{\frac{1}{2} + \varepsilon_0}$ when $n \rightarrow \infty$. Hence we find that

$$\varepsilon_0 = \frac{\ln\left(\frac{c}{c-1} \cdot (0.5\sqrt{n} + 1)\right)}{\ln n} \text{ when } n \rightarrow \infty, \varepsilon_0 \rightarrow 0. \text{ Therefore, we can assume that } \varepsilon > \varepsilon_0, \text{ where } \varepsilon \text{ is a random small number. And here we find that } \frac{c}{c-1} \cdot (0.5\sqrt{n} + 1) \ll \left(n^{\frac{1}{2} + \varepsilon}\right) \text{ when } n \rightarrow \infty.$$

Proposition 6 is proved.

1.2). **A determination the values of coefficient c .**

1.2.1). Then we can write that according to the properties of Mobius function- $\mu(n) = 1$, then $n = 1$; $\mu(n) = (-1)^k$, where k the number of prime factors of the number $n = p_1 \cdot p_2 \cdot \dots \cdot p_k$ and $\mu(n) = 0$, when n is the multiple of p^m for $m \geq 2$ that

$$\left((1 + T^n) - (\Pi + T^n)\right) - c \cdot (1 + T^n - (\Pi + T^n)) < (0.5 \cdot \sqrt{n} + 1),$$

Then

$$M(n) < \frac{c}{1-c} \cdot (0.5 \cdot \sqrt{n} + 1); |M(n)| < \frac{c}{c-1} \cdot (0.5 \cdot \sqrt{n} + 1).$$

From the expression $K(n) + (n - \sqrt{n} + 1) + \frac{3}{2} \cdot \sqrt{n} > 0$, using the properties of Möbius function, it can be written that

$$M(n) < 1.5 \cdot \sqrt{n} - 1.$$

And from the expression $N - \sqrt{n} > 0$ we find that

$$M(n) > \sqrt{n}.$$

This coincides with the results [5]. Then we can find the extent to which the coefficient c is located. From the double inequality

$$\sqrt{n} < \frac{c}{c-1} \cdot (0.5 \cdot \sqrt{n} + 1) < 1.5 \cdot \sqrt{n}, \text{ we find that } 2 < \frac{c}{c-1} < 3. \text{ And here we find that } 1.5 < c < 2.$$

1.2.2). Using a more precise value $M(n)$, we find that

$$\sqrt{n} < \frac{c}{c-1} (0.25\sqrt{n} + \sqrt[3]{n} - 1.25\sqrt[4]{n} + 1) < 1.25 \cdot \sqrt{n}.$$

And here we find that $4 < \frac{c}{c-1} < 5$ and from the double inequality we find that the coefficient c will be in the range $1.25 < c < 1.33$.

1.3). Theorem 3. The series $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ converges if $\sigma = \frac{1}{2} + \varepsilon > \frac{1}{2}$ and $|M(n)| < \frac{c}{c-1} (0.5\sqrt{n} + 1)$ where ε is a random small number.

Corollary of Theorem 3 (the Riemann's hypothesis). All non-trivial zeros of the zeta-function have a real part equal to $\sigma = \frac{1}{2}$.

Proof. A necessary and sufficient condition for the validity of the Riemann's hypothesis is the convergence of the series $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ when $\sigma > \frac{1}{2}$ ([4], p. 114). We find the convergence of the series, when

$$M(n) = O\left(\frac{c}{c-1} (0.5\sqrt{n} + 1)\right) \text{ and } \sigma = \frac{1}{2}.$$

$$\begin{aligned} \frac{1}{\zeta\left(\frac{1}{2}\right)} &= \sum_{n=1}^{\infty} \frac{M(n)}{n^{\frac{1}{2}}} = \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^{\frac{1}{2}}} = \sum_{n=1}^{\infty} M(n) \cdot \left(\frac{1}{n^{\frac{1}{2}}} - \frac{1}{(n+1)^{\frac{1}{2}}}\right) \\ &= \sum_{n=1}^{\infty} M(n) \left(\frac{(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}}{n^{\frac{1}{2}} \cdot (n+1)^{\frac{1}{2}}}\right) \leq \sum_{n=1}^{\infty} \frac{M(n)}{2n\sqrt{n}} \\ &= \sum_{n=1}^{\infty} \frac{M(n)}{2n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{c}{c-1} \frac{(0.5\sqrt{n} + 1)}{2n\sqrt{n}} = \frac{0.5 \cdot c}{c-1} \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \infty \end{aligned}$$

the series diverges.

And when $\sigma = \frac{1}{2} + \varepsilon > \frac{1}{2}$ we have

$$\begin{aligned} \frac{1}{\zeta\left(\frac{1}{2} + \varepsilon\right)} &= \sum_{n=1}^{\infty} \frac{M(n)}{n^{\frac{1}{2} + \varepsilon}} \leq \frac{0.5 \cdot c}{c-1} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \leq \frac{0.5 \cdot c}{c-1} \int_1^{\infty} \frac{1}{n^{1+\varepsilon}} dn \\ &= \frac{0.5 \cdot c}{c-1} = 0.25 \cdot \frac{c}{\varepsilon \cdot (c-1)} \end{aligned}$$

the series converges, where ε is an arbitrary small number.

Therefore, the series $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ converges uniformly for $\sigma = \frac{1}{2} + \varepsilon > \frac{1}{2}$, and since it is a function $\frac{1}{\zeta(s)}$ if $\sigma > 1$, for the theorem of analytic continuation, it is also at its $\frac{1}{2} < \sigma \leq 1$. Therefore, the Riemann's hypothesis is

true.

The theorem is proved.

2. Theorem 4. All Non-Trivial Zeros of Dirichle's Function

$L(s, \chi)$ Have a Real Part That Is Equal to $\sigma = \frac{1}{2}$

Proof. Let's consider the Dirichle's series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s = \sigma + it, \quad (18)$$

where χ is the character of modulus m .

There is $\varphi(m)$ of such series where φ is the Euler's function. Since $|\chi(n)| \leq 1$, the series (18) converges when $\sigma > 1$, as can be seen from a comparison of this series with the series $\sum \frac{1}{n^s}$. We denote it by the sum through series $L(1, \chi)$. For various characters χ , we obtain different functions $L(s, \chi)$. They are called L is the Dirichle's functions. In studying the properties of these functions it is convenient to distinguish the cases where χ is the main character χ_1 and when $\chi \neq \chi_1$.

2.1) If $\chi \neq \chi_1$ than the series (18) converges in the half-plane $\sigma > 0$. Let us show from the beginning, that the partial sums $\sum_{n < x} \chi(n)$ are limited. We divide

the integer number from 1 to $[x]$ into classes of deductions by $\text{mod } m$ and write $[x] = m \cdot q + r$, $0 \leq r \leq m - 1$. Then

$$\sum_{n < x} \chi(n) = \sum_{n=1}^{[x]} \chi(n) = \left(\sum_1^m + \sum_{m+1}^{2m} + \dots + \sum_{m(n-1)+1}^{mq} \right) \cdot \chi(n) + \sum_{mq+1}^{mq+r} \chi(n).$$

Because of the orthogonality relations

$$\sum_{n(\text{mod } m)} \chi(n) = \begin{cases} \varphi(m), & \text{than } \chi = \chi_1 \\ 0, & \text{than } \chi \neq \chi_1 \end{cases}$$

we have

$$\sum_{n < x} \chi(n) = \sum_{mq+1}^{mq+r} \chi(n),$$

hence

$$\left| \sum_{n < x} \chi(n) \right| \leq \sum_{mq+1}^{mq+r} |\chi(n)| \leq r < m.$$

Since $n^{-\sigma}$ at $\sigma > 0$ decreases monotonically and tends to zero when $n \rightarrow \infty$, then the series $\sum \chi(n)/n^s$ converges for real $s = \sigma > 0$, and, consequently, for all s in the half-plane $\sigma > 0$ when $\chi \neq \chi_1$. If, however, $\sigma < 0$, then this the series obviously diverge. It's abscissa converges $\sigma_0 = 0$ and the abscissa of absolute convergence $\bar{\sigma} = 1$. By the theorem 4, The Dirichle's series $\sum_{n=1}^{\infty} a_n \cdot n^{-s}$ in the half-plane of the convergence is a regular analytic

function from s , the successive derivatives of which are obtained by the term differentiation of this the series ([6], p. 153), the function $L(s, \chi)$, $\chi \neq \chi$ is a regular analytic function from s when $\sigma > 0$.

2.2) If $\chi = \chi_1$ we use

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\mu(n) \cdot \chi(n)}{n^s}, \text{ Re } s \geq \frac{1}{2}. \tag{19}$$

From the theorem 3 it follows that $L(s, \chi) \neq 0$ when $\sigma \geq \frac{1}{2}$. If χ_1 is the main character by mod m , then

$$\chi_1(a) = \begin{cases} 1, & \text{then } (a, m) = 1 \\ 0, & \text{then } (a, m) > 1 \end{cases}$$

Using the condition $|\chi(n) \cdot n^{-s}| \leq n^{-\sigma}$ the function (19) can be written as $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\mu(n) \chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$, when $\chi(n) = 1$ and $\sigma \geq \frac{1}{2}$.

Using the results of the theorem 3, it can be argued that the generalized Riemann’s hypothesis is true, and accordingly to it: “All non-trivial zeros of the Dirichle’s functions have a real part equal to $\sigma = \frac{1}{2}$ ”.

The theorem 4 is proved.

3. Appendix. Disproof of the Mertens Hypothesis

The refutation of the Mertens hypothesis can be found on the basis of the proof of the Riemann hypothesis given in this paper. Take the series

$$f_1 = (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n}) < \left(n^{\frac{1}{\sqrt{n}}} \cdot (n - \sqrt{n} + 1) + (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n}) \right)$$

and it can be written in the form

$$\begin{aligned} & \left(n^{\frac{1}{\sqrt{n}}} \cdot (n - \sqrt{n} + 1) + (\sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n}) \right) \\ & < n + 2.5 \cdot \sqrt[2]{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n} - \sqrt{n} + 1 \\ & = n + 1.5 \cdot \sqrt{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n} + 1, \end{aligned}$$

where

$$f_2(n) = \sqrt{n} + \sqrt[3]{n} + \dots + \sqrt[n]{n} < 2.5\sqrt{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n}.$$

Then we can write

$$N + 2.5 \cdot \sqrt{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n} - \sqrt{n} + 1 > 0$$

or

$$N > -1.5 \cdot \sqrt{n} - 2 \cdot \sqrt[3]{n} + 2.5 \cdot \sqrt[4]{n} - 1 \tag{20}$$

The value N from the expression 3 is substituted instead of N , we obtain

$$N = 1 + \Pi + T^n + T^n + K^k + K(n)$$

From the expression (20) we obtain

$$1 + \Pi + T^n + T^n + K^k + K(n) > -1.5 \cdot \sqrt{n} - 2 \cdot \sqrt[3]{n} + 2.5 \cdot \sqrt[4]{n} - 1. \quad (21)$$

The properties of Mobius function ([5], p. 3) will be applied to the expression (21) and we obtain that

$$M(n) > -1.5 \cdot \sqrt{n} - 2 \cdot \sqrt[3]{n} + 2.5 \cdot \sqrt[4]{n} - 1$$

or

$$M(n) < 1.5 \cdot \sqrt{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n} + 1.$$

It will be the smallest value of the function of Mertens $M(n)$ and the biggest value for the function of Mertens $M(n)$. From the expression 6

$$\phi_1\left(n^{\frac{1}{n}}\right) = 1 + n^{\frac{1}{n}} + \dots + n^{\frac{1}{2}}$$

we find that

$$N - \sqrt{n} > 0, \quad (22)$$

When $(\sqrt[n]{n} - 1) \approx \frac{2}{\sqrt{n}}$ and $n \rightarrow \infty$ ([3], p. 67).

We write the expression 22 in the form

$$1 + \Pi + T^n + T^n + K^k + K(n) > \sqrt{n}. \quad (23)$$

Let us apply the properties of Mobius function to the expression (23) and we obtain

$$M(n) > \sqrt{n}.$$

Then we can state that the function of Mertens $M(n)$ is within

$$\sqrt{n} < M(n) < 1.5 \cdot \sqrt{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n} + 1$$

And it rejects the hypothesis of Mertens.

4. Conclusion

In the article, based on the finite exponential functional series and the finite exponential functional progressions, we prove the generalized Riemann's hypothesis, as well as the Riemann hypothesis. It is shown that in the Riemann's hypothesis $\sigma = \frac{1}{2}$. In the annex to the article, the Mertens hypothesis is refuted.

In the refuted Mertens hypothesis it is shown that the Mertens function is within $\sqrt{n} < M(n) < 1.5 \cdot \sqrt{n} + 2 \cdot \sqrt[3]{n} - 2.5 \cdot \sqrt[4]{n} + 1$.

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