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Chaos, Mixing, Weakly Mixing and Exactness

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Abstract

In this paper, new definitions of chaos, exact chaos, mixing chaos, and weak mixing chaos called θ-chaos, θ exact chaos, θ-mixing chaos are introduced and extended to topological spaces. Our purpose is to investigate other types of transitivity, chaos and mixing, because when we confirm not existence of θ-transitivity we confirm not existence of other types of transitive functions and its absence cannot find other types of transitivity. So the author must confirm the existence of this type of transitivity. We have proved that these chaotic definitions are all preserved under θr-conjugation.

Subject Areas

Topological Dynamics

Keywords

θ-Chaotic Maps, θ-Mixing, Topological Exactness, Weak Mixing

1. Introduction

In this research paper, new types of minimal systems, transitivity and exactness are introduced and studied. This is intended as a survey article on transitivity and chaoticity of a discrete system given by θ-irresolute self-map of a topological space. On one hand, it introduces postgraduate students to the study of new types of exactness, minimal systems and chaotic maps and gives an overview of results on the topic, but, on the other hand, it covers some of the recent developments of dynamics, technology, electronic and computer science. We denote the interior and the closure of a subset A of X by $Int(A)$ and $Cl(A)$ respectively. By a space X, we mean a topological space (X,τ). A point $x \in X$ is called a θ-adherent point of A [1], if $A \cap Cl(U) \neq \phi$ for every open set U containing x. The set of all θ-adherent points of a subset A of X is called the θ-closure of A and is denoted by $Cl_\theta(A)$. A subset A of X is called θ-closed if

$A = Cl_\theta(A)$. Dontchev and Maki [2] have shown that if A and B are subsets of a space X, then $Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$ and that $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$. Recall that a space (X, τ) is Hausdorff if and only if every compact set is θ-closed. The complement of a θ-closed set is called a θ-open set. The family of all θ-open sets forms a topology on X and is denoted by τ^θ. This topology is coarser than τ and that a space (X, τ) is regular if and only if $\tau = \tau^\theta$ [3]. Note also that the θ-closure of a given set needs not be a θ-closed set. Our purpose is to investigate some new types of transitivity, because when we confirm not existence of θ-transitivity we can't confirm the existence of other types of transitive functions and then we can't study chaos theory, *i.e.* theta-transitivity absence cannot find other types of transitive maps therefore cannot find chaotic maps. For more knowledge about transitivity and chaotic maps see references [4] and [5].

2. Preliminaries and Definitions

Definition 2.1

A point $x \in X$ is said to be A θ-interior point of A, if there exists an open set U containing x such that $U \subset Cl(U) \subset A$.

The set of all θ-interior points of A is said to be the θ-interior of A, and is denoted by $Int_\theta(A)$. It is obvious that an open set U in X is θ-open if $Int_\theta(A) = U$.

Definition 2.2

1) A map $h: X \to Y$ is a homeomorphism if it is continuous, bijective and has a continuous inverse.

2) A map $h: X \to Y$ is θr-homeomorphism if it is bijective and thus invertible and both h and h^{-1} are θ-irresolute.

3) The systems $f: X \to X$ and $g: Y \to Y$ are topologically conjugate or conjugate if there is a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$

4) The systems $f: X \to X$ and $g: Y \to Y$ are topologically θr conjugate or θr conjugate if there is θr-homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$

Definition 2.3 A map f is said to be transitive (resp., θ-transitive [6]) if for any non-empty open (resp., θ-open) sets U and V in X, there exists $n \in \mathbf{N}$ such that $f^n(U) \cap V \neq \phi$.

Kaki definition 2.4 A map f *is* said to be 1-transitive, if for every $x, y \in X$, there exists $n \in \mathbf{N}$ such that $f^n(x) = y$.

Theorem 2.5 every 1-transitive implies transitive.

Proof:

We have to prove that for any non-empty open sets U and V in X, there exists $n \in \mathbf{N}$ such that $f^n(U) \cap V \neq \phi$. Now, since U and V are non-empty so there is $x \in U$ and $y \in V$ so there is $n \in \mathbf{N}$ such that $f^n(x) = y$ since f is 1-transitive and $y = f^n(x) \in f^n(U)$ but $y \in V$, we have $y \in f^n(U) \cap V$ *i.e.* $f^n(U) \cap V \neq \phi$.

Definition 2.6

The points x is called a non-wandering point if for every open set U containing x there is an integer $n>0$ such that $f^n(U)\cap U\neq\phi$.

Definition 2.7

The non-wandering set of a map *f*, $NW(f)$, includes the points x such that for every open set U containing x there is an integer $n>0$ such that $f^n(U)\cap U\neq\phi$.

Definition 2.8

The points x is called theta-non-wandering point if for every θ-open set U containing x there is an integer $n>0$ such that $f^n(U)\cap U\neq\phi$.

Definition 2.9

The theta-non-wandering set of a map *f*, $NW_\theta(f)$, includes the points x such that for every θ-open set U containing x there is an integer $n>0$ such that $f^n(U)\cap U\neq\phi$.

Proposition 2.10 Every non-wandering point is a theta-nonwandering point but not conversely.

3. Action of a Group on a Topological Space

If G is a group and X is a topological space, then a *group action* φ of G on X is a function $\varphi: G\times X\to X$ such that $\varphi(g,x)=\varphi_g(x)$ that satisfies the following three axioms [7] and [8]:

1) φ_g is continuous, for all g in G

2) Identity: $\varphi_g(x)=x$ for all x in X. (Here, e denotes the neutral element of the group G.)

3) Compatibility $\varphi_{gh}(x)=\varphi_g(\varphi_h(x))$ for all g, h in G and all x in X.

The group G is said to act on X (on the left). The set X is called a (*left*) *G-set*.

Definition 3.1

1) The action of G on X is called 1-t*ransitive* if X is non-empty and if for each pair x, y in X there exists a g in G such that $\varphi_g(x)=y$.

2) The action of G on X is called topologically-t*ransitive* if X is non-empty and if for each non-empty pair $U,V\subset X$, there exists a g in G such that $\varphi_g(U)\cap V\neq\phi$.

Theorem 3.2 Every 1-transitive implies topologically-t*ransitive*.

Proof: The same technique of theorem 2.5.

Definition 3.3 The action of G on X is called n-t*ransitive* if X is non-empty and if for any two ordered sets of n different points $\{x_1,\cdots,x_n\}$ and $\{y_1,\cdots,y_n\}$ *in* X there exists a g in G such that $\varphi_g(x_i)=y_i$ for $i=1,\cdots,n$.

Definition 3.4 The action of G on X is called topologically n-t*ransitive* if X is non-empty and if for each non-empty pair $U_i,V_i\subset X$, such that $U_i\cap U_j=\phi$ and $V_i\cap V_j=\phi$ for $i\neq j$ there exists a g in G such that $\varphi_g(U_i)\cap V_i\neq\phi$ for $i=1,\cdots,n$.

Definition 3.5 Relative to ϕ_g a point $x\in X$ is called wandering if there is an open U containing x such that $\phi_g(U)\cap U=\phi$ for all $g\in G$.

Definition 3.6

1) A point $x \in X$ is θ-recurrent if, for every θ-open set U containing x, infinitely many $n \in \mathbf{N}$ satisfy $g^n(x) \in U$.

2) Let X be a topological space, $g: X \to X$ be θ-irresolute map, then the function g is called topologically θ-strongly mixing if, given any nonempty θ-open subsets $U, V \subseteq X$ $\exists N \geq 1$ such that $g^n(U) \cap V \neq \phi$ for all $n > N$.

3) A subset B of X is g-invariant if $g(B) \subset B$. A non-empty θ-closed invariant subset B of X is θ-minimal, if $Cl_\theta(O_g(x)) = B$ for every $x \in B$. A point $x \in X$ is θ-minimal if it is contained in some θ-minimal subset of X

Theorem 3.7 if g is topologically θ-strongly mixing then it is also θ-transitive but not conversely.

Definition 3.8

1) The function f is θ-exact if, for every nonempty θ-open set U, there exists some $n \in \mathbf{N}$ such that $g^n(U) = X$.

2) The function g is (topological) θ-transitive (resp., θ-mixing) if for any two nonempty θ-open sets $U, V \subset X$, there exists some $n \in \mathbf{N}$ such that $g^n(U) \cap V \neq \phi$. (resp. $g^m(U) \cap V \neq \phi$ for all $m \geq n$).

3) The function g is weak θ-mixing if $g \times g$ is θ-transitive on $X \times X$

4) The θ-mixing function $g: X \to X$ is pure θ-mixing if and only if there exists θ-open set $U \subset X$ such that $g^n(U) \neq X$.

5) The function $g: X \to X$ is θ-chaotic if g is θ-transitive on X and the set of periodic points of g is θ-dense in X.

6) The function g is called θ-exact chaos (resp., θ-mixing chaos and weakly θ-mixing chaos) if g is θ-exact (resp., θ-mixing and weakly θ-mixing) and θ-chaotic function on the space X.

Theorem 3.9 Theorem 3.9 topological θ-exactness implies θ-mixing implies weakly θ-mixing implies θ-transitivity

Theorem 3.10 If $f: X \to X$ and $g: Y \to Y$ are topologically θr-conjugate by a θr-homeomorphism $h: X \to Y$. Then

1) The map f is θ-exact if and only if g is θ-exact

2) The map f is θ-mixing if and only if g is θ-mixing

3) The map f is θ-chaotic if and only if g is θ-chaotic

4) The map f is weakly θ-mixing if and only if g is weakly θ-mixing.

Theorem 3.11 if $f: X \to X$ and $g: Y \to Y$ are θr-conjugate via $h: X \to Y$. Then

1) T is θ transitive subset of $X \Leftrightarrow h(T)$ is θ-type transitive subset of Y;

2) T is θ-mixing subset of $X \Leftrightarrow h(T)$ is θ-mixing subset of Y.

Corollary A subset B in the space X is a chaotic set $\Leftrightarrow h(B)$ is a chaotic set in the space Y.

Proof (1)

Assume that $f: X \to X$ and $g: Y \to Y$ are topological systems which are topologically θr-conjugated by $h: X \to Y$. Thus, h is θr-homeomorphism (that is, h is bijective and thus invertible and both h and h^{-1} are θ-irresolute) and $h \circ f = g \circ h$

Suppose T is θ-type transitive subset of X. Let A, B be θ-open subsets of Y with $B\cap h(T)\neq\phi$ and $A\cap h(T)\neq\phi$ (to show $g^n(A)\cap B\neq\varphi$ for some $n>0$). $U=h^{-1}(A)$ and $V=h^{-1}(B)$ are θ-open subsets of X since h is an θ-irresolute.

Then there exists some $n>0$ such *that* $f^n(U)\cap V\neq\varphi$ *since* T is θ-type transitive subset of X, with $U\cap T\neq\phi$ and $V\cap T\neq\phi$. Thus (as $f\circ h^{-1}=h^{-1}\circ g$ implies $f^n\circ h^{-1}=h^{-1}\circ g^n$).

$\phi\neq f^n\left(h^{-1}(A)\right)\cap h^{-1}(B)=h^{-1}\left(g^n(A)\right)\cap h^{-1}(B)$

Therefore, $h^{-1}\left(g^n(A)\cap B\right)\neq\phi$ implies $g^n(A)\cap B\neq\phi$ since h^{-1} invertible. So $h(T)$ is θ-type transitive subset of Y.

Proof (2)

We only prove that if T is topologically θ-mixing subset of Y then $h^{-1}(T)$ is also topologically θ-mixing subset of X. Let U, V be two θ-open subsets of X with $U\cap h^{-1}(T)\neq\phi$ and $V\cap h^{-1}(T)\neq\phi$. We have to show that there is $N>0$ such that for any $n>N$, $f^n(U)\cap V\neq\phi$. $h^{-1}(U)$ and $h^{-1}(V)$ are two θ-open sets since h is θ-irresolute with $h^{-1}(V)\cap T\neq\phi$ and $h^{-1}(U)\cap T\neq\phi$. If the set T is topologically λ-mixing then there is $N>0$ such that for any $n>M$, $g^n\left(h^{-1}(U)\right)\cap h^{-1}(V)\neq\phi$. So $\exists x\in g^n\left(h^{-1}(U)\right)\cap h^{-1}(V)$. That is $x\in g^n\left(h^{-1}(U)\right)$ and $x\in h^{-1}(V)$ $\Leftrightarrow$ $x=g^n(y)$ for $y\in h^{-1}(U)$. $h(x)\in V$. Thus, since $h\circ g^n=f^n\circ h$, so that $h(x)=h\left(g^n(y)\right)=f^n\left(h(y)\right)\in f^n(U)$ and we have $h(x)\in V$ that is $f^n(U)\cap V\neq\phi$. So, $h^{-1}(T)$ is θ-mixing set.

Theorem 3.12 Let (X,f) be a topological system and A be a nonempty θ-closed set of X. Then the following conditions are equivalent.

1) A is a θ-transitive set of (X,f).

2) Let V be a nonempty θ-open subset of A and U be a nonempty θ-open subset of X with $U\cap A\neq\phi$. Then ther exists $n\in\mathbf{N}$ such that $V\cap f^{-n}(U)\neq\phi$.

3) Let U be a nonempty θ-open set of X with $U\cap A\neq\phi$. Then $\bigcup_{n\in\mathbf{N}}f^{-n}(U)$ is θ-dense in A.

Theorem 3.13 *Let (X,f) be topological system and A be a nonempty θ-closed invariant set of X. Then A is a θ-type transitive set of (X,f) if and only if (A,f) is θ-type transitive set.*

Proof:

$\Rightarrow$) Let V_1 and U_1 be two nonempty θ-open subsets of A. For a nonempty θ-open subset U_1 of A, there exists a θ-open set U of X such that $U_1=U\cap A$ Since A is a θ-type transitive set of (X,f), there exists $n\in\mathbf{N}$ such that $f(V_1)\cap U\neq\phi$. Moreover, A is invariant, *i.e.*, $f(A)\subset A$, which implies that $f(A)\subset A$. Therefore, $f(V_1)\cap A\cap U\neq\phi$, *i.e.* $f(V_1)\cap U_1\neq\phi$. This shows that (A,f) is θ-type transitive.

$\Leftarrow$) Let V_1 be a nonempty θ-open set of A and U be a nonempty θ-open set of X with $A\cap U\neq\phi$, Since U is an θ-open set of X and $A\cap U\neq\phi$, it follows that $U\cap A$ is a nonempty θ-open set of A. Since (A,f) is topologically θ-type transitive, there exists $n\in\mathbf{N}$ such that $f(V_1)\cap(A\cap U)\neq\phi$, which implies that $f(V_1)\cap U\neq\phi$. This shows that A is a θ-type transitive set of

(X, f).

We now introduce four definitions of dense orbit and transitivity as follows:

Definition 3.14 Let X be a topological space.

1) A map f *is* said to have θ-dense *orbit* if there exists $x \in X$ such that $Cl_\theta(O_f(x)) = X$;

2) A map f *is* said to have strictly dense orbit (resp., strictly θ*-dense orbit),* if there exists $x \in X$ such that $Cl(O_f(f(x))) = X$ (resp. $Cl_\theta(O_f(f(x))) = X$);

3) A map f is said to be ω*-transitive* if there exists $x \in X$ such that $\omega(x, f) = X$;

4) A map f is said to be transitive (resp., θ*-transitive)* if for any non-empty open (resp., θ-open) sets U and V in X, there exists $n \in \mathbf{N}$ such that $f^n(U) \cap V \neq \phi$.

Now, we will discuss the relations between these orbits and transitivity maps in the theorem below:

Theorem 3.15

1) For any topological space X, each strictly θ-dense orbit is α θ-dense orbit, and each ω-transitive map has strictly dense orbit and it is transitive.

2) For any topological space X, each continuous map which has strictly dense orbit is ω-transitive and transitive.

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