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T_3 Tree and Its Traits in Understanding Integers

Xingbo Wang[1,2]

[1]Department of Mechatronics, Foshan University, Foshan City, China
[2]Guangdong Engineering Center of Information Security for Intelligent Manufacturing System, Foshan City, China
Email: xbwang@fosu.edu.cn; 153668@qq.com

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Abstract

Valuated binary tree is emerging its magical characteristics in study of integers. As the father tree of all valuated binary subtrees, T_3 tree plays a very important role in studying the odd integers for its owning both the common properties of a general valuated binary tree and its own traits of covering all the odd integers bigger than 1. The article investigates the properties of the T_3 tree as well as the multiplications on the tree. It exhibits the distribution of a node's multiples, the distribution of the product of two nodes and the distribution of the square of a node. Some other contents such as fundamental properties of the T_3 tree and path connecting a product to its divisor-nodes are also investigated with detail mathematical proofs. The theorems and corollaries proved in the article lay a foundation for future work.

Keywords

Number Theory, Valuated Binary Tree, Multiplication

1. Introduction

Number theory is an ancient subject in mathematics and it has attracted people's interests because it has continuously left kinds of amazing problems for human being. Classical approach of studying number theory can be seen to put integers on a line called *number axis*. It is of course a thinking in one dimensional space. In 2016, article [1] proposed a new approach to study integers by putting odd integers bigger than 1 on a full binary tree from top to bottom and from left to right and called the new approach a *valuated binary tree* approach. The approach is of course a two-dimensional thinking. Following the article's study, articles [2] and [3] further investigated and proved some properties of the valuated binary tree and its subtrees, article [4] revealed the genetic properties of odd integers and obtained a new approach to factorize odd integers with the help of the valuated binary tree, and article [5] developed a parallel method to factorize odd integers by applying the new factoring approach on parallel computing systems. Undoubtedly, the approach of the valuated binary tree is emerging its characteristic in study of integers.

The T_3 tree, namely, the 3-rooted valuated binary tree, is the father tree of all subtrees. Owning both the common properties of a general valuated binary tree and its own traits of covering all the odd integers bigger than 1, it obviously plays a very important role in understanding the integers. This article introduces fundamental properties of the T_3 tree as well as multiplication related properties of the tree so as for people to know more about the valuated binary in study of integers.

2. Preliminaries

2.1. Definitions and Notations

A valuated binary tree $\boldsymbol{T}$ is such a binary tree that each of its nodes is assign a value. An odd-number N-rooted tree, denoted by T_N is a recursively constructed valuated binary tree whose root is the odd number N with $2N-1$ and $2N+1$ being the root's left and right sons, respectively. The left son is said to have a left attributive and the right son is to have a right attributive. Each son is connected with its father with a *path*, but there is no path between the two sons. T_3 tree, or briefly called T_3, is the case $N=3$. The root of the T_3 is assigned a right attributive. For convenience, symbol $N_{(k,j)}$ is by default the node at the j^{th} position on the k^{th} level of T_3, where $k \ge 0$ and $0 \le j \le 2^k - 1$; symbol $T_{N_{(k,j)}}$ denotes a subtree whose root is $N_{(k,j)}$ and symbol $N_{(i,\omega)}^{N_{(k,j)}}$ denotes the node at the ω^{th} position on the i^{th} level of $T_{N_{(k,j)}}$. Level m is said to be higher than level n if $m<n$ and it is said to be lower than level n if $m>n$. Symbol $P(A,B)$ is the path from node A to node B and the length of a path is calculated by the number of total nodes on the path subtracting by 1. Symbol $N_{\left(N_{(m,\alpha)} \ll \chi\right)}$ is the node where $N_{(m,\alpha)}$ slides down χ levels along the leftmost path of subtree $T_{N_{(m,\alpha)}}$ and symbol $N_{\left(N_{(m,\alpha)} \gg \chi\right)}$ is that $N_{(m,\alpha)}$ slides down χ levels along the rightmost path of subtree $T_{N_{(m,\alpha)}}$.

An odd interval $[a,b]$ is a set of consecutive odd numbers that take a as lower bound and b as upper bound, for example, $[3,11]=\{3,5,7,9,11\}$. Intervals in this whole article are by default the odd ones unless particularly mentioned. Symbol $\lfloor x \rfloor$ is the floor function, an integer function of real number χ that satisfies inequality $x-1<\lfloor x \rfloor \le x$, or equivalently $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.

2.2. Lemmas

Lemma 1 (Subtree Transition Law, See in [2]) *Let* $T_{N_{(m,s)}}$ *and* $T_{N_{(k,j)}}$ ($m \ge k$) *be two subtrees of* $\boldsymbol{T}$; *then it holds*

$$N_{(i,\omega)}^{N_{(m,s)}} = 2^i \left(N_{(m,s)} - N_{(k,j)}\right) + N_{(i,\omega)}^{N_{(k,j)}},$$
$$i = 1,2,\cdots;\omega = 0,1,\cdots,2^i - 1$$

Lemma 2 (See in [4]) *Let* $N_{(0,0)}$ *be an odd integer bigger than* 1 *and* $T_{N_{(0,0)}}$ *be the* $N_{(0,0)}$ *-rooted valuated binary tree; then the following statements hold*

1) There are 2^k nodes on the k^{th} level with $k = 0,1,2,\cdots$;

2) Node $N_{(k,j)}^{N_{(0,0)}}$ is computed by

$$N_{(k,j)}^{N_{(0,0)}} = 2^k N_{(0,0)} - 2^k + 2j + 1$$
$$k = 0,1,2,\cdots; j = 0,1,\cdots,2^k - 1$$

3) Two position-symmetric nodes, $N_{(i,\omega)}^{N_{(0,0)}}$ and $N_{(i,2^i-1-\omega)}^{N_{(0,0)}}$, satisfy

$$N_{(i,\omega)}^{N_{(0,0)}} + N_{(i,2^i-1-\omega)}^{N_{(0,0)}} = 2^{i+1} N_{(0,0)}$$

4) Level i ($i \ge 0$)of subtree $T_{N_{(k,j)}^{N_{(0,0)}}}$ ($k \ge 0$) is the level $(k+i)$ of $T_{N_{(0,0)}}$ and it contains 2^i nodes. Node $N_{(i,\omega)}^{N_{(k,j)}^{N_{(0,0)}}}$ of $T_{N_{(k,j)}^{N_{(0,0)}}}$ ($0 \le \omega \le 2^i - 1$) is corresponding to node $N_{(k+i,2^i j+\omega)}^{N_{(0,0)}}$ of $T_{N_{(0,0)}}$ by

$$N_{(i,\omega)}^{N_{(k,j)}^{N_{(0,0)}}} = N_{(k+i,2^i j+\omega)}^{N_{(0,0)}} = 2^i N_{(k,j)}^{N_{(0,0)}} - 2^i + 2\omega + 1$$
$$j = 0,1,\cdots,2^k - 1; i = 0,1,\cdots; \omega = 0,1,\cdots,2^i - 1$$

Lemma 3 (See in [1] & [6]) Let p be a positive odd integer; then among p consecutive positive odd integers there exists one and only one that can be divisible by p. Let q be a positive odd number and S be a finite set that is composed of consecutive odd numbers; then S needs at least $(n-1)q+1$ elements to have n multiples of q.

3. Fundamental Properties of T_3

Theorem 1. The T_3 Tree has the following fundamental properties.

(**P0**) Every node is an odd integer and every odd integer bigger than 1 must be a node of the T_3 tree. For an arbitrary positive integer k > 0, the odd integer of the form $4k+1$ is a left node while the odd integer of the form $4k+3$ is a right node.

(**P1**) On the k^{th} level with $k = 0,1,\cdots$, there are 2^k nodes starting by $2^{k+1}+1$ and ending by $2^{k+2}-1$, namely, $N_{(k,j)} \in \left[2^{k+1}+1, 2^{k+2}-1\right]$ with $j = 0,1,\cdots,2^k - 1$.

(**P2**) $N_{(k,j)}$ is calculated by

$$N_{(k,j)} = 2^{k+1} + 1 + 2j, j = 0,1,\cdots,2^k - 1$$

(**P3**) Nodes $N_{(k+1,2j)}$ and $N_{(k+1,2j+1)}$ on the $(k+1)^{th}$ level are respectively the left son and the right son of node $N_{(k,j)}$ on the k^{th} level. The descendants of $N_{(k,j)}$ on the $(k+i)^{th}$ level are $N_{(k+i,2^i j+\omega)}$ ($0 \le \omega \le 2^i - 1$), which are

$$N_{(k+i,2^i j)}, N_{(k+i,2^i j+1)}, N_{(k+i,2^i j+2)}, \cdots, N_{(k+i,2^i j+2^i-1)}$$

Particularly,

$$N_{(N_{(m,\alpha)} \ll \chi)} = 2^{\chi}\left(N_{(m,\alpha)} - 1\right) + 1$$

and

$$N_{\left(N_{(m,\alpha)} \gg \chi\right)} = 2^{\chi}\left(N_{(m,\alpha)} + 1\right) - 1$$

(**P4**) The biggest node on the k^{th} level of the left branch is $2^{k+1} + 2^k - 1$ whose position is $j = 2^{k-1} - 1$, and the smallest node on the k^{th} level of the right branch is $2^{k+1} + 2^k + 1$ whose position is $j = 2^{k-1}$.

(**P5**) Node $N_{(i,\omega)}$ and node $N_{\left(i,2^i-1-\omega\right)}$ are at the symmetric positions on the i^{th} level and it holds

$$N_{(i,\omega)} + N_{\left(i,2^i-1-\omega\right)} = 2^{i+1} \times 3$$

(**P6**) Multiplication of arbitrary two nodes of T_3, say $N_{(m,\alpha)}$ and $N_{(n,\beta)}$, is a third node of T_3. That is,

$$N_{(m,\alpha)} \times N_{(n,\beta)} = \left(2^{m+1} + 1 + 2\alpha\right) \times \left(2^{n+1} + 1 + 2\beta\right) \in T_3$$

(**P7**) The binary representation of node $N_{(k,j)}$ takes $k+2$ bits with the least significant bit and the most significant bit being 1, that is

$$N_{(k,j)} = (1\underbrace{n_k n_{k-1} \cdots n_1}_{k+2}1)_2$$

(**P8**) Odd integer N with $N > 1$ lies on level $\lfloor \log_2 N \rfloor - 1$.

Proof. Omit the proofs for (**P0**) to (**P6**). (**P7**) can be proved by mathematical induction. Here only prove (**P8**). By (**P1**) it yields $k + 1 \le \log_2 N_{(k,j)} < k + 2$, that is

$$\log_2 N_{(k,j)} - 2 < k \le \log_2 N_{(k,j)} - 1$$

By definition of the floor function, it knows $k = \left\lfloor \log_2 N_{(k,j)} \right\rfloor - 1$.

4. Multiples and Divisors on T_3

This section presents relationship between a node and its multiples in T_3 tree.

Theorem 2. The following claims are true.

1) On the same level of T_3, there is not a node that is a multiple of another one.

2) An arbitrary subtree of T_3 contains a multiple of 3 on level 2, a multiple of 5 on level 3, a multiple of 7 on level 4 and so on. Or for a given integer $n > 1$ a subtree of T_3 must contain a multiple of $2n - 1$ on level n counted from the subtree's root that stands on level 0.

3) For integer $k \ge 1$, a node of T_3 on level k must have at least 2^{k-1} multiples on level $2k + 1$ of T_3; for integer $k \ge 0$, a node of T_3 on level k must have at least 2^{2k} multiples on level $2(k+1)$ of T_3.

Proof. The first claim is seen in [1]. Here only prove (2) and (3). The n level of an arbitrary subtree contains 2^n consecutive odd numbers. Since an arbitrary integer $n > 1$ yields $2^n > 2n - 1$, by Lemma 3 it knows that the claim (2) holds. Note that a node $N_{(k,j)}$ on level k of T_3 satisfies $2^{k+1} + 1 \le N_{(k,j)} \le 2^{k+2} - 1$ and there are respectively 2^{2k+1} nodes and $2^{2(k+1)}$

nodes on level $2k+1$ and level $2(k+1)$. Now consider the case $k\ge 1$. Since $2^{2k+1}=2^{k-1}\left(2^{k+2}-1\right)+2^{k-1}$, it can see that on level $2k+1$, there are at least 2^{k-1} consecutive sections each of which contain $2^{k+2}-1$ odd integers. By Lemma 3 it knows that there are at least 2^{k-1} multiples of $N_{(k,j)}$ on level $2k+1$. Similarly, when $k\ge 0$ it holds $2^{2(k+1)}=2^k\left(2^{k+2}-1\right)+2^k$, it knows that the level $2(k+1)$ contains at least 2^k multiples of $N_{(k,j)}$. □

Theorem 3. Among all nodes that lie on level k or a higher level in subtree $T_{N_{(k,j)}}$ with $k\ge 0$, there is not a node that is a multiple of $N_{(k,j)}$. Or node $N_{(k,j)}$ of T_3 cannot divide its direct descendants that lie on level $2k$ or a higher level in terms of T_3.

Proof. Use proof by contradiction. Without loss of generality, assume $N_{(k,j)}$ can divide $N_{(m,s)}^{N_{(k,j)}}$ with $0<m\le k$ and $0\le s\le 2^m-1$, namely, $N_{(m,s)}^{N_{(k,j)}}=\alpha N_{(k,j)}$ with α being an odd integer; then by $N_{(m,s)}^{N_{(k,j)}}=\alpha N_{(k,j)}$ and

$$N_{(m,s)}^{N_{(k,j)}}=2^m N_{(k,j)}-2^m+2s+1$$

it leads to

$$\alpha N_{(k,j)}=2^m N_{(k,j)}-2^m+2s+1$$

namely

$$\left(2^m-\alpha\right)N_{(k,j)}=2^m-2s-1$$

which indicates $N_{(k,j)}$ can divide $\vartheta=2^m-2s-1$ or $N_{(k,j)}=\vartheta=2^m-2s-1$ in the case $2^m-\alpha=1$. Since the minimal value of s is 0, it yields

$$\vartheta\le 2^m-1$$

which shows that, due to $0<m\le k$, $\vartheta\le 2^m-1<2^{k+1}+1\le N_{(k,j)}$. This means that ϑ is impossible to be divisible by $N_{(k,j)}$ and neither holds $N_{(k,j)}=2^m-2s-1$. Hence contradiction occurs and the first part of the theorem holds. The second part surely holds by Lemma 2(4). □

Theorem 3 is called a ***law of inbreeding avoidance*** and it can be intuitionally described with Figure 1, in which the ***inbred area*** has no multiple of $N_{(k,j)}$.

Theorem 4. If node $N_{(k,j)}$ of T_3 tree is composite, it must have a divisor that lies on level $\left\lfloor \frac{k}{2}\right\rfloor$ or a higher level.

Proof. Use proof by contradiction. Assume p_1 and p_2 are two divisors of

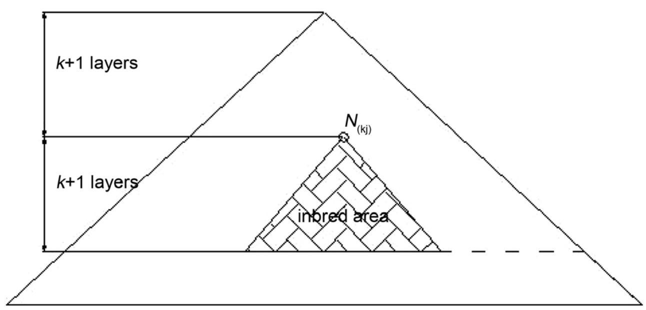

Figure 1. Inbreeding avoidance.

$N_{(k,j)}$ and each of the two lies on level $\left\lfloor \frac{k}{2} \right\rfloor + 1$ or lower levels; then $p_1 \geq 2^{2+\left\lfloor \frac{k}{2} \right\rfloor} + 1$ and $p_2 \geq 2^{2+\left\lfloor \frac{k}{2} \right\rfloor} + 1$; thus

$$p_1 p_2 \geq \left(2^{2+\left\lfloor \frac{k}{2} \right\rfloor} + 1 \right)^2 = 2^{4+2\left\lfloor \frac{k}{2} \right\rfloor} + 2^{3+\left\lfloor \frac{k}{2} \right\rfloor} + 1$$

Referring to (**P31**) in [6], it knows $2 + 2\left\lfloor \frac{k}{2} \right\rfloor \geq k + 1$ for an arbitrary integer $k \geq 0$; hence

$$p_1 p_2 > 2^{2+k+1} + 1 > 2^{k+3} + 1$$

which is contradictory to the fact that $N_{(k,j)}$ lies on level k.

Corollary 1* If a node $N_{(k,j)}$ ($k > 0$) of T_3 is a composite number and it has a divisor d on level $\left\lfloor \frac{k}{2} \right\rfloor$; then another factor $\frac{N_{(k,j)}}{d}$ must lie on level $\left\lfloor \frac{k}{2} \right\rfloor$ or level $\left\lfloor \frac{k}{2} \right\rfloor - 1$ or level $\left\lfloor \frac{k}{2} \right\rfloor - 2$.

Proof. Considering $N^2_{\left(\left\lfloor \frac{k}{2} \right\rfloor, 0\right)}$ and $N^2_{\left(\left\lfloor \frac{k}{2} \right\rfloor, 2^{\left\lfloor \frac{k}{2} \right\rfloor} - 1\right)}$, the square of the smallest and the largest nodes on level $\left\lfloor \frac{k}{2} \right\rfloor$ respectively, and $N_{\left(\left\lfloor \frac{k}{2} \right\rfloor, 0\right)} \times N_{\left(\left\lfloor \frac{k}{2} \right\rfloor + 1, 0\right)}$, multiplication of the two smallest nodes on level $\left\lfloor \frac{k}{2} \right\rfloor$ and $\left\lfloor \frac{k}{2} \right\rfloor + 1$ respectively, it yields

$$2^{k+1} + 1 < N^2_{\left(\left\lfloor \frac{k}{2} \right\rfloor, 0\right)} = \left(2^{\left\lfloor \frac{k}{2} \right\rfloor + 1} + 1 \right)^2 < 2^{k+3} - 1$$

and

$$2^{k+2} + 1 < N^2_{\left(\left\lfloor \frac{k}{2} \right\rfloor, 2^{\left\lfloor \frac{k}{2} \right\rfloor} - 1\right)} = \left(2^{\left\lfloor \frac{k}{2} \right\rfloor + 2} - 1 \right)^2 < 2^{k+4} - 1$$

Similarly, it holds

$$2^{k+2} + 1 < N_{\left(\left\lfloor \frac{k}{2} \right\rfloor, 0\right)} \times N_{\left(\left\lfloor \frac{k}{2} \right\rfloor + 1, 0\right)} < 2^{k+4} - 1$$

These inequalities show that, multiplication of any two small nodes on level $\left\lfloor \frac{k}{2} \right\rfloor$ lies on level k or level $k+1$ or even level $k+2$ and even the two smallest nodes one level $\left\lfloor \frac{k}{2} \right\rfloor$ and $\left\lfloor \frac{k}{2} \right\rfloor + 1$ respectively cannot have a multiplication that lies on level k since the multiplication lies on level $k+1$ or level $k+2$.

Corollary 1. A node $N_{(k,j)}$ of T_3 on level k is not divisible by every node on level $\left\lfloor \frac{k}{2} \right\rfloor$ and higher levels is a prime number. Furthermore, $N_{(k,j)}$ of T_3 on level k is not divisible by each prime-node on level $\left\lfloor \frac{k}{2} \right\rfloor$ or higher levels is a prime number.

5. Multiplications on T_3 Tree

This section presents basic law of multiplication of two nodes and properties of the square of a node in T_3 tree. Path connecting a multiple to its divisors are also investigated in this section.

5.1. Basic Law of Multiplication

Theorem 5. Multiplication of two left nodes or two right nodes results in a left node; multiplying a left node with a right node results in a right node.

Proof. (Omitted) □

Theorem 6. Suppose $m \ge 0$ and $n \ge 0$; then product $N_{(m,\alpha)} \times N_{(n,\beta)}$ ($\alpha = 0,1,\cdots,2^m - 1$; $\beta = 0,1,\cdots,2^n - 1$) lies on level $m+n+2=2$ in the case $m=n=0$; otherwise, it may lie from the position $J_{00} = 2^m + 2^n$ on level $m+n+1$ to the position $J_{mn} = 2^{m+n+2} - 2^{m+1} - 2^{n+1}$ on level $m+n+2$, totally $2^{m+n+2} - 2^{m+1} - 2^{n+1} - 2^m - 2^n$ possible positions.

Proof.

$$\begin{aligned}
N_{(m,\alpha)} \times N_{(n,\beta)} &= \left(2^{m+1} + 1 + 2\alpha\right) \times \left(2^{n+1} + 1 + 2\beta\right) \\
&= 2^{m+n+2} + 2^{m+1}\left(1+2\beta\right) + 2^{n+1}\left(1+2\alpha\right) + 4\alpha\beta + 2\alpha + 2\beta + 1 \\
&= 2^{m+n+2} + 1 + 2\left(2^m\left(1+2\beta\right) + 2^n\left(1+2\alpha\right) + 2\alpha\beta + \alpha + \beta\right)
\end{aligned}$$

Let $J = 2^m\left(1+2\beta\right) + 2^n\left(1+2\alpha\right) + 2\alpha\beta + \alpha + \beta$; then

$$N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J$$

Hence when $0 \le J \le 2^{m+n+1} - 1$, $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n,J)}$ lies on level $m+n+1$ of T_3 and when $J > 2^{m+n+1} - 1$, $N_{(m+n,J)}$ lies on level $m+n+2$ or some lower level. Now consider the following cases.

Case (*i*) $m=n=0$. This time $\alpha = \beta = 0$ and thus $J = \left(1+2\beta\right) + \left(1+2\alpha\right) + 2\alpha\beta + \alpha + \beta = 2 > 1$. Since on level $m+n+1=1$, the maximal number of position is 1, it knows $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J$ lies on level 2, which is level $m+n+2$.

Case (*ii*) $m=0$ plus $n>0$, or $m>0$ plus $n=0$. Since *J* is a *symmetric expression* of m,n,α, and β, here only consider $m=0$ plus $n>0$. This time, $\alpha = 2^m - 1 = 0$ thus $J = \left(1+2\beta\right) + 2^n + \beta$ takes its minimal value at $\beta = 0$ and maximal value at $\beta = 2^n - 1$. Let the minimal value and the maximal value be respectively J_{00} and J_{mn}; direct calculation shows

$$J_{00} = 2^n + 1,\quad J_{mn} = 2^{n+2} - 2$$

Since $J_{00} \le 2^{m+n+1} - 1$ (the equality holds only if $n=1$), the minimal value of

$N_{(m,\alpha)} \times N_{(n,\beta)}$, which is $2^{m+n+2} + 1 + 2J_{00}$, lies on level $m+n+1$.

Meanwhile, by $J_{mn} = 2^{n+2} - 2$ and $m = 0$ it knows

$$J_{mn} = 2^{n+2} - 2 = 2\left(2^{n+1} - 1\right) > 2^{n+1} - 1 = 2^{m+n+1} - 1$$

and

$$J_{mn} = 2^{n+2} - 2 < 2^{m+n+2} - 1$$

That is to say, $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n} + 1 + 2J_{mn}$ lies on level $m+n+2$.

Case (*iii*) $m > 0$ and $n > 0$. This time, $J = 2^m\left(1+2\beta\right) + 2^n\left(1+2\alpha\right) + 2\alpha\beta + \alpha + \beta$ takes its minimal value at $\alpha = \beta = 0$ and maximal value at $\alpha = 2^m - 1, \beta = 2^n - 1$. Direct calculation shows

$J_{00} = 2^m + 2^n$, $J_{mn} = 2^{m+n+2} + 2^{m+n+1} - 2^{m+1} - 2^{n+1}$

Note that, $m > 0$ and $n > 0$ yield $\frac{J_{00}}{2^{m+n+1}} = \frac{1}{2^{m+1}} + \frac{1}{2^{n+1}} < 1$, namely, $4 \le J_{00} = 2^m + 2^n \le 2^{m+n+1} - 1 < 2^{m+n+1}$; hence $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J_{00}$ lies on level $m+n+1$.

On the other hand, by $J_{mn} = 2^{m+n+2} + 2^{m+n+1} - 2^{m+1} - 2^{n+1}$, it holds

$$\begin{aligned}
&2^{m+n+2} + 1 + 2J_{mn} \\
&= 2^{m+n+2} + 1 + 2 \times 2^{m+n+2} + 2^{m+n+2} - 2^{m+2} - 2^{n+2} \\
&= 2^{m+n+3} + 1 + 2\left(2^{m+n+2} - 2^{m+1} - 2^{n+1}\right)
\end{aligned}$$

hence when $m \ge 1$ and $n \ge 1$, $0 \le \frac{2^{m+n+2} - 2^{m+1} - 2^{n+1}}{2^{m+n+2}} = 1 - \frac{1}{2^{m+1}} - \frac{1}{2^{n+1}} < 1$

and thus

$$0 \le 2^{m+n+2} - 2^{m+1} - 2^{n+1} \le 2^{m+n+2} - 1$$

Consequently,

$$\begin{aligned}
2^{m+n+3} + 1 \le N_{(m,\alpha)} \times N_{(n,\beta)} &= 2^{m+n+2} + 1 + 2J_{mn} \\
&= 2^{m+n+3} + 1 + 2\left(2^{m+n+2} - 2^{m+1} - 2^{n+1}\right) \\
&< 2^{m+n+3} + 1 + 2\left(2^{m+n+2} - 1\right)
\end{aligned}$$

which shows that $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J_{mn}$ lies on level $m+n+2$.

Now it can summarize that the smallest possible value of $N_{(m,\alpha)} \times N_{(n,\beta)}$ is $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J_{00}$, which lies on level $m+n+1$ at position $J_{00} = 2^m + 2^n$, and its maximal possible value is $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J_{mn}$, which lies on level $m+n+2$ at position $J_{mn} = 2^{m+n+2} - 2^{m+1} - 2^{n+1}$. Since there are totally 2^{m+n+1} nodes on level $m+n+1$, it knows that there are $2^{m+n+1} - 1 - 2^m - 2^n = 2^{m+n+1} - 2^m - 2^n - 1$ nodes from position J_{00} to position $2^{m+n+1} - 1$. Similarly, there are on level $m+n+2$ totally $2^{m+n+2} - 2^{m+1} - 2^{n+1} + 1$ nodes from position 0 to position J_{mn}. It knows that, $N_{(m,\alpha)} \times N_{(n,\beta)}$ has $2^{m+n+2} - 2^{m+1} - 2^{n+1} + 1 + 2^{m+n+1} - 2^m - 2^n - 1 = 2^{m+n+2} - 2^{m+1} - 2^{n+1} - 2^m - 2^n$ possible positions. □

Corollary 2. Suppose $m \ge 0$ and $n \ge 0$; then the product $N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J$ with $J = 2^m(1+2\beta) + 2^n(1+2\alpha) + 2\alpha\beta + \alpha + \beta$ and $0 \le \alpha \le 2^m - 1$ plus $0 \le \beta \le 2^n - 1$ lies on level $m+n+1$ and fits $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+1,J)}$ when $J < 2^{m+n+1}$, whereas it lies on level $m+n+2$ and satisfies $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+2,\chi)}$ with $\chi = J - 2^{m+n+1}$ when $J \ge 2^{m+n+1}$.

Proof. The case $J < 2^{m+n+1}$ has already been proved in the proof of Theorem 6. Now consider the case $J \ge 2^{m+n+1}$. Note that there are in all $N_{(m,\alpha)} \times N_{(n,\beta)} - 1$ integers from 2 to $N_{(m,\alpha)} \times N_{(n,\beta)}$. Eliminating all the even integers remains $\left(N_{(m,\alpha)} \times N_{(n,\beta)} - 1\right)/2$ odd ones. By property of full binary tree, if putting these odd integers on the nodes of T_3 from top to bottom and from the left to the right, the preceding $m+n+1$ levels contain $2^{m+n+2} - 1$ ones and the left $\left(N_{(m,\alpha)} \times N_{(n,\beta)} - 1\right)/2 - \left(2^{m+n+2} - 1\right)$ ones will be placed on the level $m+n+2$. Counting from position 0 to the position χ yields

$$\chi = \left(N_{(m,\alpha)} \times N_{(n,\beta)} - 1\right)/2 - \left(2^{m+n+2} - 1\right) - 1 = \left(N_{(m,\alpha)} \times N_{(n,\beta)} - 1\right)/2 - 2^{m+n+2}.$$

Substituting $N_{(m,\alpha)} \times N_{(n,\beta)}$ by $2^{m+n+2} + 1 + 2J$ yields

$$\begin{aligned}\chi &= \left(2^{m+n+2} + 1 + 2J - 1\right)/2 - 2^{m+n+2} \\ &= 2^{m+n+1} + J - 2^{m+n+2} + 2 \\ &= J - 2^{m+n+1}\end{aligned}$$ □

Corollary 3. For positive integer m and n, the node of T_3 at position $J_{00} = 2^m + 2^n$ on level $m+n+1$ is in left branch of T_3 but it cannot a leftmost node; the node at position $J_{mn} - 2^{m+n+1}$ on level $m+n+2$ is in right branch but it cannot be a rightmost node.

Proof. The maximal node on level $m+n+1$ in the left branch of T_3 is $2^{m+n+2} + 1 + 2\left(2^{m+n} - 1\right)$ with position $\vartheta^l_{(m+n+1,\max)} = 2^{m+n} - 1$. Note that

$$J_{00} - \vartheta^l_{(m+n+1,\max)} = 2^m + 2^n - 2^{m+n} + 1 = 2^{m+n}\left(\frac{1}{2^m} + \frac{1}{2^n} - 1\right) + 1$$

it knows that, $mn = 1$ yields $J_{00} - \vartheta^l_{(m+n+1,\max)} = 1$ and $mn > 1$ yields $J_{00} - \vartheta^l_{(m+n+1,\max)} < 0$, which says that the position $J_{00} = 2^m + 2^n$ is on level $m+n+1$ in the left branch of T_3. Since the leftmost node is at position 0 whereas $J_{00} = 2^m + 2^n > 0$, the position $J_{00} = 2^m + 2^n$ is surely not a leftmost one.

To prove the second conclusion, let $\vartheta = J_{mn} - 2^{m+n+1}$. Since the smallest node on level $m+n+2$ in the right branch of T_3 is $2^{m+n+3} + 1 + 2 \times 2^{m+n+1}$, it merely needs to prove $2^{m+n+1} \le \vartheta < 2^{m+n+2} - 1$. In fact,

$$\begin{aligned}\vartheta - 2^{m+n+1} &= J_{mn} - 2^{m+n+2} = 2^{m+n+1} - 2^{m+1} - 2^{n+1} \\ &= 2^{m+n+1}\left(1 - \frac{1}{2^m} - \frac{1}{2^n}\right) \ge 0\end{aligned}$$

and

$$\begin{aligned}\vartheta - \left(2^{m+n+2} - 1\right) &= J_{mn} - 2^{m+n+1} - 2^{m+n+2} + 1 \\ &= 2^{m+n+2} + 2^{m+n+1} - 2^{m+1} - 2^{n+1} - 2^{m+n+1} - 2^{m+n+2} + 1 \\ &= -2^{m+1} - 2^{n+1} + 1 < 0\end{aligned}$$ □

By Corollary 2 and Corollary 3, the product $N_{(m,\alpha)} \times N_{(n,\beta)}$ can be intuitionally depicted with **Figure 2**.

Corollary 4. Let $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ be two nodes of T_3 with $0 \le m \le n$; then the product $N_{(m,\alpha)} \times N_{(n,\beta)}$ cannot fall in $T_{N_{(n,\beta)}}$ on level $m+1$ or a higher level.

Proof. Let $N_{(m,\alpha)} = 2^{m+1} + 1 + 2\alpha$ and $N_{(n,\beta)} = 2^{n+1} + 1 + 2\beta$ with $\alpha = 0,1,\cdots,2^m - 1$ and $\beta = 0,1,\cdots,2^n - 1$. Since the maximal node on level $m+1$ of $T_{N_{(n,\beta)}}$ is $2^{m+1}\left(N_{(n,\beta)} + 1\right) - 1$, direct calculation yields

$$\begin{aligned} N_{(m,\alpha)} \times N_{(n,\beta)} &= \left(2^{m+1} + 1 + 2\alpha\right) \times N_{(n,\beta)} \\ &= 2^{m+1} \times N_{(n,\beta)} - 2^{m+1} + 2^{m+1} + \left(1 + 2\alpha\right) N_{(n,\beta)} \\ &= \underbrace{2^{m+1}\left(N_{(n,\beta)} + 1\right) - 1}_{N_{(n,\beta)} \gg (m+1)} - 2^{m+1} + 1 + \left(1 + 2\alpha\right) N_{(n,\beta)} \end{aligned}$$

Note that

$$\begin{aligned} & -2^{m+1} + 1 + \left(1 + 2\alpha\right) N_{(n,\beta)} \\ &= -2^{m+1} + 1 + \left(1 + 2\alpha\right)\left(2^{n+1} + 1 + 2\beta\right) \\ &\ge -2^{m+1} + 1 + 2^{n+1} + 1 + 2\beta > 0 \end{aligned}$$

hence $N_{(m,\alpha)} \times N_{(n,\beta)} > 2^{m+1}\left(N_{(n,\beta)} + 1\right) - 1$, which says $N_{(m,\alpha)} \times N_{(n,\beta)}$ cannot fall on level $m+1$ or a higher level of $T_{N_{(n,\beta)}}$. □

Corollary 4 can be equivalently stated as that, if $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ are two nodes of T_3 with $m \le n$, then the product $N_{(m,\alpha)} \times N_{(n,\beta)}$ can only fall in the shadow area in **Figure 3**.

Corollary 5. Let $N_{(m,\alpha)}$ and $N_{(m,\beta)}$ be two nodes of T_3 with $\alpha < \beta$; then the product $N_{(m,\alpha)} \times N_{(m,\beta)}$ cannot lie in $T_{N_{(m,\alpha)}}$ or $T_{N_{(m,\beta)}}$ on level $m+1$ or a higher level.

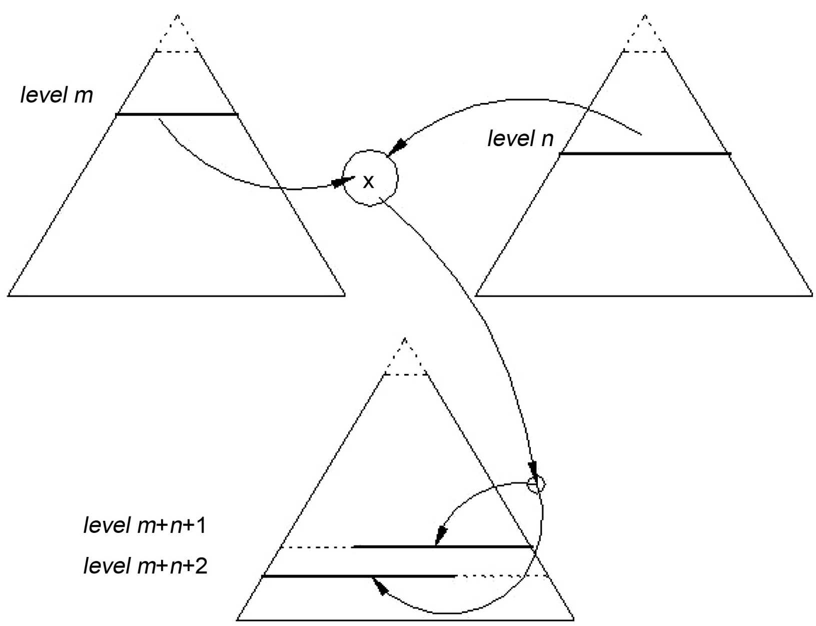

Figure 2. Possible positions of product $N_{(m,\alpha)} \times N_{(n,\beta)}$.

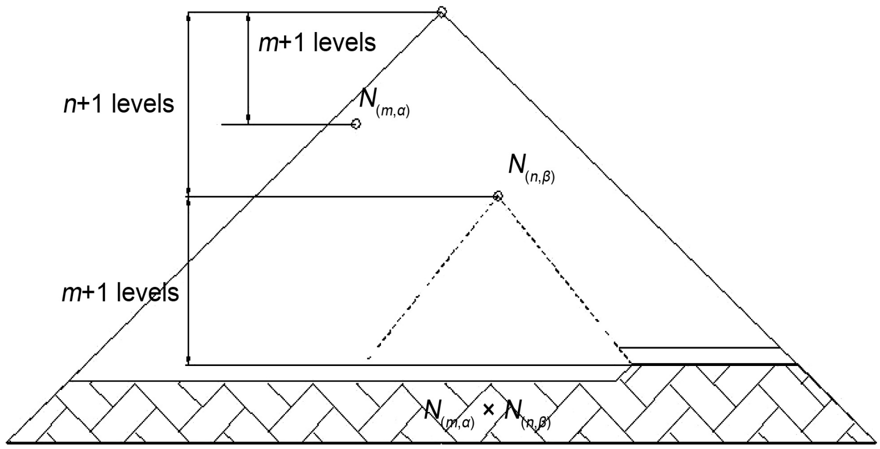

Figure 3. $N_{(m,\alpha)} \times N_{(n,\beta)}$ can merely lie in area of shadow.

Proof. (Omitted) □

Corollary 5 can be equivalently stated as that, if $N_{(m,\alpha)}$ and $N_{(m,\beta)}$ are two nodes of T_3 with $\alpha < \beta$, then the product $N_{(m,\alpha)} \times N_{(m,\beta)}$ can only fall in shadow area in **Figure 4**.

5.2. Square of A Node

Theorem 7. Product $N_{(m,\alpha)} \times N_{(m,\alpha)} = N^2_{(m,\alpha)}$ is a left node of T_3, it lies in T_3 on level $2m+1$ or $2m+2$ and there are $2\left(2^m+1\right)\alpha + 2\alpha^2$ nodes from $N^2_{(m,\alpha)}$ to $N_{\left(N_{(m,\alpha)} \gg (m+1)\right)}$.

Proof. Direct calculation shows

$$N^2_{(m,\alpha)} = \left(2^{m+1}+1+2\alpha\right)N_{(m,\alpha)}$$
$$= \underbrace{2^{m+1}\left(N_{(m,\alpha)}+1\right)-1}_{N_{(m,\alpha)}\gg(m+1)} - 2^{m+1}+1+\left(1+2\alpha\right)N_{(m,\alpha)}$$

Let $\xi = -2^{m+1}+1+\left(1+2\alpha\right)N_{(m,\alpha)}$; then

$$\xi = -2^{m+1}+1+\left(1+2\alpha\right)\left(1+2\alpha+2^{m+1}\right)$$
$$= -2^{m+1}+1+1+2\alpha+2^{m+1}+2\alpha+4\alpha^2+2^{m+2}\alpha$$
$$= 2\left(1+2\alpha+2\alpha^2+2^{m+1}\alpha\right)$$

and $\Delta = 2\alpha + 2\alpha^2 + 2^{m+1}\alpha$ is the number of nodes from $N^2_{(m,\alpha)}$ to $N_{\left(N_{(m,\alpha)} \gg (m+1)\right)}$. □

Corollary 6. For arbitrary integer $m \ge 0$, it holds $N_{(m,2^m-1)} \times N_{(m,2^m-1)} = N_{\left(N_{(m,2^m-1)} \ll (m+2)\right)}$ and $N_{(m,0)} \times N_{(m,0)} = N_{\left(N_{(m,1)} \ll (m+1)\right)}$.

Proof. By (**P4**), it yields

$$N_{\left(N_{(m,\alpha)} \ll (m+2)\right)} = 2^{m+2}\left(N_{(m,\alpha)}-1\right)+1 = 2^{m+2}\left(2^{m+1}+1+2\alpha-1\right)+1$$
$$= 2^{m+2}\left(2^{m+1}+2\alpha\right)+1 = 2^{2m+3}+2^{m+3}\alpha+1$$

Note that

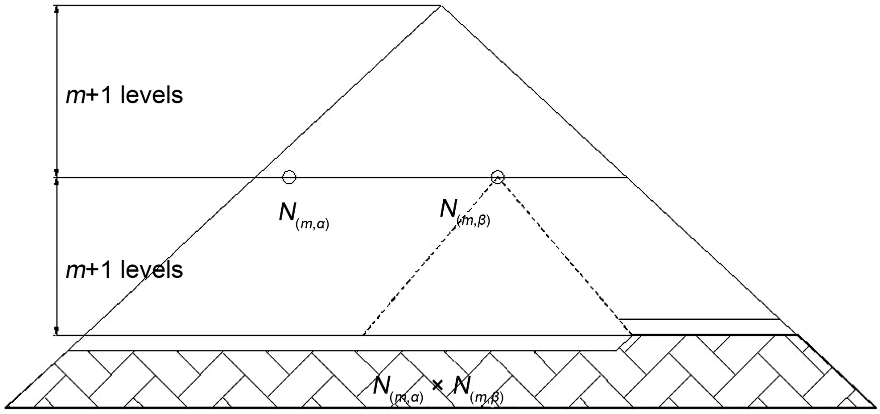

Figure 4. $N_{(m,\alpha)}\times N_{(n,\beta)}$ can merely lie in area of shadow.

$$\begin{aligned} N_{(m,\alpha)}^2 &= \left(2^{m+1}+1+2\alpha\right)\left(2^{m+1}+1+2\alpha\right) \\ &= 2^{2m+2}+2^{m+1}+2^{m+2}\alpha+2^{m+1}+1+2\alpha+2^{m+2}\alpha+2\alpha+4\alpha^2 \\ &= 2^{2m+2}+2^{m+2}+2^{m+3}\alpha+4\alpha+4\alpha^2+1 \end{aligned}$$

Hence

$$N_{\left(N_{(m,\alpha)}\ll(m+2)\right)}-N_{(m,\alpha)}^2=2^{2m+2}-2^{m+2}-4\alpha-4\alpha^2$$

Taking $\alpha=2^m-1$ results in

$$\begin{aligned} &N_{\left(N_{(m,\alpha)}\ll(m+2)\right)}-N_{(m,\alpha)}^2 \\ &=2^{2m+2}-2^{m+2}-4\left(2^m-1\right)-4\left(2^m-1\right)^2 \\ &=2^{2m+2}-2^{m+2}-2^{m+2}+4-2^2\left(2^{2m}-2^{m+1}+1\right) \\ &=2^{2m+2}-2^{m+2}-2^{m+2}+4-2^{2m+2}+2^{m+3}-4 \\ &=-2^{m+2}-2^{m+2}+2^{m+3}=0 \end{aligned}$$

Thus $N_{(m,\alpha)}\times N_{(m,\alpha)}=N_{\left(N_{(m,\alpha)}\ll(m+2)\right)}$.

Now directly calculating $N_{\left(N_{(m,1)}\ll(m+1)\right)}$ and $N_{(m,0)}\times N_{(m,0)}$ yields

$$N_{(m,0)}^2=\left(2^{m+1}+1\right)^2=2^{2m+2}+2^{m+2}+1$$

and

$$\begin{aligned} N_{\left(N_{(m,1)}\ll(m+1)\right)} &= 2^{m+1}\left(N_{(m,1)}-1\right)+1=2^{m+1}\left(2^{m+1}+1+2-1\right)+1 \\ &= 2^{m+1}\left(2^{m+1}+2\right)+1=2^{2m+2}+2^{m+2}+1 \end{aligned}$$

which says $N_{(m,0)}\times N_{(m,0)}=N_{\left(N_{(m,1)}\ll(m+1)\right)}$. □

Corollary 7 For arbitrary integer $m\ge 0$, $N_{\left(m,2^m-1\right)}^2$ lies in $T_{N_{\left(m,2^m-1\right)}}$ whereas $N_{(m,0)}^2$ lies in $T_{N_{(m,1)}}$.

Proof. (Omitted) □

Corollary 7 can be equivalently and intuitively stated as that, the square of a rightmost node of T_3 is a descendant of the node itself, whereas the square of a leftmost node of T_3 is a descendant of the node's elder brother.

Corollary 8. If $m \ge 0$ and $0 \le \alpha < 2^m - 1$, then $N^2_{(m,\alpha)}$ does not lie in $T_{N_{(m,\alpha)}}$. Or equivalently, $N^2_{(m,\alpha)}$ does not lie in $T_{N_{(m,\alpha)}}$ unless $m = 0$ or $m > 0$ plus $\alpha = 2^m - 1$.

Proof. By Theorem 7, $N^2_{(m,\alpha)}$ lie in T_3 on level $2m+1$ or $2m+2$. Since by Lemma 2 the level $m+1$ of $T_{N_{(m,\alpha)}}$ is the level $2m+1$ of T_3, $N^2_{(m,\alpha)}$ can never lie in $T_{N_{(m,\alpha)}}$ on a level lower than $m+2$. By Corollary 5, $N^2_{(m,\alpha)}$ cannot lie in $T_{N_{(m,\alpha)}}$ on level $m+1$ or a higher level. Now it needs to prove that it cannot either lie in $T_{N_{(m,\alpha)}}$ on level $m+2$. In fact, the smallest node of $T_{N_{(m,\alpha)}}$ on level $m+2$ is $N_{\left(N_{(m,\alpha)} \ll (m+2)\right)}$. By Theorem 7 and Corollary 6 it knows that when $\alpha = 2^m - 1$, $N^2_{(m,\alpha)}$ reaches it maximal value that is also the minimal node of $T_{N_{(m,\alpha)}}$ on level $m+2$. Hence it cannot lie in $T_{N_{(m,\alpha)}}$ under the condition $m \ge 0$ and $0 \le \alpha < 2^m - 1$. □

5.3. Path Connecting Multiple to Divisor

Theorem 8. Let $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ be two nodes of T_3 with $0 \le m \le n$; then there is a path connecting $N_{(m,\alpha)} \times N_{(n,\beta)}$ to $N_{(m,\alpha)}$ and the length of the path is less than $2m+n+2$; there is also a path connecting $N_{(m,\alpha)} \times N_{(n,\beta)}$ to $N_{(n,\beta)}$ and the length of the path is less than $m+2n+2$.

Proof. By Theorem 6 and Corollary 7 and Corollary 8, it knows that $N_{(m,\alpha)} \times N_{(n,\beta)}$ must lie in $T_{N_{(m,\alpha)}}$, $T_{N_{(n,\beta)}}$ or some other subtree of T_3. First consider the case that $N_{(m,\alpha)} \times N_{(n,\beta)}$ lies in $T_{N_{(n,\beta)}}$. By Corollary 2 and Lemma 2 it knows $P\left(N_{(m,\alpha)} \times N_{(n,\beta)}, N_{(n,\beta)}\right) \le m+1 < m+n+2$. Likewise, if $N_{(m,\alpha)} \times N_{(n,\beta)}$ lies in $T_{N_{(m,\alpha)}}$ it holds $P\left(N_{(m,\alpha)} \times N_{(m,\alpha)}, N_{(m,\alpha)}\right) \le n+1 < m+n+2$.

Now consider the case that $N_{(m,\alpha)} \times N_{(n,\beta)}$ lies in neither $T_{N_{(m,\alpha)}}$ nor $T_{N_{(n,\beta)}}$; then there must be a γ and a λ with $0 \le \gamma \le 2^m - 1$ plus $\gamma \ne \alpha$ and $0 \le \lambda \le 2^n - 1$ plus $\lambda \ne \beta$ such that $N_{(m,\alpha)} \times N_{(n,\beta)}$ lies in $T_{N(m,\gamma)}$ or $T_{N(n,\lambda)}$. Assume it lies in $T_{N(m,\gamma)}$; then by the subtree transition law,

$$N^{N_{(m,\gamma)}}_{(i,\omega)} = 2^i \left(N_{(m,\gamma)} - N_{(m,\alpha)}\right) + N^{N_{(m,\alpha)}}_{(i,\omega)}$$

$$i = 1, 2, \cdots; \omega = 0, 1, \cdots, 2^i - 1$$

Let $N_{(l,s)}$ be the lowest common ancestor (LCA) of $N_{(m,\alpha)}$ and $N_{(m,\gamma)}$; then by properties of LCA, as introduced in [7] [8] and [9], $N_{(m,\alpha)} \times N_{(n,\beta)}$ can be connected to along path $P\left(N_{(m,\alpha)} \times N_{(n,\beta)}, N_{(m,\gamma)}\right) \to P\left(N_{(m,\gamma)}, N_{(l,s)}\right) \to P\left(N_{(l,s)}, N_{(m,\alpha)}\right)$. Since the length of $P\left(N_{(m,\alpha)} \times N_{(n,\beta)}, N_{(m,\gamma)}\right)$ is less than $m+n+2-m = n+2$, the length of $P\left(N_{(m,\gamma)}, N_{(l,s)}\right)$ is $m-l$ and the length of $P\left(N_{(l,s)}, N_{(m,\alpha)}\right)$ is $m-l$, it knows that the total length of the path is less than $2m+n+2-2l \le 2m+n+2$. Likewise, the other conclusion can be proved. □

6. Conclusion

Putting integers on binary tree can reveals many new properties of the integers, as seen in previous sections and in the references list in bibliography. This paper merely shows some traits related with multiplication of nodes on T_3 tree. In fact, there are many other new properties that have not been known till now and are required to make further study. For example, Theorem 8 indicates there exists a very fast approach to factorize an odd integer; in addition, there are evidences that there is a law of distribution of prime number on certain subtrees. Hence there is a lot of work done in the future. Hope that it is concerned by researchers of interests.

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