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The Master Equation and the Pauli Equation in the Fuzzy Time Model

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Abstract

The approach proposed in the study is based on the revision of the concept of time as a point on the real axis. It uses the concept of fuzzy time as the set of real numbers with a finite, but not equal to one, function of membership to the time set, *i.e.* the fuzzy time concept. It is postulated that in fuzzy time t the system dynamics follows from the standard variational principle of the least action and is ordinary Hamilton-Jacobi mechanics. This validates the passage to the limit from fuzzy mechanics to ordinary variational conservative mechanics. The Liouville equation is solved by the method of successive approximations in the time domain of a much larger characteristic scale of fuzziness, using interaction as a small parameter. A standard diagram technique is used. It can be shown that the defuzzification of the Liouville equation inevitably reduces the reversible part in the description to the irreversible evolutionary equation. The latter leads to the second law of thermodynamics. Generalization to the quantum case is possible, *i.e.* the so-called fuzzy Pauli equation can be drawn.

Keywords

Fuzzy Sets, Fuzzy Time, Master Equation, Second Law of Thermodynamics, Fuzzy Pauli Equation

1. Introduction

Hamiltonian mechanics (the mechanics of conservative systems) is reversible in time. The world based on Hamiltonian mechanics is either orbitally stable, or, in the presence of hyperbolic points, highly sensitive to initial conditions.

The system evolution is described in phase space by the Liouville equation as the incompressible fluid motion: the phase space element can be deformed in an arbitrarily complex manner, however the measure introduced within phase space is preserved.

At the same time, the second law of thermodynamics is valid, in accordance with which the concept of entropy is introduced: entropy in closed systems increases, reaching the maximum in a state of equilibrium. Time (sequence of events) acquires a direction: the system evolves from the "past" to the "future" from a state with lower entropy to a state with greater entropy.

The transition to another scale level of description—quantum mechanics—does not solve the problem: the Schrödinger equation is reversible in time.

It can be said that there are two kinds of system distribution functions for the phase space: the Gibbs measure $\rho(t)$, describing "conservative" evolution $\rho(t)=U(t)\rho(0)$, where $U(t)$ is some unitary operator $U(t)U(s)=U(t+s), t,s><0$, and "dissipative" evolution for the Boltzmann distribution in accordance with evolutionary equation $\overline{\rho(t)}=W(t)\overline{\rho(0)}$, that is irreversible in time $W(t)W(s)=U(t+s)$, $t,s>0$.

To connect distributions $\rho(t)$ and $\overline{\rho(t)}$ means to build a microscopic theory of irreversible processes, eliminating contradictions of a number of fundamental physical laws.

One of the options to solve this problem has been proposed in the works of I. Prigozhin and his school [1] [2]. The essence of the approach involves considering highly unstable conservative systems (so-called "K-flows"), for which the Poincaré section of the total phase space is analyzed.

Phase space regions are associated with special objects—partitions γ_n. In the particular case when the system evolution can be transformed to a discrete transformation, for example, "baker's transformation" of phase space unit square, partitions are functions that take the value of ±1 on the left and the right sides of the square.

The sequential action of evolution operator $\gamma_{n+1}=U\gamma_n$ describes the state of the system after *n* cycles of evolution. It can be shown that there is operator *T*, for which functions γ_n are eigenfunctions with infinitely singular eigenvalue *n*.

Further, existence of Hermitian operator Λ, which is a nonnegative decreasing function of *T*, is postulated. It is this operator that determines connection $\overline{\rho(t)}=\Lambda\rho(t)$ [1]. In this case it is possible to show the validity of the second law of thermodynamics from microscopic principles.

Problems and contradictions of this approach are quite obvious: firstly, the approach is realized in the phase space Poincaré section, but not in the full phase space; secondly, reducibility of all unstable systems to transformations of this kind is not obvious; finally, the main thing is that the explicit form of operator Λ is not obtained. This somewhat depreciates the approach.

In the proposed study, the author assumes a different concept. The approach is based on the revision of the concept of time as a point on the real axis to the concept of fuzzy time as a set of real numbers with a finite, but not equal to one, membership function, *i.e.* to the *fuzzy time concept*.

A fuzzy set of time $T=\{t,\mu(t)\}$ is considered, where $\mu(t)$ is the membership function, which in the future is assumed to be smooth or finitely smooth and compact $\mu(t \to \mp\infty) \to 0$.

The operation of defuzzification (weighing) with respect to measure $\mu(t)$ determines time $T = \int_{-\infty}^{\infty} t\mu(t)\mathrm{d}t$, which is hereinafter referred to as macro-time, difference $\tau = t - T$, which is called micro- or fuzzy time.

In the limiting case (classical interpretation of time), measure $\mu(t)$ is proportional to the delta function.

The characteristic scale of fuzzy time τ_0 can be defined as:
$\tau_0^2 = \int_{-\infty}^{\infty} \tau^2 \mu(\tau)\mathrm{d}\tau$.

It is postulated that in fuzzy time t the system dynamics follows from the standard variational principle of the least action and is ordinary Hamilton-Jacobi mechanics. This validates the passage to the limit from fuzzy mechanics to ordinary variational conservative mechanics at $\mu(t) \to \delta(t)$.

The Liouville equation is solved by the method of successive approximations in the time region $t \gg \tau_0$, using interaction as a small parameter. A standard diagram technique is used. It can be shown that defuzzification inevitably reduces the Liouville equation that is reversible in macro-time to the irreversible evolution equation, wherein operator $W(t)$ is expressed in terms of the interaction parameter and the fuzzy measure $\mu(t)$.

The latter leads to the second law of thermodynamics. Generalization to the quantum case is possible, *i.e.* the so-called fuzzy Pauli equation can be drawn.

Thus, time fuzziness + interaction make the world highly irreversible and "derive" the second law of thermodynamics from more general principles.

The proposed study is devoted to a consistent presentation of this approach. Another approach which is associated with the quantum delocalization of the time proposed in [3] [4] [5].

2. Elements of Theory of Fuzzy Sets

Let us consider fuzzy set $A = \{x, \mu(x), x \in \Omega\}$, where $\mu(x)$ is the measure of membership of element x to set Ω. As defined, the membership measure is positive $\mu(x) \geq 0$ and $\sup(\mu(x)) = 1$. For two sets A, B the following assertion holds: $A \in B$, if $\mu(x, A) \leq \mu(x, A)$.

Let us confine to the case of even measures $\mu(x) = \mu(-x)$, and put
$\mu(x)x^n \to 0$ for measure $\mu(x)$ for any integer powers n at $x \to \pm\infty$.

By the intersection of two odd sets A and B we mean odd set C with membership measure $\mu(z, A \cap B) = \min(\mu(x, A), \mu(x, B))$, by the union of two odd sets A and B we assume odd set C with membership measure
$\mu(z, A \cup B) = \max(\mu(x, A), \mu(x, B))$.

Other generalizations of fuzzy sets' union or intersection operators are possible, e.g. Jager *et al.* (nonparametric t and S norms) [6] [7] [8] [9] [10].

Generalization of basic classical arithmetical rules to operations with fuzzy numerical sets or *fuzzy* numbers and introduction of concepts of *fuzzy* functions and *fuzzy* relations are drawn as follows.

The sum of *fuzzy* numbers A and B is called fuzzy numerical set $C = A + B$

with membership measure: $\mu(z, A+B) = \max_{z=x+y} \min(\mu(x,A), \mu(x,B))$. The difference of *fuzzy* numbers A and B is called fuzzy numerical set $C = A - B$ with membership measure: $\mu(z, A-B) = \max_{z=x-y} \min(\mu(x,A), \mu(x,B))$. The product and the ratio of fuzzy numbers are determined in a similar way.

A fuzzy function is a one-to-one correspondence of two fields of fuzzy numbers. Appropriate measures have the following form: when multiplied by a scalar (non-fuzzy) value *a*: $\mu(ax) = \mu\left(\frac{x}{a}\right)$, raised to the power

$\mu(x^n) = \mu(\sqrt[n]{x}), n \in N$, taking exponential $\mu(\mathrm{e}^x) = \mu(\ln x)$, etc. [6].

An interesting model in the framework of the theory of uncertainty in other fields is presented in [7] [8].

If set Ω is a set of casual events, then set $A = \{x, \mu(x), x \in \Omega\}$ is its fuzzy generalization. Let $\mu(x)$ be measurable according to Brel, then

$$P(A) = \sum_{x \in \Omega} \mu(x) P(x)$$

is the expected measure of membership of event x to set Ω or, in the continuous case,

$$P(A) = \int \mu(x) \mathrm{d}P(x),$$

where $P(x)$ is the probability measure.

It is obvious that always $P(x) \le \mu(x)$.

The expected fuzzy event value (the operation of defuzzification) looks like:

$$\langle f(x) \rangle = \frac{\int f(x) \mathrm{d}\mu(x)}{\int \mathrm{d}\mu(x)}.$$

Hence an essential remark follows. If some event is likely, then it is possible, but if possible, it is not necessarily likely [6].

As a particular example of defuzzification let us consider a particle moving in fuzzy time t and not interacting with other particles. Suppose that at the initial macroscopic instant of time $T = 0$ it had non-fuzzy coordinate $x_0 = 0$ and velocity v. Let the measure of fuzzy time membership look like:

$$\mu(\tau) = \mu(t - T) = \exp\left(-\frac{\tau^2}{2\tau_0^2}\right)$$

Then, at the macroscopic instant of time T, defuzzified distribution density looks like:

$$\langle f(x,T) \rangle = \overline{f(x,T)} = \frac{\int f(x,t) \mathrm{d}\mu(x)}{\int \mathrm{d}\mu(x)} = \frac{1}{\tau_0 \sqrt{2\pi}} \exp\left(-\frac{(x - vT)^2}{2(\tau_0 v)^2}\right)$$

3. Principle of Fuzzy Causality

Let X be a fuzzy set with measure $\mu(x)$. Let us call a collective of x for which measure $\mu(x) = 1 - \eta, \min(x \in X, \mu(x) = 1) > 0$, *i.e.* the set of values of x for which the minimal x with measure $\mu(x) = 1$ is positive, an almost positive fuzzy interval with level η [7]-[12].

This definition allows us to formulate the principle of fuzzy causality. Let set X be time axis t.

Definition. Events attributed to fuzzy instants of time t_1 and t_2 at one or similar spatial points can be connected by an unconditional causal link with level η, if the fuzzy interval between them is almost positive. Otherwise they are connected by a conditional causal link with level η.

In ordinary space with non-fuzzy time, this principle goes into the ordinary principle of causality at velocities much lower than the light velocity.

The principle of fuzzy causality leads to an interesting result: for an unconditional fuzzy causality there is a classical analogue—the regular causality principle realized in fuzzy time.

A conditional causality does not have classical analogues: the "past" in it can depend on the "future", the very concept of the "past" and the "future" has no definite meaning. It is possible to talk about the "past" and the "future" only after the defuzzification process. Schematically, this principle is illustrated in **Figure 1**.

4. The Master Equation in Fuzzy Time

4.1. Defuzzification of the Liouville Equation

Let us consider a system consisting of N material points in space Ω of dimension of $d=3$. Let us assume that the points' velocities v are much smaller than the light velocity, the interaction between them is pairwise $V\left(x_j - x_n\right)$ and depends only on the distance between points j and n. If not stated otherwise, the mass of the point is assumed to be one, $m=1$, the pairwise interaction $V\left(x_j - x_n\right)$ to be small, of the order of $\varepsilon \ll 1$.

The system Hamiltonian has the standard form:

$$H\left(p_j, x_j\right) = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \varepsilon \sum_{i,j}^{N} V\left(x_j - x_n\right) = H_0 + \varepsilon V$$

where p_j, x_j is momentum and coordinate of point j, ε is small parameter.

The validity of dynamics equations in fuzzy time is postulated:

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H\left(p_j, x_j\right)}{\partial x_j}$$

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\partial H\left(p_j, x_j\right)}{\partial p_j}$$

Let $\rho\left(p_j, x_j, t\right)$ be the distribution density in $6N$-dimensional space at a fuzzy instant of time t.

The equation of evolution for density $\rho\left(p_j, x_j, t\right)$ is as follows:

$$\frac{\partial \rho\left(p_j, x_j, t\right)}{\partial t} = i\left(S_0 + \varepsilon S_1\right)\rho\left(p_j, x_j, t\right), \tag{1}$$

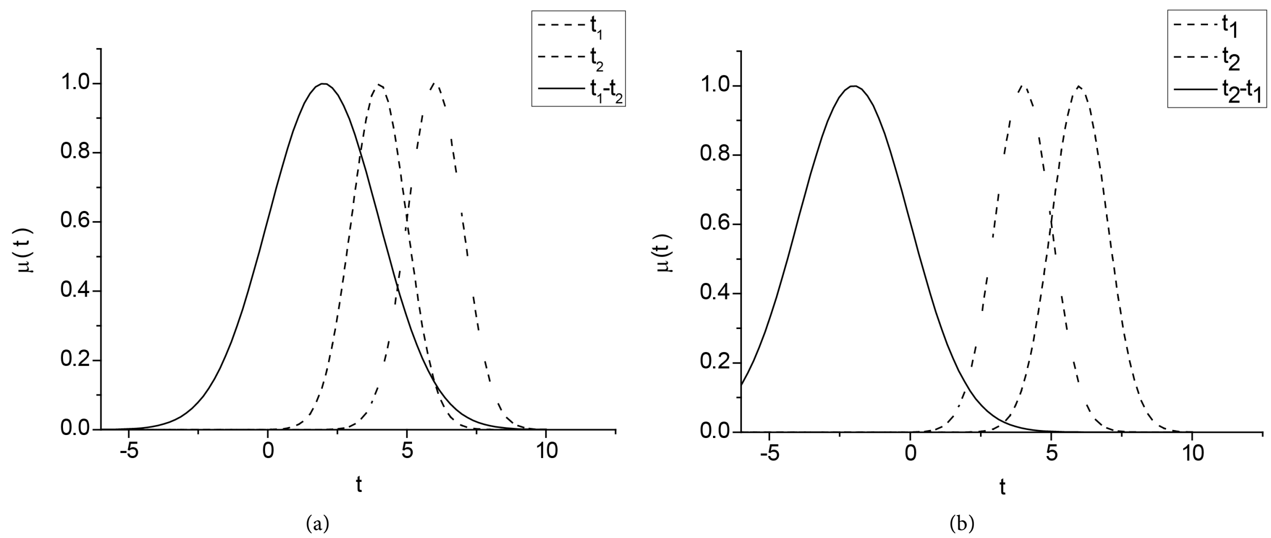


Figure 1. Unconditional and conditional causal link with the same level (~0.6): (a) An unconditional causal link—the difference between the measure maximums t_1 and t_2 is positive; (b) A conditional causal link—the difference between the measure maximums t_1 and t_2 is negative.

where $S_0 = \frac{\partial H_0}{\partial p_j}\frac{\partial}{\partial x_j}$, $S_1 = -\frac{\partial V}{\partial x_j}\frac{\partial}{\partial p_j}$.

Let us seek the solution of this equation by the small parameter expansion method ε.

Operator iS_0 is Hermitian, its own functions are:

$$S_0\varphi_{nk} = \lambda_{nk}\varphi_{nk},\ \varphi_{nk} = \frac{1}{\Omega^{N/2}}\mathrm{e}^{i\sum_n k_n x_n},$$

where k is the wave vector, n is the particle number, they are orthogonal and can be used as the orthonormal basis.

With zero $\varepsilon = 0$ approximation, solution (1) has the obvious form: $\rho(p_j, x_j, t) = \sum_n \sum_{k_n} C_{k_n}\mathrm{e}^{i\sum_n (x_n k_n + p_n k_n t)}$, where coefficients $C_{k_j} = \rho(p_j, k_j, 0)$ are Fourier transforms of initial state $\rho(p_j, x_j, 0)$.

As $\varepsilon \neq 0$, let us assume that $\rho(p_j, x_j, t) = \sum_n \sum_{k_j} \rho(p_j, k_j, t)\mathrm{e}^{i\sum_n (x_n k_n + p_n k_n t)}$ is the fuzzy time function satisfying Equation (1).

After integration over spatial variables x_j (non-fuzzy!) we have:

$$\begin{aligned}\frac{\partial\rho(p_j, k_j, t)}{\partial t} &= \varepsilon\sum_q \sum_{jn} \mathrm{e}^{i(p_n k_n + p_j k_j)t} V(q)\, iqG_{jn}\mathrm{e}^{-i(p_n(k_n - q) + p_j(k_j + q))t} \\ &\times \rho(p_j, k_j - q, k_n + q, t),\end{aligned} \tag{2}$$

where $V(q) = \frac{1}{\Omega}\int \mathrm{e}^{iq(x_j - x_n)t}$ is the Fourier transform of the interaction potential, factor 2π is included in the volume normalization Ω, $G_{jn} = \frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_n}$.

Equation (2) is a fuzzy differential equation. The fuzziness is due to the fuzziness of time t, while wave vectors and momenta are assumed to be non-fuzzy numbers.

Density $\rho\left(p_j,k_j-q,k_n+q,t\right)$ satisfies a similar equation:

$$\frac{\rho\left(p_j,k_j-q,tk_n+q,t\right)}{\partial t} = -\varepsilon V\left(q\right)iq\mathrm{e}^{-i\left(p_n\left(k_n-q\right)+p_j\left(k_j+q\right)\right)t}G_{jn}\mathrm{e}^{i\left(p_nk_n+p_jk_j\right)t}\rho\left(p_j,k_j,k_n,t\right)$$

The latter can be integrated from fuzzy zero [0] to fuzzy t and substituted into Equation (2). As a result, we obtain:

$$\begin{aligned}\frac{\partial\rho\left(p_j,k_j,t\right)}{\partial t} &= -\varepsilon^2\sum_q\sum_{jn}\mathrm{e}^{i\left(p_nk_n+p_jk_j\right)t}\left|V\left(q\right)\right|^2\left(q\right)qG_{jn}\mathrm{e}^{-i\left(p_n\left(k_n-q\right)+p_j\left(k_j+q\right)\right)t}\\ &\quad *\int_{[0]}^{t}\mathrm{e}^{-i\left(p_n\left(k_n-q\right)+p_j\left(k_j+q\right)\right)t}qG_{jn}\mathrm{e}^{i\left(p_nk_n+p_jk_j\right)t}\mathrm{d}t_1\rho\left(p_j,k_j,k_n,t_1\right)\end{aligned} \tag{3}$$

The fuzzy lower limit of integration (fuzzy zero) in this equation is indicated as [0]. In the more general case, a symmetric fuzzy number will be denoted as $\left[0,a\right]$, where a is the set of parameters characterizing the degree of measure blurring.

At that, it should be noted that if b is a non-fuzzy number, then $\left[x,a\right]b$ is a fuzzy number, except for the case of $b=0:\left[x,a\right]b=0$, if $\left[0,a\right]$ is a fuzzy zero with a measure parametrized by a value, then $\left[0,a\right]\left[0,a\right]=\left[0,a_1\right]$ is also a fuzzy zero, but with a different parameterization a_1.

Value $\rho\left(p_j,k_j,k_n,t_1\right)$ differs from $\rho\left(p_j,k_j,t\right)$ in order ε, so in (3) t can be substituted by t_1. The solution will be sought in the form of a series with respect to small parameter ε, presenting fuzzy time as a sum of macrotime T and micro(fuzzy) time τ. Let us use the standard diagram technique.

4.2. Diagram Technique

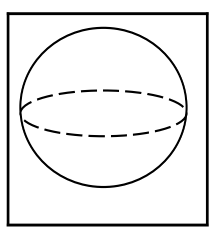
Let us consider the value of $\mathrm{e}^{i\left(p_nk_n\right)t}$ and present it in the form of a product of two prediagrams:

———————	$\mathrm{e}^{i\left(p_nk_n\right)T}$	– – – – -	$\mathrm{e}^{i\left(p_nk_n\right)\tau}$	————– – –	$\mathrm{e}^{i\left(p_nk_n\right)(T+\tau)}$

In this expression, as before, n is the particle number, k_n is the wave vector, p_n is momentum, T is macro-time and τ is fuzzy time: $t=T+\tau$.

To begin with, let us confine to integration of Equation (3) in fuzzy time for the homogeneous case, using all $k_n=0$.

The first essential term that appears in the integration is of order ε^2 and the following diagram corresponds to it:



or in an explicit form:

$$\left\langle \frac{\varepsilon^2}{\Omega^2}\sum_q\sum_{jn}\int_{[0]}^{t}\mathrm{d}t_1\int_{[0]}^{t_1}\left|V(q)\right|^2(q)qG_{jn}\mathrm{e}^{-i((p_n-p_j)q)(t_1-t_2)}\mathrm{d}t_2 qG_{jn}\right\rangle\rho\left(p_j,0\right)$$

$$=\frac{\varepsilon^2}{\Omega^2}\sum_q\sum_{jn}\int_{[0]}^{T}\mathrm{d}T_1\int_{[0]}^{T_1}\left|V(q)\right|^2(q)qG_{jn}\mathrm{e}^{-i((p_n-p_j)q)(T_1-T_2)}\mathrm{d}T_2$$

$$\times\left\langle \mathrm{e}^{-i((p_n-p_j)q)(\tau_1-\tau_2)}\right\rangle qG_{jn}\rho\left(p_j,0\right)$$

This expression reflects the fact that we are interested in the defuzzified dependence of the distribution density on macro-time $\rho\left(p_j,T\right)$, thus in the internal lines of the diagrams we can henceforward integrate over macro-time T and defuzzify (average with a weight equal to normalized measure $\mu_2(\tau_1,\tau_2)$ over microtime τ).

Considering further that the measure of the difference of two fuzzy numbers $\mu_2(\tau_1,\tau_2)$ is determined through the norm as:
$\mu_2(\tau_1,\tau_2)=\max_{\tau_1-\tau_2}\min_{\tau_1,\tau_2}\left(\mu(\tau_1),\mu(\tau_2)\right)$ and, confining to the consideration of even measures, we have: $\mu_2(\tau_1,\tau_2)=Z^{-1}\mu\left((\tau_1-\tau_2)/2\right)$, where Z is the norming quantity.

Thus:

$$\left\langle \mathrm{e}^{-i((p_n-p_j)q)(\tau_1-\tau_2)}\right\rangle = g_2\left(\left(p_n-p_j\right)q\right)$$

$$=Z^{-1}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\int_{-\infty}^{\tau_1}\mathrm{e}^{-i((p_n-p_j)q)(\tau_1-\tau_2)}\mu_2(\tau_1,\tau_2)\mathrm{d}\tau_2$$

Let us introduce discontinuous function $\theta(\tau)=1$ for $\tau>0$ and $\theta(\tau)=0$ for $\tau<0$. Its Fourier transform can be represented as:

$$\theta(\tau)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\omega\tau}}{\omega-i\gamma}\mathrm{d}\omega$$

Then expression $g_2\left(\left(p_n-p_j\right)q\right)$ can be written in the form:

$$g_2\left(\left(p_n-p_j\right)q\right)=Z^{-1}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\int_{0}^{\infty}\mathrm{e}^{-i(p_n-p_j)q\tau}\mu(\tau)\theta(\tau)\mathrm{d}\tau$$

$$=Z^{-1}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\int_{-\infty}^{\infty}\mu\left(\omega+\left(p_n-p_j\right)q\right)\delta_-(\omega)\mathrm{d}\omega$$

where $Z=\int_{-\infty}^{\infty}\mu(\tau)\mathrm{d}\tau=\mu(\omega=0)$, $\pi\delta_-(\omega)=\frac{1}{i(\omega-i\gamma)}=\pi\delta(\omega)-iP\left(\frac{1}{\omega}\right)$, $P\left(\frac{1}{\omega}\right)$ is a symbol of the principal value, $\mu(\omega)=\int_{-\infty}^{\infty}\mathrm{e}^{i\omega\tau}\mu(\tau)\mathrm{d}\tau$.

Strictly speaking, it should be written as $\mu(2\omega)$ and, in general, when defuzzification is carried out over k fuzzy variables $\mu(k\omega)$, where $\mu(\omega)$ is the Fourier transform of the fuzzy time measure. This will be assumed in the future. Finally we have:

$$g_2\left(\left(p_n-p_j\right)q\right)=\pi\frac{\mu\left(\left(p_n-p_j\right)q\right)}{\mu(0)}-i\int_{-\infty}^{\infty}\frac{\mu(\omega)}{\mu(0)}P\left(\frac{1}{\omega+\left(p_n-p_j\right)q}\right)\mathrm{d}\omega$$

Integrating over macro-time T gives:

$$\int_0^T \mathrm{d}T_1 \int_0^{T_1} \mathrm{e}^{-i(p_n-p_j)q(T_1-T_2)} \mathrm{d}T_2 = \frac{iT}{(p_n-p_j)q} + \frac{1}{\left((p_n-p_j)q\right)^2}\left(1-\mathrm{e}^{-i(p_n-p_j)qT}\right)$$

Henceforward we will be interested in behavior for large macro-times T, so the second summand in this expression can be neglected:

$$\int_0^T \mathrm{d}T_1 \int_0^{T_1} \mathrm{e}^{-i(p_n-p_j)q(T_1-T_2)} \mathrm{d}T_2 \approx \frac{iT}{(p_n-p_j)q} = T\Delta_2\left((p_n-p_j)q\right)$$

This expression is invariant under replacement of $p_n \to -p_n$, $T \to -T$, *i.e.* it still describes reversible processes.

Finally we have:

$$\frac{\varepsilon^2}{\Omega^2} T \sum_q \sum_{jn} \left|V(q)\right|^2 qG_{jn}\Delta_2\left((p_n-p_j)q\right) g_2\left((p_n-p_j)q\right) qG_{jn}\rho(p_j,0)$$

It is important that for large Ω, more precisely for $\Omega \to \infty$, and simultaneously $N \to \infty$, but on retention of $\frac{\Omega}{N} \to \text{const}$, the summation over q can be replaced with integration $\int \mathrm{d}q$, but then

$$\Delta_2\left((p_n-p_j)q\right) \to \delta_-\left((p_n-p_j)q\right) = \pi\delta\left((p_n-p_j)q\right) - iP\left(\frac{1}{(p_n-p_j)q}\right).$$

This leads to violation of the diagram invariance with simultaneous replacement of $p_n \to -p_n$, $T \to -T$, *i.e.* to transition from the reversible Liouville equation to the irreversible basic kinetic equation. This is the basis for the approach to describing irreversible behavior in one of the models of [1].

In a system of finite dimensions the replacement of summation with integration is impossible: the approach of Prigozhin does not work, but within the framework of the fuzzy time concept irreversibility in (3) arises from factor $g_2\left((p_n-p_j)q\right)$, which is non-invariant relative of this transformation.

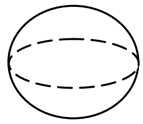
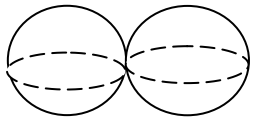
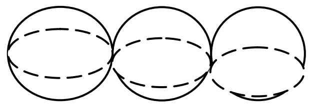
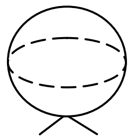
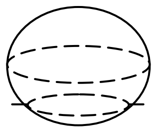
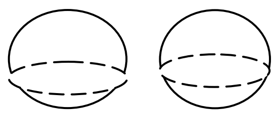
Let us consider the following orders with respect to ε of perturbation theory. Some diagrams and their orders are presented in **Table 1**. The value of $C = \frac{N}{\Omega}$ is the concentration of particles. Since we are interested in behavior for large macro-times T, we will keep the terms having the maximal order with respect to macro-time T and the minimal order with respect to small parameter ε. If these orders are equal, then we leave the terms of minimal order with respect to C.

In other words, we keep diagram 2, omitting 3 (the same order with respect to ε, but different with respect to C). Further, we keep diagram 4, omitting 5, etc. In addition, it is necessary to take into account iterations of diagrams 1 - 5, *i.e.* diagram 6 and those similar to it.

Taking into account (3), let us write in an explicit form, for example, the contribution due to defuzzification into a diagram of type 2:

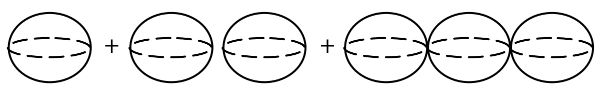
$$\begin{aligned} &g_3\left((p_n-p_j)q,(p_n-p_j)q_1\right) \\ &= \int_{-\infty}^{\infty} \frac{\mu(\omega)}{\mu(0)} \delta_-\left(\omega+(p_n-p_j)q,\right)\delta_-\left(\omega+(p_n-p_j)q_1\right)\mathrm{d}\omega \end{aligned}$$

Table 1. Diagrams of integration in fuzzy time of Equation (2). The type of the diagram, its order with respect to perturbation parameter ε, number of particles N, concentration C and the macro-time T are indicated.

	Diagram	Order
1		$\varepsilon^2 NCT$
2		$\varepsilon^3 NCT$
3		$\varepsilon^4 NCT$
4		$\varepsilon^3 NC^2T$
5		$\varepsilon^4 NC^2T$
6		$\left(\varepsilon^2 NCT\right)^2$

However it is obvious that the series obtained in **Table 1** is a series expansion of the exponential function of macro-time T, thus the equation for the defuzzified distribution density $\overline{\rho\left(p_j,T\right)}$ takes the following form:

$$\frac{\partial \overline{\rho\left(p_j,T\right)}}{\partial T} =$$



(4)

Or in an explicit form, for example, in order ε^2:

$$\frac{\partial \overline{\rho\left(p_j,T\right)}}{\partial T} = -\frac{\varepsilon^2}{\Omega^2}\sum\nolimits_q \sum\nolimits_{jn} \left|V(q)\right|^2 qG_{jn}\Delta_2\left(\left(p_n - p_j\right)q\right)$$
$$\times g_2\left(\left(p_n - p_j\right)q\right)qG_{jn}\overline{\rho\left(p_j,T\right)}$$

This is the basic kinetic equation in fuzzy time after defuzzification. The major difference is that non-invariance with respect to replacement of $T \to -T$, $p_j \to -p_j$ (*i.e.* irreversibility) is a consequence of not an infinite number of degrees of freedom, but of the fuzziness of time through the factor $g_2\left(\left(p_n - p_j\right)q\right)$.

For heterogeneous systems $k_j \neq 0$ the results are similar. This can be shown by simple but cumbersome transformations.

Operator $\Lambda(k) = g_k(p_n, p_j, q_1, \cdots, q_k)$ connecting Gibbs and Boltzmann distribution functions in fuzzy time: $\bar{\rho} = \Lambda\rho$, henceforward, let us call it the operator of dissipative projection of order *k*.

4.3. Quantum Case. Fuzzy Pauli Equation

For a quantum mechanical system it is necessary to use a half-set of phase variables and a density operator instead of distribution density $\rho(p_j, k_j, t)\rho$.

Let us again consider systems of N interacting particles with a pair potential, which depends on the distance between them $V(x_j - x_n)$ and assume ε to be a small parameter. Operator ρ satisfies the evolution equation of the following form:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H\rho],$$

where $H = \sum_j \frac{\hbar^2}{2m}\nabla_j^2 + \varepsilon \sum_{jn} V(x_j - x_n)$ is the Hamiltonian operator, $\hbar$ is the Planck's constant.

The transition to the interaction representation is carried out by means of transformation $\rho_1 = \mathrm{e}^{iH_0t/\hbar}\rho\mathrm{e}^{-iH_0t/\hbar}$, and the equation for operator ρ_1 takes the form as follows:

$$\frac{\partial \rho_1}{\partial t} = -\varepsilon\frac{i}{\hbar}[V\rho_1]$$

The only difference is in the fuzziness of time t and, as a consequence, the need to defuzzify this expression.

In momentum space this equation can be written as:

$$\frac{\partial \rho_1(k_j, k_n)}{\partial t} = -\varepsilon\frac{i}{\hbar}\sum_q \sum_{jn} V(q)\mathrm{e}^{\frac{i}{\hbar}\left(E(k_j)+E(k_n)-E(k_j+q)-E(k_n-q)\right)t} \times\left(\rho_1(k_j, k_n) - \rho_1(k_j+q, k_n-q)\right)$$

In this expression $E(k_j) = \frac{\hbar^2}{2m}k_j^2$ is the energy of a free particle. In the quantum case, we will explicitly write out the mass of the particle. Because of space homogeneity transformations are possible: $k_n \to k_n + \frac{q}{2}$, $k_j \to k_i - \frac{q}{2}$, and then, since $E(k_j) + E(k_n) - E(k_j+q) - E(k_n-q) = \frac{\hbar^2}{2m}q(k_n - k_j)$, we have:

$$\frac{\partial \rho_1(k_j, k_n)}{\partial t} = -\varepsilon\frac{i}{\hbar}\sum_q \sum_{jn} V(q)\mathrm{e}^{i\left(\frac{\hbar}{2m}q(k_n-k_j)\right)t} \times\left(\rho_1\left(k_j+\frac{q}{2}, k_n-\frac{q}{2}\right) - \rho_1\left(k_j-\frac{q}{2}, k_n+\frac{q}{2}\right)\right) \tag{5}$$

For matrix element $\rho_1\left(k_j+\frac{q}{2}, k_n-\frac{q}{2}\right)$, by analogy with the results obtained above, the following expression holds:

$$\frac{\partial\rho_1\left(k_j+\frac{q}{2},k_n-\frac{q}{2}\right)}{\partial t}$$
$$=-\varepsilon\frac{i}{\hbar}V(q)\mathrm{e}^{i\left(\frac{\hbar}{2m}q\left(k_n-k_j\right)\right)t}\left(\rho_1\left(k_j+q,k_n-q\right)-\rho_1\left(k_j,k_n\right)\right)$$

By integration of this equation with fuzzy time and substitution in (4) we obtain:

$$\frac{\partial\rho_1\left(k_j,k_n\right)}{\partial t}=-\frac{\varepsilon^2}{\hbar^2}\sum_q\sum_{jn}\left|V(q)\right|^2\mathrm{e}^{i\left(\frac{\hbar}{2m}q\left(k_n-k_j\right)\right)t}\int_{[0]}^{t}\mathrm{e}^{i\left(\frac{\hbar}{2m}q\left(k_n-k_j\right)\right)t_1}$$
$$\times\left(\rho_1\left(k_j+q,k_n-q\right)-\rho_1\left(k_j,k_n,t_1\right)\right)\mathrm{d}t_1$$

Again we split fuzzy time t into macroscopic T and fuzzy component τ and, by analogy with (3), draw the evolution equation for the defuzzified density operator $\overline{\rho_1\left(k_j,T\right)}$:

$$\frac{\partial\overline{\rho_1\left(k_i,T\right)}}{\partial T}=\frac{-\varepsilon^2}{\hbar^2}\sum_q\sum_{jn}\left|V(q)\right|^2\Delta_2(T)g_2\left(\left(k_n-k_j\right)q\right)$$
$$\times\left(\overline{\rho_1\left(k_j,k_n,T\right)}-\overline{\rho_1\left(k_j+q,k_n-q,T\right)}\right) \tag{6}$$

In this expression:

$$\Delta_2(T)=\frac{1}{E\left(k_j\right)+E\left(k_n\right)-E\left(k_j+q\right)-E\left(k_n-q\right)},$$

$$g_2\left(\left(k_n-k_j\right)q\right)=\int_{-\infty}^{\infty}\frac{\mu(\omega)}{\mu(0)}\delta_-\left(\omega+\frac{\hbar}{2m}\left(k_n-k_j\right)q\right)\mathrm{d}\omega$$

In quantum mechanics momentum is $p_j=\hbar k_j$, therefore Equation (6) describes the momentum distribution density evolution in macro-time.

Higher-order summands are drawn in a similar way. All considerations concerning orders of diagrams in analyzing the system's behavior for large T remain valid. Let us call defuzzified Equation (6) the fuzzy Pauli equation. It is important that in systems of a finite size non-invariance with respect to the replacement of $T\to-T$, $p_j\to-p_j$ is again a consequence of time fuzziness.

5. The Second Law of Thermodynamics

Let us consider the classical model. Let us define entropy for a homogeneous system (the distribution density depends only on N momenta p_j) in a standard way:

$$S=-\int\overline{\rho\left(p_j,T\right)}\ln\overline{\rho\left(p_j,T\right)}\mathrm{d}p_1\cdots\mathrm{d}p_N,$$

Its derivative with respect to macro-time is:

$$\frac{\mathrm{d}S}{\mathrm{d}T}=-\frac{\varepsilon^2}{\Omega^2}\int\cdots\int\mathrm{d}p_1\cdots\mathrm{d}p_N\left(1+\ln\overline{\rho\left(p_j,T\right)}\right)$$
$$\times\left(\sum_q\sum_{jn}\left|V(q)\right|^2qG_{jn}L_{jn}qG_{jn}\overline{\rho\left(p_j,T\right)}\right)$$

Here we introduce designation $L_{jn} = \Delta_2\left(\left(p_n - p_j\right)q\right) g_2\left(\left(p_n - p_j\right)q\right)$.

If it is assumed that when $p_j \to \mp\infty$, $\overline{\rho\left(p_j, T\right)} \to 0$, $\dfrac{\partial \overline{\rho\left(p_j, T\right)}}{\partial p_j} \to 0$, then after integration by parts we have:

$$\frac{\mathrm{d}S}{\mathrm{d}T} = \frac{\varepsilon^2}{\Omega^2} \int \cdots \int \mathrm{d}p_1 \cdots \mathrm{d}p_N \sum_q \sum_{jn} \left|V(q)\right|^2 L_{jn} \left(qG_{jn} \overline{\rho\left(p_j, T\right)}\right)^2$$

It is obvious that sign $\dfrac{\mathrm{d}S}{\mathrm{d}T}$ is determined by parity or oddity L_{jn} during replacement $p_j \to -p_j$.

For a system of a finite size, taking into account definition of $\Delta_2\left(\left(p_n - p_j\right)q\right)$ and $g_2\left(\left(p_n - p_j\right)q\right)$: $L_{jn} \sim \dfrac{1}{\left(p_n - p_j\right)q} P\left(\dfrac{1}{\left(p_n - p_j\right)q}\right)$ is even and therefore the integral is positive, entropy increases with the system's evolution in macrotime: $\dfrac{\mathrm{d}S}{\mathrm{d}t} \geq 0$.

It is essential that there is no requirement to the infinity of system $\Omega \to \infty$ provided that $C = \dfrac{N}{\Omega} = \text{const}$, *i.e.* the second law of thermodynamics is a consequence of just the fuzziness of time in a system with interactions. In the quantum case the situation is similar.

6. Discussion

Let us consider correction $g_2\left(\left(p_n - p_j\right)q\right)$ for the model in fuzzy time. Corrections of higher orders and heterogeneity of k_j will not lead to fundamental changes. Let us denote $\omega(n-j) = \omega = qP_{nj}$, where P_{nj} is the relative momentum of particles *n* and *j*.

In expression $g_2(\omega)$ frequency ω will inevitably come in combination with time dimension value τ_0, which is naturally identified with the characteristic scale of the fuzzification of time and evaluated as:

$$\tau_0^2 = \int_{-\infty}^{\infty} \tau^2 \mu(\tau) \mathrm{d}\tau$$

It follows from definition $g_2(\omega)$, that when we turn to the classical approach (a non-fuzzy model) $\mu(t) \to \delta(t)$ and because of the oddness of the principal value $P\left(\dfrac{1}{\omega}\right)$: $g_2(\omega\tau_0) \to 1$.

Hence for small τ_0 it is possible to expand in a series:

$$g_2(\omega\tau_0) = 1 + g_2'(\omega\tau_0 = 0)\omega\tau_0 + \cdots$$

The value of ω in this expression is of the order of reverse characteristic time of the system: $\omega \sim \dfrac{1}{t_c}$, thus the characteristic scale of macrotime *T*, where experimental fixation of these effects is possible: $T \sim \dfrac{t_c^2}{\tau_0}$.

Of course, irreversibility effects as a consequence of defuzzification arise in systems with interaction of order ε^2.

A natural question arises about the value scale estimation and, as a consequence, about the possibility of experimental verification of fuzzy time hypothesis.

At present, a number of fundamental constants is known, the only combination of which gives a value with time dimension. This is Planck's constant $\hbar$, gravitational constant γ and light velocity *c*.

Their combination: $\tau_0 = \frac{1}{c^2}\sqrt{\frac{\hbar\gamma}{c}} \approx 0.5\times10^{-43}$ s is a possible candidate for characteristic scale of time blurring.

From estimate $T \sim \frac{t_c^2}{\tau_0}$ we have characteristic time $T\sim$ equal to 10^{15} - 10^{17} s both for frequencies of the order of 10^{17} - 10^{18} s^{-1} (X-ray radiation) $T\sim 2(10^7$ - $10^9)$ s and for optical scales with frequencies of the order of $\sim10^{14}$ s^{-1}.

In the quantum case, the effect should be observed for high-energy processes: $\tau_0 \sim \frac{\hbar}{2m}\left(k_n - k_j\right)q$.

Finally, it is necessary to note one more aspect that is related to the results obtained. It can be shown that the algebra of non-commuting operators is isomorphic to the algebra of fuzzy numbers, so an alternative (or more familiar) approach is to consider time as the Hermitian operator that does not commute with the operator of evolution. The results will be similar.

7. Conclusions

1) The principle of fuzzy causality is formulated, generalizing the principle of causality to the system evolution in fuzzy time.

2) The master equation for the fuzzy time model is obtained, both in the classical and in the quantum cases.

3) It is shown that the second law of thermodynamics is a consequence of the system evolution in fuzzy time.

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