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Some Uniqueness Results of Q-Shift Difference Polynomials Involving Sharing Functions*

Xuexue Qian, Yasheng Ye

Department of Mathematics, College of Sciences, University of Shanghai for Science and Technology, Shanghai, China
Email: 1714774700@qq.com, yashengye@aliyun.com

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Abstract

In this paper, we mainly study the uniqueness of specific q-shift difference polynomials $f^n(z)\prod_{j=1}^{d} f(q_j z+c_j)^{v_j}$ and $g^n(z)\prod_{j=1}^{d} g(q_j z+c_j)^{v_j}$ of meromorphic functions, which share a common small function and get the corresponding results. In addition, we also investigate the problem of value distribution on q-shift difference polynomials of entire functions.

Keywords

Value Distribution, Meromorphic Functions, Difference Polynomials, Uniqueness

1. Introduction

In recent years, many Scholars have been interested in value distribution of difference operators of meromorphic functions (see [1]-[6]). Furthermore, a large number of papers have studied and obtained the uniqueness results of difference polynomials of meromorphic functions, their shifts and difference operators (see [7]-[12]). Our purpose in the paper is to study the value distribution for q-shift polynomials of transcendental meromorphic with zero order, and some results about entire functions.

For a meromorphic function f, we always assume that f is meromorphic in the complex plane $\mathbb{C}$. We use standard notations of the Nevanlinna Value Distribution Theory (see [13]), such as $m(r,f)$, $N(r,f)$, $\bar{N}(r,f)$, $T(r,f)$, $S(r,f)$, and define $N_2\left(r,\frac{1}{f}\right)$ as the counting function of zero of f, such that simple zero is counted once and multiple zeros are counted twice. We denote any quantity by $S(r,f)$, if it satisfies $S(r,f)=o(T(r,f))$, as $r\to\infty$

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outside of a possible exceptional set of r with finite logarithmic measure. In addition, the notation $\rho(f)$ is the order of growth of f. Let meromorphic function α be a common small function of $f(z)$ and $g(z)$, suppose that $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ have the same zeros counting multiplicities (ignoring multiplicities), then we say that f and g share $\alpha(z)$ CM(IM).

In this paper, we define a q-shift difference product of meromorphic function $f(z)$ as follows.

$$F(z)=f^{n}(z)\prod_{j=1}^{d}f\left(q_{j}z+c_{j}\right)^{v_{j}} \tag{1}$$

$$F_{1}(z)=P_{n}\left(f(z)\right)\prod_{j=1}^{d}f\left(q_{j}z+c_{j}\right)^{v_{j}} \tag{2}$$

where $c_j\in\mathbb{C}$ $\left(c_j\neq 0, j=1,2,3,\cdots,d\right)$ are distinct constants, $q_j\left(j=1,2,\cdots,d\right)$ be non-zero finite complex constants, let $P_n(z)=\alpha_n z^n+\alpha_{n-1}z^{n-1}+\cdots+\alpha_1 z+\alpha_0$ be a non-zero polynomial, where $\alpha_n(\neq 0),\alpha_{n-1},\cdots,\alpha_0$ are small functions of f. Let $n,d,v_j\left(j=1,2,\cdots,d\right)$ are positive integers and $\sigma=v_1+v_2+\cdots+v_d$.

Recently, Liu *et al.* [14] have considered and proved the uniqueness of q-shift difference polynomials of meromorphic functions.

Theorem A. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f)=\rho(g)=0$. Let q and η be two non-zero finite complex constants. If $f^n(z)f(qz+\eta)$ and $g^n(z)g(qz+\eta)$ share 1 CM, then either $f(z)=tg(z)$ or $f(z)g(z)=t$, where $n\left(\in N^*\right)\geq 14$ satisfying $t^{n+1}=1$.

Theorem B. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f)=\rho(g)=0$. Let q and η be two non-zero finite complex constants. If $f^n(z)f(qz+\eta)$ and $g^n(z)g(qz+\eta)$ share 1 IM, then either $f(z)=tg(z)$ or $f(z)g(z)=t$, where $n\left(\in N^*\right)\geq 26$ satisfying $t^{n+1}=1$.

First, we will prove the following theorems on value sharing results of q-shift difference polynomials extend the Theorem A, B, as follows:

Theorem 1.1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f)=\rho(g)=0$, and let $\alpha(z)(\not\equiv 0)$ be a common small function of $f(z)$ and $g(z)$. If $F(z)$ and $G(z)$ share $\alpha(z)$ CM, then $f(z)=tg(z)$, where $n\geq 4\min(2d,\sigma)+\sigma+9$ satisfying $t^{n+\sigma}=1$.

Theorem 1.2. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f)=\rho(g)=0$, and let $\alpha(z)(\not\equiv 0)$ be a common small function of $f(z)$ and $g(z)$. If $F(z)$ and $G(z)$ share $\alpha(z)$ IM, then $f(z)=tg(z)$, where $n\geq 4\min(2d,\sigma)+\sigma+6d+15$ satisfying $t^{n+\sigma}=1$.

Liu *et al.* [14] also considered some properties of q-shift difference polynomials of entire functions, as follow:

Theorem C. Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f)=\rho(g)=0$, and let q and η are two non-zero finite complex constants, and let $P_n(z)=\alpha_n z^n+\alpha_{n-1}z^{n-1}+\cdots+\alpha_1 z+\alpha_0$ be a non-zero polynomial, where $\alpha_n(\neq 0),\alpha_{n-1},\cdots,\alpha_0$, are constants, and let m be the number of the distinct zero of $P_n(z)$. If $P_n(f(z))f(qz+\eta)$ and $P_n(g(z))g(qz+\eta)$ share 1 CM, then only one of the following two cases holds:

a) $f(z)=tg(z)$, where $n>2m+1$, and k is greatest common divisor of $(\lambda_0,\lambda_1,\cdots,\lambda_n)$, satisfying $t^k=1$. When $\alpha_i=0$, then $\lambda_i=n+1$, otherwise $\lambda_i=i+1$. $i=0,1,\cdots,n$.

b) $f(z)$ and $g(z)$ satisfy a algebraic equation $Q(f(z),g(z))=0$, where

$$Q(w_1,w_2)=P_n(w_1)w_1(qz+c)-P_n(w_2)w_2(qz+c) \tag{3}$$

Next, it is easy to derive that $P_n(f(z))f(qz+\eta)$ in Theorem C can be replaced by $P_n(f(z))\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}$, as follows

Theorem 1.3. Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f)=\rho(g)=0$, and let $\alpha(z)$ be a common small function of $f(z)$ and $g(z)$, and let k be the number of distinct zeros of $P_n(z)$. If $F_1(z)$ and $G_1(z)$ share $\alpha(z)$ CM, then only one of the following results holds:

a) $f(z)=tg(z)$ for a constant t such that $t^m=1$, where $n>2k+2d+\sigma$ and m is greatest common divisor of $(n+\sigma,n+\sigma-1,\cdots,n+\sigma-i,\cdots,\sigma+1)$, $\alpha_{n-i}\neq 0$, $i=0,1,\cdots,n-1$.

b) $f(z)$ and $g(z)$ satisfy a algebraic equation $Q(f,g)\equiv 0$, where

$$Q(w_1,w_2)=P_n(w_1)\prod_{j=1}^{d}w_1(q_jz+c_j)^{v_j}-P_n(w_2)\prod_{j=1}^{d}w_2(q_jz+c_j)^{v_j}. \tag{4}$$

2. Some Lemmas

Lemma 2.1. (see [15]) Let $n(\geq 1)$ be a positive integer, and let $f(z)$ be a transcendental meromorphic function, and let $\alpha_i(i=0,1,\cdots,n)$ be small meromorphic functions of f. If

$$P_n(f(z))=\alpha_n f^n(z)+\alpha_{n-1}f^{n-1}(z)+\cdots+\alpha_1 f(z)+\alpha_0, \tag{5}$$

then

$$T(r,P_n(f(z)))=nT(r,f(z))+S(r,f(z)) \tag{6}$$

Lemma 2.2. (see [9]) Let q and η be two non-zero finite complex numbers, and let $f(z)$ be a nonconstant meromorphic function with $\rho(f)=0$, then

$$m\left(r,\frac{f(qz+\eta)}{f(z)}\right)=S(r,f). \tag{7}$$

on a set of logarithmic density 1.

Lemma 2.3. (see [12]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $f(z)$ and $g(z)$ share 1 IM and

$$L=\frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1} \tag{8}$$

If $L\not\equiv 0$, then

$$\begin{aligned} & T(r,f)+T(r,g) \\ & \le 2\left(N_2(r,f)+N_2(r,g)+N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)\right) \\ & \quad +3\left(\overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right)+S(r,f)+S(r,g) \end{aligned} \tag{9}$$

Lemma 2.4. (see [16]) Let f and g be two non-constant meromorphic functions. If f and g share 1 CM, then only one of the following results holds:

(a) $$\begin{aligned} & \max\{T(r,f),T(r,g)\} \\ & \le N_2(r,f)+N_2(r,g)+N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g) \end{aligned} \tag{10}$$

(b) $f\equiv g$;

(c) $fg\equiv 1$.

Lemma 2.5. (see [14]) Let q and η be two non-zero finite complex constants, and let f be a non-constant meromorphic function with $\rho(f)=0$, then

$$T(r,f(qz+\eta))\le T(r,f(z))+S(r,f) \tag{11}$$

on a set of logarithmic density 1.

Lemma 2.6. (see [14]) Let q and η be two non-zero finite complex constants, and let f be a nonconstant meromorphic function of zero order, then

$$\begin{aligned} & \overline{N}(r,f(qz+\eta))\le \overline{N}(r,f(z))+S(r,f) \\ & \overline{N}\left(r,\frac{1}{f(qz+\eta)}\right)\le \overline{N}\left(r,\frac{1}{f(z)}\right)+S(r,f) \\ & N(r,f(qz+\eta))\le N(r,f(z))+S(r,f) \\ & N\left(r,\frac{1}{f(qz+\eta)}\right)\le N\left(r,\frac{1}{f(z)}\right)+S(r,f). \end{aligned} \tag{12}$$

Lemma 2.7. Let $f(z)$ be a non-constant meromorphic function of zero order, and $F_1(z)$ be defined as in (2). Then

$$(n-\sigma)T(r,f)+S(r,f)\le T(r,F_1)\le (n+\sigma)T(r,f)+S(r,f) \tag{13}$$

Proof. Combining Lemma 2.1 with Lemma 2.5, we obtain

$$\begin{aligned} T(r,F_1) & \le T(r,P_n(f(z)))+T\left(r,\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}\right)+S(r,f) \\ & \le nT(r,f(z))+\sum_{j=1}^{d} T\left(r,f(q_jz+c_j)^{v_j}\right)+S(r,f) \\ & \le (n+\sigma)T(r,f(z))+S(r,f) \end{aligned} \tag{14}$$

In addition, by Lemma 2.1 and Lemma 2.5, we also get

$$\begin{aligned}
&(n+\sigma)T(r,f(z)) \le T\left(r,P_n(f(z))f^{\sigma}\right)+S(r,f)\\
&=m\left(r,P_n(f(z))f^{\sigma}\right)+N\left(r,P_n(f(z))f^{\sigma}\right)+S(r,f)\\
&\le m\left(r,\frac{F_1(z)f^{\sigma}}{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right)+N\left(r,\frac{F_1(z)f^{\sigma}}{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right)+S(r,f) \qquad (15)\\
&\le m(r,F_1)+N(r,F_1)+T\left(r,\frac{f^{\sigma}}{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right)+S(r,f)\\
&\le T(r,F_1)+2\sigma T(r,f)+S(r,f)
\end{aligned}$$

which is equivalent to

$$(n-\sigma)T(r,f)+S(r,f)\le T(r,F_1) \qquad (16)$$

Therefore, we get Lemma 2.7.

Lemma 2.8. Let $f(z)$ be an entire function with $\rho(f)=0$, and $F_1(z)$ be stated as in (2). Then

$$T(r,F_1)=(n+\sigma)T(r,f)+S(r,f) \qquad (17)$$

Proof. Using the same method as the Lemma 2.7, we can easily to prove.

3. Proof of Theorem

3.1. Proof of Theorem 1.1

Set $F^*(z)=\frac{F(z)}{\alpha(z)}$, $G^*(z)=\frac{G(z)}{\alpha(z)}$, than $F^*(z)$ and $G^*(z)$ share 1 CM. Thus by Nevanlinna second fundamental theory, Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned}
&(n-\sigma)T(r,f)+S(r,f)\le T\left(r,F^*(z)\right)\\
&\le \bar{N}\left(r,F^*(z)\right)+\bar{N}\left(r,\frac{1}{F^*(z)}\right)+\bar{N}\left(r,\frac{1}{F^*(z)-1}\right)+S\left(r,F^*(z)\right)\\
&\le \bar{N}\left(r,f^n\right)+\bar{N}\left(r,\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}\right)+\bar{N}\left(r,\frac{1}{f^n}\right) \qquad (18)\\
&+\bar{N}\left(r,\frac{1}{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right)+\bar{N}\left(r,\frac{1}{G^*(z)-1}\right)+S(r,f)\\
&\le (2d+2)T(r,f)+(n+\sigma)T(r,g)+S(r,g)+S(r,f)
\end{aligned}$$

Then

$$(n-2d-\sigma-2)T(r,f)\le (n+\sigma)T(r,g)+S(r,g)+S(r,f) \qquad (19)$$

Similarly,

$$(n-2d-\sigma-2)T(r,g)\le (n+\sigma)T(r,f)+S(r,f)+S(r,g) \qquad (20)$$

It follows that $S(r,f)=S(r,g)$.

Then by Lemma 2.4, we consider three subcases.

Case 1. Suppose that

$$\max\left\{T\left(r,F^*(z)\right),T\left(r,G^*(z)\right)\right\}\le N_2\left(r,F^*(z)\right)+N_2\left(r,\frac{1}{F^*(z)}\right)+N_2\left(r,G^*(z)\right)$$
$$+N_2\left(r,\frac{1}{G^*(z)}\right)+S\left(r,F^*(z)\right)+S\left(r,G^*(z)\right)$$

holds.

Through simple calculation, we have

$$\begin{aligned}N_2\left(r,F^*(z)\right)&\le N_2\left(r,f^n\right)+N_2\left(r,\prod_{j=1}^{d}f\left(q_jz+c_j\right)^{v_j}\right)\\&\le\left\{2+\min(2d,\sigma)\right\}T(r,f)+S(r,f)\end{aligned}\tag{21}$$

In the same way,

$$\begin{aligned}N_2\left(r,\frac{1}{F^*(z)}\right)&\le\left\{2+\min(2d,\sigma)\right\}T(r,f)+S(r,f)\\N_2\left(r,G^*(z)\right)&\le\left\{2+\min(2d,\sigma)\right\}T(r,g)+S(r,g)\\N_2\left(r,\frac{1}{G^*(z)}\right)&\le\left\{2+\min(2d,\sigma)\right\}T(r,g)+S(r,g)\end{aligned}\tag{22}$$

Combining Lemma 2.4, Lemma 2.7, Equations ((21) and (22)), we obtain that

$$\begin{aligned}&(n-\sigma)\left(T(r,f)+T(r,g)\right)\le T\left(r,F^*(z)\right)+T\left(r,G^*(z)\right)\\&\le 2N_2\left(r,F^*(z)\right)+2N_2\left(r,\frac{1}{F^*(z)}\right)+2N_2\left(r,G^*(z)\right)\\&\quad+2N_2\left(r,\frac{1}{G^*(z)}\right)+S\left(r,F^*(z)\right)+S\left(r,G^*(z)\right)\\&\le 4\left[2+\min(2d,\sigma)\right]\left(T(r,f)+T(r,g)\right)+S(r,f)+S(r,g)\end{aligned}\tag{23}$$

Then

$$\left(n-\sigma-8-4\min(2d,\sigma)\right)\left(T(r,f)+T(r,g)\right)\le S(r,f)\tag{24}$$

Which is impossible, since $n\ge 4\min(2d,\sigma)+\sigma+9$.

Case 2. Suppose that $F^*(z)\equiv G^*(z)$ holds, we obtain

$$f^n(z)\prod_{j=1}^{d}f\left(q_jz+c_j\right)^{v_j}=g^n(z)\prod_{j=1}^{d}g\left(q_jz+c_j\right)^{v_j}.\tag{25}$$

We assume that $h(z):=\dfrac{f(z)}{g(z)}$. If $h(z)\equiv\mathbf{C}$ (constant), then $f=tg$, and by substituting $f=tg$ into (25), we obtain that

$$g^n\prod_{j=1}^{d}g\left(q_jz+c_j\right)^{v_j}\left[t^{n+\sigma}-1\right]=0.\tag{26}$$

Since g is a transcendental meromorphic function, than

$g^n\prod_{j=1}^{d}g\left(q_jz+c_j\right)^{v_j}\not\equiv 0$. It follows that $t^{n+\sigma}=1$.

Suppose that $h(z)\not\equiv \mathbf{C}$ (constant), then using (25), we deduce that $h^n(z)=\prod_{j=1}^{d}\frac{1}{h\left(q_jz+c_j\right)^{v_j}}$,

So

$$nT\left(r,h(z)\right)=T\left(r,\prod_{j=1}^{d}\frac{1}{h\left(q_jz+c_j\right)^{v_j}}\right)\le \sigma T\left(r,h(z)\right)+S\left(r,h(z)\right) \tag{27}$$

We get a contradiction, since $n\ge 4\min(2d,\sigma)+\sigma+9$.

Case 3. Suppose that $F^*(z)G^*(z)\equiv 1$ holds, then

$f^n(z)\prod_{j=1}^{d}f\left(q_jz+c_j\right)^{v_j}\cdot g^n(z)\prod_{j=1}^{d}g\left(q_jz+c_j\right)^{v_j}=\alpha^2(z)$.

We define $h_1(z)=f(z)\cdot g(z)$, we easily get $h_1^n(z)=\prod_{j=1}^{d}\frac{\alpha^2(z)}{h_1\left(q_jz+c_j\right)^{v_j}}$ is non-constant, hence

$$nT\left(r,h_1(z)\right)=T\left(r,\prod_{j=1}^{d}\frac{\alpha^2(z)}{h_1\left(q_jz+c_j\right)^{v_j}}\right)\le \sigma T\left(r,h_1(z)\right)+S\left(r,h_1(z)\right) \tag{28}$$

We get a contradiction, since $n\ge 4\min(2d,\sigma)+\sigma+9$. This implies that $h_1(z)$ is a constant, which is impossible.

3.2. Proof of Theorem 1.2

Set $F^*(z)=\frac{F(z)}{\alpha(z)}$, $G^*(z)=\frac{G(z)}{\alpha(z)}$, So $F^*(z)$ and $G^*(z)$ share 1 IM.

Using the same arguments as in Theorem 1.1, we prove that (18)-(22) holds.

We can easily get

$$\begin{aligned}
&\bar{N}\left(r,F^*(z)\right)\le(1+d)T(r,f)+S(r,f)\\
&\bar{N}\left(r,\frac{1}{F^*(z)}\right)\le(1+d)T(r,f)+S(r,f)\\
&\bar{N}\left(r,G^*(z)\right)\le(1+d)T(r,g)+S(r,g)\\
&\bar{N}\left(r,\frac{1}{G^*(z)}\right)\le(1+d)T(r,g)+S(r,g)
\end{aligned} \tag{29}$$

Let

$$L(z)=\frac{F^{*\prime\prime}(z)}{F^{*\prime}(z)}-2\frac{F^{*\prime}(z)}{F^*(z)-1}-\frac{G^{*\prime\prime}(z)}{G^{*\prime}(z)}+2\frac{G^{*\prime}(z)}{G^*(z)-1} \tag{30}$$

If $L\not\equiv 0$, combining Lemma 2.3, (21), (22) with (29), we obtain

$$\begin{aligned}
&(n-\sigma)\left(T(r,f)+T(r,g)\right)\le T\left(r,F^*(z)\right)+T\left(r,G^*(z)\right)\\
&\le\left[14+6d+4\min(2d,\sigma)\right]\left(T(r,f)+T(r,g)\right)+S(r,f)+S(r,g)
\end{aligned} \tag{31}$$

Then,

$$\left(n-\sigma-14-6d-4\min\left(2d,\sigma\right)\right)\left(T\left(r,f\right)+T\left(r,g\right)\right)\le S\left(r,f\right)+S\left(r,g\right) \quad (32)$$

that is impossible, since $n\ge 4\min\left(2d,\sigma\right)+\sigma+6d+15$. Hence, we get $L\equiv 0$.

By integrating L twice, we obtain that

$$F^{*}=\frac{\left(b+1\right)G^{*}+\left(a-b-1\right)}{bG^{*}+\left(a-b\right)} \quad (33)$$

which yields $T\left(r,F^{*}\right)=T\left(r,G^{*}\right)+O\left(1\right)$. From Lemma 2.8, we deduced that $T\left(r,f\right)=T\left(r,g\right)+S\left(r,f\right)$. Next, we will consider the following three subcases.

Case 1. $b\ne 0$ and $b\ne -1$. Suppose that $a-b-1\ne 0$, by (33), we get

$$\overline{N}\left(r,\frac{1}{F^{*}}\right)=\overline{N}\left(r,\frac{1}{G^{*}-\frac{a-b-1}{b+1}}\right) \quad (34)$$

Combining the second fundamental theory with Lemma 2.5, Lemma 2.7, (29), and (34), we have

$$\begin{aligned}(n-\sigma)T\left(r,g\right)&\le T\left(r,G^{*}\left(z\right)\right)+S\left(r,g\right)\\&\le \overline{N}\left(r,G^{*}\left(z\right)\right)+\overline{N}\left(r,\frac{1}{G^{*}\left(z\right)}\right)+\overline{N}\left(r,\frac{1}{G^{*}-\frac{a-b-1}{b+1}}\right)+S\left(r,g\right)\\&\le \overline{N}\left(r,G^{*}\left(z\right)\right)+\overline{N}\left(r,\frac{1}{G^{*}\left(z\right)}\right)+\overline{N}\left(r,\frac{1}{F^{*}}\right)+S\left(r,g\right)\\&\le\left(2+2d\right)T\left(r,g\right)+\left(1+d\right)T\left(r,f\right)+S\left(r,g\right)\\&\le\left(3+3d\right)T\left(r,g\right)+S\left(r,g\right)\end{aligned} \quad (35)$$

which is impossible, since $n\ge 4\min\left(2d,\sigma\right)+\sigma+6d+15$. Therefore, $a-b-1=0$, so

$$F^{*}=\frac{\left(b+1\right)G^{*}}{bG^{*}+1} \quad (36)$$

Then, $\overline{N}\left(r,\frac{1}{F^{*}}\right)=\overline{N}\left(r,\frac{1}{G^{*}+1/b}\right)$. Similarly, we have

$$\begin{aligned}(n-\sigma)T\left(r,g\right)&\le \overline{N}\left(r,G^{*}\left(z\right)\right)+\overline{N}\left(r,\frac{1}{G^{*}\left(z\right)}\right)+\overline{N}\left(r,\frac{1}{G^{*}+1/b}\right)+S\left(r,g\right)\\&\le \overline{N}\left(r,G^{*}\left(z\right)\right)+\overline{N}\left(r,\frac{1}{G^{*}\left(z\right)}\right)+\overline{N}\left(r,\frac{1}{F^{*}}\right)+S\left(r,g\right)\\&\le\left(2+2d\right)T\left(r,g\right)+\left(1+d\right)T\left(r,f\right)+S\left(r,g\right)\\&\le\left(3+3d\right)T\left(r,g\right)+S\left(r,g\right)\end{aligned} \quad (37)$$

Which is impossible, since $n\ge 4\min\left(2d,\sigma\right)+\sigma+6d+15$.

Case 2. If $b=0$ and $a=1$, then $F^{*}\equiv G^{*}$ obviously. From the proof of case 2 in theorem 1.1, we get $f\left(z\right)=tg\left(z\right)$, where $t^{n+\sigma}=1$. Therefore, we consider $b=0$ and $a\neq 1$. Then from (33), we obtain

$$F^{*}=\frac{G^{*}+a-1}{a}. \tag{38}$$

Using the same discuss as Case 1, we get contradiction.

Case 3. If $b=-1$ and $a=-1$, then $F^{*}G^{*}\equiv 1$ obviously. Thus from the proof of case 3 in theorem 1.1, we get a contradiction. Therefore, we consider $b=-1$ and $a\neq -1$. From (33), we get

$$F^{*}=\frac{a}{a+1-G^{*}}. \tag{39}$$

Which is impossible, using the similar method as Case 1.

3.3. Proof of Theorem 1.3

We use the similar method as [14]. By the theorem condition that $F_1(z)-\alpha(z)$ and $G_1(z)-\alpha(z)$ share 0 CM, hence there exist an entire function $u(z)$, than

$$\frac{F_1(z)-\alpha(z)}{G_1(z)-\alpha(z)}=\mathrm{e}^{u(z)}. \tag{40}$$

Since $\rho(f)=\rho(g)=0$, than $\mathrm{e}^{u(z)}\equiv\eta$ is a constant.

Rewriting (40)

$$G_1(z)=F_1(z)+(\eta-1)\alpha(z) \tag{41}$$

If $\eta\neq 1$, we can use Nevanlinnas two fundamental theorems, Lemma 2.5 and Lemma 2.8 to get a contradiction, since $n>\sigma+2k+2d$.

So we get $\eta=1$. Rewriting (40)

$$\begin{aligned}&P_n(f(z))\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}\\&=P_n(g(z))\prod_{j=1}^{d}g(q_jz+c_j)^{v_j}.\end{aligned} \tag{42}$$

Set $h(z):=\dfrac{f(z)}{g(z)}$, suppose that $h(z)\equiv\mathbf{C}$ (constant), then $f=tg$. Then we take $f=tg$ into (42) and get

$$\prod_{j=1}^{d}g(q_jz+c_j)^{v_j}\left[\alpha_n g^n\left(t^{n+\sigma}-1\right)+\alpha_{n-1}g^{n-1}\left(t^{n+\sigma-1}-1\right)+\cdots+\alpha_1 g\left(t^{\sigma+1}-1\right)\right]\equiv 0. \tag{43}$$

where α_n is a non-zero complex constant. And $\prod_{j=1}^{d}g(q_jz+c_j)^{v_j}\not\equiv 0$, since g is transcendental meromorphic function. So $h^m=1$, where m is greatest common divisor of $(n+\sigma,n+\sigma-1,\cdots,n+\sigma-i,\cdots,\sigma+1)$, $\alpha_{n-i}\neq 0$ ($i=0,1,\cdots,n-1$).

Suppose that $h(z)\not\equiv\mathbf{C}$ (constant), Equation (43) imply that $f(z)$ and $g(z)$ satisfy a algebraic equation $Q(f,g)\equiv 0$, where

$$Q(w_1,w_2)=P_n(w_1)\prod_{j=1}^{d}w_1(q_jz+c_j)^{v_j}-P_n(w_2)\prod_{j=1}^{d}w_2(q_jz+c_j)^{v_j}. \tag{44}$$

4. Conclusion

In this paper, we obtain some important results about the uniqueness of specific q-shift difference polynomials of meromorphic functions by Nevanlinna and value distribution theory and extend previous results. In addition, we also investigate the problem of value distribution on q-shift difference polynomials of entire functions.

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