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Crystallography in Spaces E^2, E^3, E^4, E^5 … Study of Three Crystal Families of Space E^5

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Abstract

In two previous papers, we explained the classification of all crystallographic point groups of n-dimensional space with $n \le 6$ into different isomorphism classes and we describe some crystal families. This paper mainly consists in the study of three crystal families of space E^5, the (di-iso hexagons)-al, the hypercube 5 dim and the (hypercube 4 dim)-al crystal families. For each studied family, we explain their name, we describe their cell and we list their point groups which are classified into isomorphism classes. Then we give a WPV symbol to each group. (WPV means Weigel Phan Veysseyre). Our method is based on the description of the cell of the holohedry of each crystal family and of the results given by the Software established by one of us. The advantage to classify the point groups in isomorphism classes is to give their mathematical structures and to compare their WPV symbols. So the study of all crystal families of space E^5 is completed. Some crystal families of space E^5 can be used to describe di incommensurate structures and quasi crystals.

Keywords

Crystallographic Point Groups, Isomorphism Classes, Hyper Cube and (di iso Hexagons)-al Crystal Families, Cubic and iso Cubic Point Groups

1. Introduction

The crystal families of fourth-dimensional space E^4 have been studied in the paper [1].

The crystal families of five-dimensional space E^5 have been studied in different papers:

- the names of the 32 crystal families together with the WPV symbols of their holohedries are listed in paper [2].

- families numbered I, II, III, IV, V, VI, VII, XII, XIII, XVI, XVII in paper [3].
- families numbered I to XV together with all point groups in paper [4].
- families numbered XII, XVI, XVII, XIX, XX, XXI in paper [5].
- families numbered XVIII, XXII, XXVI, XXVII in paper [6].
- families numbered XXIII, XXIV, XXV, XXX, XXXII in paper [7].

The studied families have been reassembled thanks to the geometric nature of their cell. Their numbers appear in different papers as for instance in the paper of Plesken [8].

To end the study of all crystal families of space E^5, we list the properties and the crystallographic point groups of the three following families: the (di iso hexagons)-al family (N°XXIX), the hyper cube 5 dim. family (N°XXXI) and the (hyper cube 4 dim)-al family (N°XXVIII). Their names are explained in different paragraphs.

From now, we use cr for crystallographic.

For each family, we use the results given by the Scientific Software established by H. Veysseyre [9] (SS E5 is the given name of this software) *i.e.* the point operation list and the sub group list of the holohedries of the three studied families. These results have been completed by the geometric nature of the cell. The given symbols are in agreement with the "Hermann-Mauguin" symbols of the cr point groups of spaces E^2 and E^3 and respect the International Sub-Commission of the Nomenclature recommendations [10]. However some symmetry operations appear in spaces E^4 and E^5 as double rotations or reflection-double rotations and we introduce some new notations explained in different paragraphs.

The cr point groups of a given family are rearranged in "isomorphism classes" which have been explained in the previous two papers [7], [11]. We only give the examples of the classes C5 and C5 × C2 then D5 and D5 × C2 or D10.

The isomorphism class C5 (generated by a cyclic group of order 5) has the identity and one double rotation of order 5 generated by the operation $5^1 5^3$, *i.e.* 4 elements of order 5, so we write this list of elements 4(5) 1. Only one cr point group belongs to this class, the group denoted [5] (space E^4) of the crystal family decadic-al (N°XXIV, [7]). Then, we consider the isomorphism class C5 × C2 or C10 (generated by a cyclic group of order 10) with 4(10) 4(5) 1(2) 1 for elements, this list means 4 elements of order 10, 4 elements of order 5, one element of order 2, and the identity. These elements are obtained as the Cartesian product of the elements of groups C5 and C2 (4(5) 1) × (1(2) 1) = 4(10) 4(5) 1(2) 1. To obtain this result, we do the direct product of one element of order 5 and one element 2 which gives one element of order 10, therefore 4 elements of order 10, then the product of the 4 elements of order 5 with the identity give 4 elements of order 5. We repeat this process with the second element of group C2 *i.e.* the identity. We finally obtain 4(10) 4(2) 4(5) 1 Three cr point groups of class C5 × C2 belong to the family N°XXIV [7], they have for WPV symbols [10], $\left[\bar{\bar{5}}\right]$ and [5] ⊥ m. The isomorphism class D5 dihedral group of order 10 has 4(5) 5(2) 1 for elements and 2 groups of family N°XXIV [7] belong to this class. Their WPV symbols are [5] 2 (space E^4), $[5]\bar{1}$.

Then, we consider the isomorphism class D5 × C2 or D10 (dihedral group of order 20) with 4(10) 4(5) 11(2) 1 for elements; these elements are obtained as the Cartesian product of the elements of groups D5 and C2. Four cr point groups of class D5 × C2 belong to family N°XXI, they are [10] 2 2 (space E^4), $\left[10\right]\bar{1}\,\bar{1}$, $\left[\bar{\bar{5}}\right]2\,\bar{1}$ and ([5] 2) ⊥ m.

In these two examples, we note that a group g_4 (space E^4) of a class C, gives a group g_4 ⊥ m in the class C × C2. We find this general property for any class C (see Tables 1-5).

Other examples of direct product of arrangements are given in paragraph 2.2.

2. Some Properties of the Element List of a Finite Point Group

2.1. Remark

The number of elements of order two is an odd number whereas the one of elements of any order different of two is always an even number.

2.2. Relations between the Element Numbers of Different Orders

We begin with an example, the direct product of the two groups Q8 and C2.

Elements of group Q8: 6(4) 1(2) 1, of group C2: 1(2) 1.

Elements of the group Q8 × C2 (6(4) 1(2) 1) × (1(2) 1) = 12(4) 3(2) 1 (Table 4).

Indeed, the direct product of the 6 elements of order 4 with 1 element of order 2 gives 6 elements of order 4 and with the identity 6 elements of order 4 therefore 12 elements of order 4. The three elements of order 2 are easily found.

It is possible to generalize this process for any Cartesian products but it is not valid for a semi direct product. For instance if n_2 is the number of elements of order 2 in the group C, n_2' the one in the group C × C2, then $n_2' = 2n_2 + 1$

Now, we give the example of the direct product D3 × D3 (Table 1).

Elements of dihedral group D3 (order 6) 2(3) 3(2) 1.

Elements of the group D3 × D3

$$(2(3)\ 3(2)\ 1) \times (2(3)\ 3(2)\ 1) = 4(3)\ 6(6)\ 2(3)\ 6(6)\ 9(2)\ 3(2)\ 2(3)\ 3(2)\ 1$$
$$= 12(6)\ 8(3)\ 15(2)\ 1.$$

3. General Introduction about the Three Studied Families

1) The (di iso hexagons)-al family splits into two sub-families, the primitive sub-family (N°XXIX) and the centered sub-family (N°XXIXa). The holohedry symbols of these two families are respectively (([12] 2 2).6 mm) ⊥ m (order 24 × 12 × 2 = 576) and $\left(\left(63\ \ 2\ \ 2\right)\bullet 3\mathrm{m}\right)\times\bar{1}_5$ (order 12 × 6 × 2 = 144). These families have respectively 89 and 15 cr point groups.

2) The hyper cube 5 dim. family (N°XXXI) has for holohedry the cr point group $\left(\left(\left[5\right]\ \ 2\right)\bullet\left[8\right]\bullet\left(42\ \ 3\ \ \bar{1}_4\right)\right)\times\bar{1}_5$ (order 10 × 8 × 24 × 2 = 3840). This family is an irreducible crystal family of space E^5, [12], therefore all these 13 point groups belong to space E^5.

3) The (hyper cube 4 dim.)-al family splits into two sub-families, the primitive sub-family (N˚XXVIIIa) and the (hyper cube 4 dim. Z centered)-al sub-family (N˚XXVIII). The holohedry symbols of these two families are respectively $\left([8]\bullet \mathrm{m}\bar{3}\mathrm{m}\right)\perp \mathrm{m}$ (order 8 × 48 × 2 = 768) and $\left(\left([12]-[8]\right)\bullet \bar{4}3\mathrm{m}\right)\perp \mathrm{m}$ (order 48 × 24 × 2 = 2304). These two families have 90 and 51 cr point groups.

The name of these crystal families, the building of their cells and some notations used in different WPV symbols are explained below.

4. (Di iso Hexagons)-al Crystal Families (N˚XXIX)

4.1. Geometric Study of the Cell

As the name suggested it, the cell of the primitive family is a right hyper prism (suffix "al"), the basis of which is constituted by two equal hexagons belonging to two orthogonal planes, it is the reason why the words di and iso appear in the family name. This cell is the one of the family N˚XXI of space E^4 (system 29) described in paper [1].

The metric tensor of the quadratic form defining the cell of this family is as follows (matrix N˚1):

Matrix N˚1 associated with the cell of the (di iso hexagons)-al family in space E^5

$$\begin{pmatrix} a & -a/2 & 0 & 0 & 0 \\ -a/2 & a & 0 & 0 & 0 \\ 0 & 0 & a & -a/2 & 0 \\ 0 & 0 & -a/2 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}$$

Caption Let be denoted e_i (i =1, … , 5) the five axis which define the Euclidean space E^5

$$\|e_i\|^2 = a \ \forall i = 1,\cdots,4 \qquad \|e_5\|^2 = b \qquad e_1 \bullet e_2 = e_2 \bullet e_1 = e_3 \bullet e_4 = e_4 \bullet e_3 = -a/2$$

This tensor depends on two length parameters: the side of the two hexagons (a) and the length of the hyper prism (b).

4.2. Properties of the Point Groups and of the Isomorphism Classes of the Two Families N˚XXIX

The orders of all cr point groups of these two families and of the isomorphism classes are multiples of 12, *i.e.* of the form p × 12 (with p integer) except for one class of order 18 The p values are 1, 2, 3, 4, 6, 8, 12, 24 and 48 for the holohedry of the primitive family (48 × 12 = 576). Indeed, the family cell is built from two equal hexagons belonging to two orthogonal planes. As a matter of fact, the cr point group of the hexagon is 6 mm of order 12 and the one of the equilateral triangle is 3 m of order 6; these two groups belong to the cr point groups of the two families N˚XXIX. It is the reason why the orders of the cr point groups are on the form p × 12 and why two groups of order 18 appear into the list of the groups.

Table 1 lists the 15 cr point groups of family XXIXa and Table 2 the 89 cr point groups of family XXIX

Table 1. Cr point groups of the centered (di iso hexagons)-al family (N°XXIXa).

Classes	WVP Symbols of the the cr point groups	Arrangements	Orders
D3 × C3	$(33\ 2)\times 3$ E^4; $(33\ \bar{1})\times 3$	6(6) 8(3) 3(2)	18
D3 × D3	$(33\ 2)\times(3\ 2)$ E^4; $(33\ \bar{1})\times(3\ 2)$; $(33\ \bar{1})\times(3\ \bar{1})$	12(6) 8(3) 15(2)	36
D3 × C3 × C2	$((33\ 2)\times 3)\times\bar{1}_5$	20(6) 8(3) 7(2)	36
(C3 × C3).C4	$(3\perp 3)\bullet\bar{4}$ E^4; $(3\perp 3)\bullet 42$	18(4) 8(3) 9(2)	36
D3 × D3 × C2	$(33\ 2)\times(3\ 2)\times\bar{1}_5$	32(6) 8(3) 31(2)	72
((C3 × C3).C4) × C2	$((3\perp 3)\bullet\bar{4})\times\bar{1}_5$	8(6) 36(4) 8(3) 19(2)	72
D6.D3	$(63\ 2\ 2)\bullet 3m$ E^4; $(63\ 2\ 2)\bullet(3\ \bar{1}_4)$ $(\overline{\overline{36}}\ 2\ \bar{1})\bullet(3\ \bar{1}_4)$; $(\overline{\overline{36}}\ 2\ \bar{1})\bullet 3m$	24(6) 18(4) 8(3) 21(2))	72
(D6.D3) × C2	$((63\ 2\ 2)\bullet 3m)\times\bar{1}_5$	56(6) 36(4) 8(3) 43(2))	144

Table 2. Cr point groups of the primitive (di iso hexagons)-al family (N°XXIX).

Classes	WVP Symbols of the the Point Groups	Arrangements	Orders
Q12	$44\bullet 33$ E^4, $\overline{\overline{44}}\bullet 33$	2(6) 6(4) 2(3) 1(2)	12
Q12 × C2	$(44\bullet 33)\perp m$	6(6) 12(4) 2(3) 3(2)	
C6.C4	$63\bullet 44$ E^4; $63\bullet\overline{\overline{44}}$; $\overline{\overline{36}}\bullet 44$; $\overline{\overline{36}}\bullet\overline{\overline{44}}$	6(6) 6(4) 2(3) 9(2)	24
D3 × C4	$(33\ 2)\times 44$ E^4; $(33\ 2)\times\overline{\overline{44}}$ $(33\ \bar{1})\times 44$; $(33\ \bar{1})\times\overline{\overline{44}}$	4(12) 2(6) 8(4) 2(3) 7(2)	24
D4 × C3	$(44\ 2\ 2)\times 33$ E^4; $(44\ \bar{1}\ \bar{1})\times 33$; $(\overline{\overline{44}}\ 2\ \bar{1})\times 33$	4(12) 10(6) 2(4) 2(3) 5(2)	24
D3 × C6	$(33\ 2)\times 63$ E^4; $(33\ \bar{1})\times\overline{\overline{36}}$	20(6) 8(3) 7(2)	36
D3 × C3 × C2	$((33\ 2)\times 3)\perp m$	20(6) 8(3) 7(2)	36
Q12 × C3	$(44\bullet 33)\times 3$ E^4; $(\overline{\overline{44}}\bullet 33)\times 3$	12(12) 8(6) 6(4) 8(3) 1(2)	36
D4 × D3	$(44\ 2\ 2)\times(33\ 2)$ E^4; $(44\ 2\ 2)\times(33\ \bar{1})$ $(\overline{\overline{44}}\ 2\ \bar{1})\times(33\ 2)$; $(\overline{\overline{44}}\ 2\ \bar{1})\times(33\ \bar{1})$, $(44\ \bar{1}\ \bar{1})\times(33\ 2)$; $(44\ \bar{1}\ \bar{1})\times(33\ \bar{1})$	4(12) 10(6) 8(4) 2(3) 23(2)	48
(C6.C4) × C2	$(63\bullet 44)\perp m$	14(6) 12(4) 2(3) 19(2)	48
D3 × C4 × C2	$((33\ 2)\times 44)\perp m$	8(12) 6(6) 16(4) 2(3) 15(2)	48
D4 × C3 × C2	$((44\ 2\ 2)\times 33)\perp m$	8(12) 22(6) 4(4) 2(3) 11(2)	48
D6 × D3	$(63\ 2\ 2)\times(33\ 2)$ E^4; $(63\ \bar{1}\ \bar{1})\times(33\ 2)$ $(\overline{\overline{36}}\ 2\ \bar{1})\times(33\ 2)$	32(6) 8(3) 31(2)	72
D3 × D3 × C2	$((33\ 2)\times(3\ 2))\perp m$	32(6) 8(3) 31(2)	72
C12.C6	$[12]\bullet 63$ E^4; $[12]\bullet\overline{\overline{36}}$; $[\overline{\overline{12}}]\bullet 63$; $[\overline{\overline{12}}]\bullet\overline{\overline{36}}$	12(12) 20(6) 6(4) 8(3) 25(2)	72
D6.D3	$(63\ 2\ 2)\bullet(3\ \bar{1})$ E^4; $(63\ 2\ 2)\bullet(3\ 2)$ $(\overline{\overline{36}}\ 2\ \bar{1})\bullet(3\ \bar{1})$, $(\overline{\overline{36}}\ 2\ \bar{1})\bullet(3\ 2)$	24(6) 18(4) 8(3) 21(2)	72

Continued

((C3 × C3).C4) × C2	$\left((3\perp 3)\bullet\bar{4}\right)\times\bar{1}_4$ E^4; $\left((3\perp 3)\bullet 42\right)\times\bar{1}_4$; $\left((3\perp 3)\bullet\bar{4}\right)\perp m$	8(6) 36(4) 8(3) 19(2)	72
C12.D3	$[12]\bullet(3\ 2)$ E^4; $\left[\overline{\overline{12}}\right]\bullet(3\ 2)$; $[12]\bullet(3\ \bar{1})$	24(12) 8(6) 12(4) 8(3) 19(2)	72
(C6.C4) × C3	$(63\bullet 44)\times 3$ E^4; $(\overline{\overline{36}}\bullet\overline{\overline{44}})\times 3$; $(63\bullet\overline{\overline{44}})\times 3$; $(\overline{\overline{36}}\bullet 44)\times 3$	12(12) 36(6) 6(4) 8(3) 9(2)	72
D3 × C6 × C2	$\left((33\ 2)\times 63\right)\perp m$	48(6) 8(3) 15(2)	72
Q12 × C3 × C2	$\left((44\bullet 33)\times 3\right)\perp m$	24(12) 24(6) 12(4) 8(3) 3(2)	72
D4 × D3 × C2	$\left((44\ 2\ 2)\times(33\ 2)\right)\perp m$	8(12) 22(6) 16(4) 2(3) 47(2)	96
D12.C6	$([12]\ 2\ 2)\bullet 63$ E^4; $([12]\ 2\ 2)\bullet\overline{\overline{36}}$; $([12]\ \bar{1}\ \bar{1})\bullet 63$ $([12]\ \bar{1}\ \bar{1})\bullet\overline{\overline{36}}$; $\left(\left[\overline{\overline{12}}\right]\ 2\ \bar{1}\right)\bullet 63$; $\left(\left[\overline{\overline{12}}\right]\ 2\ \bar{1}\right)\bullet\overline{\overline{36}}$	24(12) 48(6) 12(4) 8(3) 51(2)	144
(C12.C6) × C2	$\left([12]\bullet 63\right)\perp m$	24(12) 48(6) 12(4) 8(3) 51(2)	144
((C3 × C3).C4) × C2) × C2	$\left(\left((3\perp 3)\bullet\bar{4}\right)\times\bar{1}_4\right)\perp m$	24(6) 72(4) 8(3) 39(2)	144
(C6 × C6).C4	$(6\perp 6)\bullet\bar{4}$ E^4; $(6\perp 6)\bullet 42$; $(66\times\bar{3})\bullet\bar{4}$	24(6) 72(4) 8(3) 39(2)	144
(D6.D3) × C2	$\left((63\ 2\ 2)\bullet 3m\right)\times\bar{1}_4$ E^4; $\left((63\ 2\ 2)\bullet 3m\right)\perp m$ $\left((63\ 2\ 2)\bullet(3\ 2)\right)\times\bar{1}_4$; $\left((63\ 2\ 2)\bullet(3\ 2)\right)\times\bar{1}_5$ $\left((\overline{\overline{36}}\ 2\ \bar{1})\bullet(3\ 2)\right)\times\bar{1}_4$; $\left((\overline{\overline{36}}\ 2\ \bar{1})\bullet 3m\right)\times\bar{1}_4$ $\left((63\ 2\ 2)\bullet(3\ \bar{1})\right)\perp m$	56(6) 36(4) 8(3) 43(2)	144
D6 × D3 × C2	$\left((63\ 2\ 2)\times(3\ 2)\right)\perp m$	72(6) 8(3) 63(2)	144
C12.D6	$[12]\bullet\bar{3}m$ E^4; $\left[\overline{\overline{12}}\right]\bullet\bar{3}m$ $[12]\bullet(62\ 2\ 2)$; $\left[\overline{\overline{12}}\right]\bullet(62\ 2\ 2)$	24(12) 32(6) 48(4) 8(3) 31(2)	144
(C12.D3) × C2	$\left([12]\bullet(3\ 2)\right)\perp m$	48(12) 24(6) 24(4) 8(3) 39(2)	144
(C6.C4) × C3 × C2	$\left((63\bullet 44)\times 3\right)\perp m$	24(12) 80(6) 12(4)8(3)19(2)	144
D12.D6	$([12]\ 2\ 2)\bullet 6mm$ E^4, $([12]\ 2\ 2)\bullet 622$, $([12]\ 2\ 2)\bullet\bar{3}m$ $\left(\left[\overline{\overline{12}}\right]\bar{1}\ 2\right)\bullet 6mm$, $\left(\left[\overline{\overline{12}}\right]\bar{1}\ 2\right)\bullet 622$, $\left(\left[\overline{\overline{12}}\right]\bar{1}\ 2\right)\bullet\bar{3}m$	24(12) 96(6) 84(4) 8(3) 75(2)	288
(D12.C6) × C2	$\left(([12]\ 2\ 2)\bullet 63\right)\perp m$	48(12) 104(6) 24(4) 8(3) 103(2)	288
((C6 × C6).C4) × C2	$\left((6\perp 6)\bullet\bar{4}\right)\perp m$	56(6) 144(4) 8(3) 79(2)	288
((D6.D3) × C2) × C2	$\left(\left((63\ 2\ 2)\bullet 3m\right)\times\bar{1}_4\right)\perp m$	120(6) 72(4) 8(3) 87(2)	288
(C12.D6) × C2	$\left([12]\bullet\bar{3}m\right)\perp m$	48(12) 72(6) 96(4) 8(3) 63(2)	288
(D12.D6) × C2	$\left(([12]\ 2\ 2)\bullet 6mm\right)\perp m$	48(12) 200(6) 168(4) 8(3) 151(2)	576

Caption of the two Table 1 and Table 2: First column: Mathematical symbols of the isomorphism classes. Second column: WPV symbols of the cr point groups of the centered (di iso hexagons)-al crysal family (Table 1), of the primitive (di iso hexagons)-al crystal family (Table 2), the cr point groups of space E^4 are pointed out. Third column: List of the symmetry elements with their numbers of every isomorphism class. Fourth column: Order of these classes.

4.3. Remarks about Some Notations of the WPV Symbols

1) We recall the well-known symbols used in the WPV symbols; the cross $\times$ for a direct product, the point. for a semi-direct product, the geometric symbol $\perp$ for a geometric and direct product.

2) A great number of cr point groups are isomorphic to dihedral groups. Some of them are well known as 3 m (isomorphic to group D3 of order 6), 4 mm (isomorphic to group D4 of order 8), 6 mm (isomorphic to group D6 of order 12). In **Table 1**, another dihedral groups appear:

The groups (63 2 2), (66 2 2) isomorphic to dihedral group D6 (order 12), 63 is the symbol of a double rotation of order 6 generated by a rotation of order 6 and a rotation of order 3 into two orthogonal planes, 66 has a similar property.

The group ([8] 2 2) isomorphic to group D8 (order 16),

The groups ([12] 2 2), ([12] $\bar{1}$ $\bar{1}$), ([$\bar{\bar{12}}$] 2 $\bar{1}$) isomorphic to group D12 (order 24).

As previously, the first number of the symbol *i.e.* 63, 8, 12 for instance, gives the order of the rotation generating the dihedral group and the following numbers are operations of order two. We note that the order of the group is the double of the order of the first cr. point operation.

4.4. Summary

Among the 15 + 89 = 104 cr point groups of the two families N˚XXIX, 22 cr point groups g_4 belong to space E^4 and define 22 isomorphic classes that we denote Cg_4. These 22 groups appear in space E^5 under the form $g_4 \perp$ m or $g_4 \times \bar{1}_5$.

18 cr point groups g_4 give 18 cr point groups on the form $g_4 \perp$ m, therefore 18 isomorphic classes denoted $Cg_4 \times C2$.

4 cr point groups give 2 groups $g_4 \perp$ m and $g_4 \times \bar{1}_5$ which belong to the same isomorphic class, therefore 4 isomorphism classes. However 2 groups $g_4 \perp$ m belong to another classes defined by another group g'_4.

Two arrangements define two different isomorphism classes.

Then, the result is that the 104 cr point groups of the two families N˚XXIX belong to 22 + 18 + (4 − 2) + 2 = 44 different isomorphic classes.

The 13 cr point groups.

5. (Hyper Cube 5 dim) Crystal Family (N° XXXI)

5.1. Geometrical and Analytic Description of the Hyper Cube 5 dim. (H5) of Space E5

The cell of the family N˚XXXI is a regular hyper cube, one of the five regular polytopes of the Euclidean five-dimensional space [13]. It is the generalization of the square (space E^2), of the cube (space E^3). The hyper cube of space E^4, one of the six regular polytopes of this Euclidean space can be built by translating a cube (of space E^3) along a line orthogonal to space E^3 of a length equal to the side of the cube. This polytope is called "Hyper cube 4 dim." or H4 for short. The description of its cell and the list of its cr point groups are in [14]. Then, we repeat the same method from the hyper cube H4 in order to obtain the hypercube H5. We recall that the square has 4 vertices and 4 sides, the cube 8 vertices, 12 sides and 6 faces (equal squares); the hyper cube H4 has 16 vertices, 32 sides and 24 faces, it is bounded by 16 equal cubes or volumes. The hyper cube H5 has 32 vertices, 80 sides and 24 faces, it is bounded by 16 equal cubes or volumes, it is bounded by 20 equal cubes and 10 equal hyper cubes H4.

In Euclidean space E^3, the characteristic numbers of a regular polytope verify the Euler relation:

Number of vertices − number of sides + number of faces = 2 (8 − 12 + 6 = 2 for the cube).

This relation becomes Number of vertices + number of faces = number of sides + number of volumes (16 + 24 = 32 + 8) for the hyper cube H4.

The analytic description of the hyper cube H5 can be obtained by choosing an orthonormal basis denoted (O, i, j, k, l, m) where O is the center of the hyper cube H5 and (i, j, k, l, m) the names of the unit vectors of the five axes. The 32 vertices of H5 have for coordinates: ±1, ±1, ±1, ±1, ±1. The side of this hyper cube has for length 2. The 10 hyper cubes H4 which bound the hyper cube H5 belong to space (i, j, k, l), (j, k, l, m) and so on…, one coordinate of the center Oi (i = 1, …., 10) equals +1 (or −1) and the other are null for instance (0, 0, −1, 0, 0). Thanks to this analytic description, it is easy to write the matrices of all the point groups of the hyper cube H5.

The metric tensor of the quadratic form defining the cell of this family is as follows (matrix N˚2).

Matrix N˚2 associated with the cell of the (hyper cube 5 dim.) family in space E^5

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix}$$

Caption $\|e_i\|^2 = a, \ \forall i = 1, \cdots, 5 \qquad e_i \bullet e_j = 0, \ \forall i, \forall j, i \neq j$

5.2. Order of the Symmetry Group of the Holohedry of the Hyper Cube H5

The order of the symmetry group of the hyper cube H5 is 3840, this number is given by the software SS5. It can be found by the relation given in the publication to the Academic Sciences of Paris (1980) [15]. Indeed, we find that the holohedry order of the hyper cube H4 is 384. The hyper cube H5 can be built as a right hyper prism of basis the hyper cube H4 hence the holohedry order is 384 × 2 and this decomposition can be done in 5 different ways therefore the holohedry order is 384 × 2 × 5 = 3840.

5.3. Properties of the Point Groups and of the Isomorphism Classes of the Family N°XXXI

The software SS5 gives 13 point groups to the family XXXI. This family is an ir-

Table 3. Cr point groups of the (hyper cube 5 dim) crystal family (N˚XXXI).

Cr point groups of the (Hyper cube 5 dim) crystal family			
Classes	WPV Symbols	Arrangement	Order
$C5.(C2)^4$	$[5]\bullet 2222$	64(5) 15(2)	80
$(C5.(C2)^4) \times C2$	$([5]\bullet 2222)\times \bar{1}_5$	64(10) 64(5) 31(2)	160
D5. (D4 × C2)	$([5]\ 2)\bullet((42\ 2\ 2)\times \bar{1}_4)$; $([5]\ \bar{1})\bullet((\bar{4}\ 2\ \bar{1})\times \bar{1}_4)$	64(5) 60(4) 35(2)	160
$D5.(C2)^5$	$([5]\ 2)\bullet mmmmm$	64(10) 64(5) 120(4) 71(2)	320
D5.C8.C4	$([5]\ 2)\bullet[8]\bullet 42$; $([5]\ 2)\bullet[\bar{\bar{8}}]\bullet\bar{4}$	80(8) 64(5) 140(4) 35(2)	320
(D5.C8.C4) × C2	$(([5]\ 2)\bullet[8]\bullet 42)\times \bar{1}_5$	64(10) 160(8) 64(5) 280(4) 71(2)	640
D5.(A4 × C2).C4	$([5]\ 2)\bullet(2\ 62)\bullet 42$	384(5) 240(6) 180(4) 80(3) 75(2)	960
D5.C8.S4	$([5]\ 2)\bullet[8]\bullet(42\ 3\ \bar{1}_4)$; $([5]\ 2)\bullet[\bar{\bar{8}}]\bullet\bar{4}3m$	160(12) 240(8) 384(5) 400(6) 500(4) 80(3) 155(2)	1920
(D5.(A4 × C2).C4) × C2	$(([5]\ 2)\bullet(2\ 62)\bullet 42)\times \bar{1}_5$	384(10) 384(5) 560(6) 360(4) 80(3) 151(2)	1920
(D5.C8.S4) × C2	$(([5]\ 2)\bullet[8]\bullet(42\ 3\ \bar{1}4))\times \bar{1}_5$	320(12) 384(10) 480(8) 384(5) 880(6) 1000(4) 80(3) 311(2)	3840

First column: Mathematical symbols of the isomorphism classes. Second column: WPV symbols of the cr point groups of the (hyper cube 5 dim) crystal family. Third column: List of the symmetry elements with their numbers of every isomorphism class. Fourth column: Order of these classes. $(C2)^4$ is the abridged notation of C2 × C2 × C2 × C2.

reducible family of space E^5, [12], therefore all the point groups belong to space E^5, no group takes the form $g_4 \perp m$ or $g_4 \times \bar{1}_5$ where g_4 is a group of space E^4. The 13 cr point groups belong to 10 isomorphism classes. **Table 3** lists the 13 cr point groups of family N˚XXXI.

5.4. Summary

The 13 cr point groups of family N˚XXXI belong to 10 isomorphic classes:

7 classes with one point group only,

3 classes with two point groups.

6. (Hyper Cube 4 dim)-al Crystal Family (N˚XXVIII)

6.1. Geometric Study of the Cell

The suffix "al" means that the cell of this family is a right hyper prism, its basis is a regular hyper cube H4. As the hyper cube family H4 (space E^4). This family splits into two sub families: the primitive sub-family N˚XXVIIIa with $([8]\bullet m\bar{3}m)\perp m$ for holohedry symbol (order 8 × 48 × 2 = 768) and 90 cr point groups and the (hyper cube 4 dim. Z centered)-al sub-family N˚XXVIII with $(([12]-[8])\bullet\bar{4}3m)\perp m$ for holohedry (order 48 × 24 × 2 = 2304 and 51 cr point groups. We note that the order of the holohedry of the centered sub-family is greater than the one of the primitive family; as in space E^4.

The metric tensor of the quadratic form defining the cell of this family is as follows (matrix N˚3).

Matrix N˚3 associated with the cell of the (hyper cube 4 dim.)-al family in

space E^5

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}$$

Caption $\|e_i\|^2 = a,\ \forall i = 1,\cdots,4 \quad \|e_5\|^2 = b \quad e_i \bullet e_j = 0,\ \forall i, \forall j, i \neq j$

It depends on 2 parameters of length: a the side of the hypercube H4 and b the side of the hyper prism.

6.2. Properties of the Point Groups and of the Isomorphism Classes of the Two Families N°XXVIII

The orders of all cr point groups of these two families and of the isomorphism classes are multiples of 8, *i.e.* of the form p × 8 (with p integer). The p values are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144 and 288 for the holohedry (288 × 8 = 2304) of the centered family. Indeed, another name of the family N˚XXVIII, (hyper cube 4 dim.)-al could be "(di iso squares)-al because its cell is built from two equal squares in space E^4 belonging to two orthogonal planes. As a matter of fact, the cr point group of the square is 4 mm of order 8. It is the reason why the orders of the cr point groups are on the form p × 8. The hyper cube of space E^4 has been studied in paper [14].

Table 4 lists the 90 cr point groups of family XXVIIIa and **Table 5** the 51 cr point groups of family XXVIII.

6.3. Remark about One Notation of the WPV Symbols

Besides the well-known marks as the cross × for a direct product, the point. for a semi-direct product, the geometric symbol ⊥ for a geometric and direct product, another mark has been introduced into some symbol, a hyphen -.This mark is used into the symbol 44 - 44, of order 8 in space E^4, the two generators have a common element, the homothetie $\bar{1}_4$, therefore it is not a semi-direct product; in **Annex**, we show how the 8 elements of this cr point group can be obtained. This mark is used for the following point groups of order 16 [8]-[8], [8]-[$\bar{\bar{8}}$], of order 48 [12]-[8], [12]-[12], [12]-[$\bar{\bar{12}}$]. The order of the groups so generated depend of the number of common elements of the generators.

In **Annex**, we give generators of some cr point groups.

6.4. Summary

Among the (90 + 51) = 141 point groups of the two families XXVIII, 37 point groups g_4 belong to space E^4 and define 37 isomorphic classes as we denote Cg_4. All these 37 groups appear in space E^5 under the form g_4m and belong to 37 isomorphism classes denoted $Cg_4 \times C2$ and the 141 point groups of the two families XXVIII belong to 37 + 37 = 74 isomorphism classes.

Table 4. Cr point groups of the primitive (hypercube 4 dim)-al crystal family (N°XXVIIIa).

Cr point groups of the primitive crystal family (Hyper cube 4 dim)-al			
Classes	WPV Symbols	Arrangement	Order
Q8 & (C4-C4)	$44-44\ \ E^4;\ \ 44-\overline{\overline{44}}$	6(4) 1(2)	8
Q8.C2	$(44-44)\bullet 2\ \ E^4;\ \ (44-44)\bullet\bar{1}\,;$ $\left(44-\overline{\overline{44}}\right)\bullet 2\,;\ \ \left(44-\overline{\overline{44}}\right)\bullet\bar{1}$	8(4) 7(2)	16
Q8 × C2	$(44-44)\perp m$	12(4) 3(2)	16
C8.C2	$[8]\bullet 2\ \ E^4;\ \ [8]\bullet\bar{1}\,;\ \ \left[\bar{\bar{8}}\right]\bullet 2\,;\ \ \left[\bar{\bar{8}}\right]\bullet\bar{1}$	4(8) 6(4) 5(2)	16
C8-C8	$[8]-[8]\ \ E^4;\ \ [8]-\left[\bar{\bar{8}}\right];\ \ \left[\bar{\bar{8}}\right]-\left[\bar{\bar{8}}\right]$	8(8) 4(4) 3(2)	16
Q8.C3	$(44-44)\bullet 3\ \ E^4$	8(6) 6(4) 8(3) 1(2)	24
Q8.(C2 × C2)	$(44-44)\bullet 222\ \ E^4;\ \ (44-44)\bullet(2\times\bar{1})\,;\ \ \left(44-\overline{\overline{44}}\right)\bullet(2\times\bar{1})$	12(4) 19(2)	32
D8.C2	$([8]\ 2\ 2)\bullet 2\ \ E^4;\ \ ([8]\ 2\ 2)\bullet\bar{1}\,;\ \ ([8]\ \bar{1}\ \bar{1})\bullet 2\,;\ \ ([8]\ \bar{1}\ \bar{1})\bullet\bar{1}\,;$ $\left(\left[\bar{\bar{8}}\right]\ 2\ \bar{1}\right)\bullet 2\,;\ \ \left(\left[\bar{\bar{8}}\right]\ 2\ \bar{1}\right)\bullet\bar{1}$	8(8) 8(4) 15(2)	32
C4.(C4 × C2)	$\bar{4}\bullet(44\times 2)\ \ E^4;\ \ 42\bullet(44\times 2)\,;\ \ 42\bullet\left(\overline{\overline{44}}\times 2\right)$	20(4) 11(2)	32
(C8-C8).C2	$([8]-[8])\bullet 2\ \ E^4;\ \ \left(\left[\bar{\bar{8}}\right]-\left[\bar{\bar{8}}\right]\right)\bullet 2\,;\ \ \left([8]-\left[\bar{\bar{8}}\right]\right)\bullet\bar{1}$	16(8) 4(4) 11(2)	32
C8.C4	$[8]\bullet 42\ \ E^4;\ \ [8]\bullet\bar{4}\,;\ \ \left[\bar{\bar{8}}\right]\bullet 42\,;\ \ \left[\bar{\bar{8}}\right]\bullet\bar{4}$	8(8) 16(4) 7(2)	32
(Q8.C2) × C2	$((44-44)\bullet 2)\perp m$	16(4) 15(2)	32
(C8.C2) × C2	$([8]\bullet 2)\perp m$	8(8) 12(4) 11(2)	32
(C8-C8) × C2	$([8]-[8])\perp m$	16(8) 8(4) 7(2)	32
C8.D3	$[8]\bullet(3\ 2)\ \ E^4;\ \ \left[\bar{\bar{8}}\right]\bullet(3\ \bar{1})$	12(8) 8(6) 6(4) 8(3) 13(2)	48
(Q8.C3) × C2	$((44-44)\bullet 3)\perp m$	24(6) 12(4) 8(3) 3(2)	48
D8.C4	$([8]\ 2\ 2)\bullet 42\ \ E^4;\ \ ([8]\ 2\ 2)\bullet\bar{4}\,;\ \ ([8]\ \bar{1}\ \bar{1})\bullet 42$ $([8]\ \bar{1}\ \bar{1})\bullet\bar{4}\,;\ \ \left(\left[\bar{\bar{8}}\right]\ 2\,\bar{1}\right)\bullet 42\,;\ \ \left(\left[\bar{\bar{8}}\right]\ 2\,\bar{1}\right)\bullet\bar{4}$	16(8) 20(4) 27(2)	64
D4.(C4 × C2)	$\bar{4}2m\bullet(44\times 2)\ \ E^4;\ \ (42\ 2\ 2)\bullet(44\times 2)$ $(42\ 2\ 2)\bullet(44\times\bar{1})\,;\ \ \bar{4}2m\bullet(44\times\bar{1})$	36(4) 27(2)	64
(C4 × C4).C4	$(4\perp 4)\bullet\bar{4}\ \ E^4;\ \ (4\perp 4)\bullet 42\,;\ \ (\bar{4}\times\bar{4})\bullet 42$	44(4) 19(2)	64
C8.D4	$[8]\bullet\bar{4}2m\ \ E^4,\ \ [8]\bullet(42\ \bar{1}_4\ 2),\ \ \left[\bar{\bar{8}}\right]\bullet\bar{4}2m\,,\ \ \left[\bar{\bar{8}}\right]\bullet(42\ \bar{1}_4\ 2)$	16(8) 28(4) 19(2)	64
Q8.(C2 × C2 × C2	$((44-44)\bullet 222)\perp m$	24(4) 39(2)	64
(D8.C2) × C2	$(([8]\ 2\ 2)\bullet 2)\perp m$	16(8) 16(4) 31(2)	64
(C4.C4 × C2) × C2	$(\bar{4}\bullet(44\times 2))\perp m$	40(4) 23(2)	64
((C8-C8).C2) × C2	$(([8]-[8])\bullet 2)\perp m$	32(8) 8(4) 23(2)	64
(C8.C4) × C2	$([8]\bullet 42)\perp m$	16(8) 32(4) 15(2)	64
D4.A4	$(44\ 2\ 2)\bullet 23\ \ E^4$	32(6) 12(4) 32(3) 19(2)	96

Continued

(C8.D3) × C2	$([8]\bullet(3\ \ 2))\perp \mathrm{m}$	24(8) 24(6) 12(4) 8(3) 27(2)	96
D8.D4	$([8]\ 2\ 2)\bullet\bar{4}2\mathrm{m}$ E^4; $([8]\ 2\ 2)\bullet(42\ 2\ 2)$ $([\bar{\bar{8}}]\ 2\ \bar{1})\bullet\bar{4}2\mathrm{m}$ $([\bar{\bar{8}}]\ 2\ \bar{1})\bullet(42\ 2\ 2)$; $([8]\ \bar{1}\ \bar{1})\bullet\bar{4}2\mathrm{m}$; $([\bar{\bar{8}}]\ 2\ \bar{1})\bullet(42\ \mathrm{m}\ \bar{1})$	16(8) 68(4) 43(2)	128
(D8.C4) × C2	$(([8]\ 2\ 2)\bullet 42)\perp \mathrm{m}$	32(8) 40(4) 55(2)	128
D4.(C4 × C2) × C2	$(\bar{4}2\mathrm{m}\bullet(44\times 2))\perp \mathrm{m}$	72(4) 55(2)	128
((C4 × C4).C4) × C2	$((4\perp 4)\bullet\bar{4})\perp \mathrm{m}$	88(4) 39(2)	128
(C8.D4) × C2	$([8]\bullet\bar{4}2\mathrm{m})\perp \mathrm{m}$	32(8) 56(4) 39(2)	128
D8.A4	$([8]\ 2\ 2)\bullet 23$ E^4; $([\bar{\bar{8}}]\ 2\ \bar{1})\bullet 23$	48(8) 32(6) 36(4) 32(3) 43(2)	192
D4.S4	$(\bar{4}\ 2\ \bar{1})\bullet\bar{4}3\mathrm{m}$ E^4; $(42\ \bar{1}_4\ 2)\bullet(42\ \bar{1}_4\ 3)$	32(6) 84(4) 32(3) 43(2)	192
D4.(A4 × C2)	$(\bar{4}\ 2\ \bar{1})\bullet\mathrm{m}\bar{3}$ E^4; $(42\ \bar{1}_4\ 2)\bullet(2\ 62)$	96(6) 36(4) 32(3) 27(2)	192
(D4.A4) × C2	$((42\ 2\ 2)\bullet 23)\perp \mathrm{m}$	96(6) 24(4) 32(3) 39(2)	192
((C8-C8).D4) × C2	$(([8]\ \ 2\ \ 2)\bullet\bar{4}2\mathrm{m})\perp \mathrm{m}$	32(8) 136(4) 87(2)	256
C8.(S4 × C2)	$[8]\bullet\mathrm{m}\bar{3}\mathrm{m}$ E^4; $[8]\bullet(2\ 62\ 2)$; $[\bar{\bar{8}}]\bullet(\mathrm{m}\ \bar{3}\ 2)$; $[\bar{\bar{8}}]\bullet(2\ 62\ \mathrm{m})$	48(8) 96(6) 132(4) 32(3) 75(2)	384
(D8.A4) × C2	$(([8]\ 2\ 2)\bullet 23)\perp \mathrm{m}$	96(8) 96(6) 72(4) 32(3) 87(2)	384
(D4.S4) × C2	$((\bar{4}\ 2\ \bar{1})\bullet\bar{4}3\mathrm{m})\perp \mathrm{m}$	96(6) 168(4) 32(3) 87(2)	384
D4.(A4 × C2) × C2	$((\bar{4}\ 2\ \bar{1})\bullet\mathrm{m}\bar{3})\perp \mathrm{m}$	224(6) 72(4) 32(3) 55(2)	384
(C8.(S4 × C2)) × C2	$([8]\bullet\mathrm{m}\bar{3}\mathrm{m})\perp \mathrm{m}$	96(8) 224(6) 264(4) 32(3) 151(2)	768

Table 5. Cr point groups of the centered (hypercube 4 dim)-al crystal family (N˚XXVIII).

Cr point groups of the centered crystal family (Hyper cube 4 dim)-al			
Classes	WPV Symbols	Arrangement	Order
Q8.C3	$(44-44)\bullet 33$ E^4	8(6) 6(4) 8(3) 1(2)	24
Q8 × C3	$(44-44)\times 33$ E^4; $(44-\overline{\overline{44}})\times 33$	12(12) 2(6) 6(4) 2(3) 1(2)	24
C8.C3	$[8]\bullet 33$ E^4; $[\bar{\bar{8}}]\bullet 33$	4(12) 12(8) 2(6) 2(4) 2(3) 1(2)	24
C8.D3	$[8]\bullet(33\ 2)$ E^4; $[\bar{\bar{8}}]\bullet(33\ \bar{1})$	12(8) 8(6) 6(4) 8(3) 13(2)	48
C12-C8	$[12]-[8]$ E^4; $[12]-[\bar{\bar{8}}]$; $[\overline{\overline{12}}]-[8]$; $[\overline{\overline{12}}]-[\bar{\bar{8}}]$	12(12) 12(8) 2(6) 6(4) 2(3) 13(2)	48
C12-C12	$[12]-[12]$ E^4; $[\overline{\overline{12}}]-[\overline{\overline{12}}]$	16(12) 8(6) 8(4) 8(3) 7(2)	48
(Q8.C3) × C2	$((44-44)\bullet 33)\perp \mathrm{m}$	24(6) 12(4) 8(3) 3(2)	48
(Q8 × C3) × C2	$((44-44)\times 33)\perp \mathrm{m}$	24(12) 6(6) 12(4) 2(3) 3(2)	48
(C8.C3) × C2	$([8]\bullet 33)\perp \mathrm{m}$	8(12) 24(8) 6(6) 4(4) 2(3) 3(2)	48

Continued

C12.C6	$[12]\bullet 62 \quad E^4$	12(12) 26(6) 6(4) 26(3) 1(2)	72
D8.D3	$([8]\ 2\ 2)\bullet(33\ \ 2) \quad E^4$; $([8]\ \bar{1}\ \bar{1})\bullet(33\ \ 2)$ $\left(\left[\bar{\bar{8}}\right]\ 2\ \bar{1}\right)\bullet(33\ \bar{1})$	16(12) 24(8) 8(6) 8(4) 8(3) 31(2)	96
(C12-C12).C2	$([12]-[12])\bullet 2 \quad E^4$; $\left(\left[\bar{\bar{12}}\right]-\left[\bar{\bar{12}}\right]\right)\bullet\bar{1}$	48(12) 8(6) 12(4) 8(3) 19(2)	96
(C8.D3) × C2	$([8]\bullet(33\ 2))\perp m$	24(8) 24(6) 12(4) 8(3) 27(2)	96
(C12-C8) × C2	$([12]-[8])\perp m$	24(12) 24(8) 6(6) 12(4) 2(3) 27(2)	96
C8.C3.C4	$[8]\bullet 33\bullet 42 \quad E^4$; $\left[\bar{\bar{8}}\right]\bullet 33\bullet\bar{4}$	16(12) 24(8) 8(6) 32(4) 8(3) 7(2)	96
(C12-C12) × C2	$([12]-[12])\perp m$	32(12) 24(6) 16(4) 8(3) 15(2)	96
(C12-C8).C3	$([12]-[8])\bullet 3 \quad E^4$; $\left([12]-\left[\bar{\bar{8}}\right]\right)\bullet 3$	12(12) 36(8) 26(6) 6(4) 26(3) 37(2)	144
(C12.C6) × C2	$([12]\bullet 62)\perp m$	24(12) 78(6) 12(4) 26(3) 3(2)	144
D12.D4	$([12]\ 2\ 2)\bullet(42\ 2\ 2) \quad E^4$; $([12]\ 2\ 2)\bullet(\bar{4}\ \ 2\ \bar{1})$ $([12]\ \bar{1}\ \bar{1})\bullet(42\ \bar{1}\ \bar{1})$; $([12]\ \bar{1}\ \bar{1})\bullet(\bar{4}\ \ 2\ \bar{1})$	48(12) 48(8) 8(6) 36(4) 8(3) 43(2)	192
(D8.D3) × C2	$(([8]\ 2\ 2)\bullet(33\ \ 2))\perp m$	32(12) 48(8) 24(6) 16(4) 8(3) 63(2)	192
((C12-C12).C2) × C2	$(([12]-[12])\bullet 2)\perp m$	96(12) 24(6) 24(4) 8(3) 39(2)	192
(C8.C3.C4) × C2	$([8]\bullet 33\bullet 42)\perp m$	32(12) 48(8) 24(6) 64(4) 8(3) 15(2)	192
((C12-C8).C3) × C2	$(([12]-[8])\bullet 3)\perp m$	24(12) 72(8) 78(6) 12(4) 26(3) 75(2)	288
C12.A4.C2	$[12]\bullet 23\bullet 2 \quad E^4$	96(12) 80(6) 12(4) 80(3) 19(2)	288
(D12-D4) × C2	$(([12]\ 2\ 2)\bullet(42\ 2\ 2))\perp m$	96(12) 96(8) 24(6) 72(4) 8(3) 87(2)	384
(C12-C8).D6	$([12]-[8])\bullet(62\ 2\ 2) \quad E^4$; $\left([12]-\left[\bar{\bar{8}}\right]\right)\bullet(62\ \bar{1}\ \ \bar{1})$	96(12) 144(8) 80(6) 84(4) 80(3) 91(2)	576
C12.S4.C2	$[12]\bullet\bar{4}3m\bullet 2 \quad E^4$; $[12]\bullet(42\ 3\ \bar{1}_4)\bullet 2$	96(12) 272(6) 84(4) 80(3) 43(2)	576
(C12.A4.C2) × C2	$([12]\bullet 23\bullet 2)\perp m$	192(12) 240(6) 24(4) 80(3) 39(2)	576
(C12-C8).S4	$([12]-[8])\bullet\bar{4}3m \quad E^4$; $([12]-[8])\bullet 432$ $\left([12]-\left[\bar{\bar{8}}\right]\right)\bullet\bar{4}3m$	96(12) 144(8) 464(6) 228(4) 80(3) 139(2)	1152
((C12-C8).D6) × C2	$(([12]-[8])\bullet(62\ 2\ 2))\perp m$	192(12) 288(8) 240(6) 168(4) 80(3) 183(2)	1152
(C12.S4.C2) × C2	$([12]\bullet\bar{4}3m\bullet 2)\perp m$	192(12) 624(6) 168(4) 80(3) 87(2)	1152
((C12-C8).S4) × C2	$(([12]-[8])\bullet\bar{4}3m)\perp m$	192(12) 288(8) 1008(6) 456(4) 80(3) 279(2)	2304

Caption of the two Table 4 and Table 5: First column: Mathematical symbols of the isomorphism classes. Second column: WPV symbols of the cr point groups of the primitive (hyper cube 4 dim)-al crystal family (Table 4), of the centered (hyper cube 4 dim)-al crystal family (Table 5); the cr point groups of space E^4 are pointed out. Third column: List of the symmetry elements with their numbers of every isomorphism class. Fourth column: Order of these classes.

7. Conclusion

This paper brings a final term to the study of all the crystal families and of the crystallographic point groups of space E^5. This study has a mathematic interest, with the list of the point groups in isomorphism classes but it can be used for the study of the incommensurate structures and of the quasi crystals [16]. We have studied some families of space E^6 used for the tri incommensurate structures.

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Annex

In this annex, we explain how some groups can be defined from their generators. Let be denoted (x, y, z, t) the four axes of space E^4 and (x, y, z, t, u) the five axes of space E^5.

Group 44-44, isomorphism class C4-C4

The 8 elements of group 44-44 of family XXVIIIa (space E^4) can be obtained in the following way. One double rotation 44 is generated by the element $4_{xy}^{+1}4_{zt}^{+1}$ and another one by the element $4_{xz}^{+1}4_{yt}^{-1}$. The product of these two rotations is easy if we use a matrix representation as below:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 4_{xt}^{-1}4_{yz}^{-1}$$

In this way, we can obtain the 6 elements of order 4. The homothetie $\bar{1}_4$ is the square of any matrix and belongs to the two generators.

Group $44-\overline{\overline{44}}$, isomorphism class C4-C4

For the group $44-\overline{\overline{44}}$ (space E^5) of family XXVIIIa, we take for generators $4_{xy}^{+1}4_{zt}^{+1}$ and $4_{xz}^{+1}4_{yt}^{-1}m_u$. The product of these two rotations is easy if we use a matrix representation as below:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = 4_{xt}^{-1}4_{yz}^{-1}m_u$$

The homothetie $\bar{1}_4$ is the square of any matrix.

Group (44-44).2, isomorphism class (C4-C4).C2

For the group (44-44).2 (order 16, space E^5) of family XXVIIIa, we take for generators the two double rotations of order 4, $4_{xy}^{+1}4_{zt}^{+1}$, $4_{xz}^{+1}4_{yt}^{-1}$ and the rotation of order 2, 2_{xy}. The product of the two rotations $4_{xy}^{+1}4_{zt}^{+1}$ and 2_{xy} gives another double rotation 44. The other rotations of order 2 are obtained from the product of the different rotations 44 and $\bar{1}_4$ with 2_{xy}. The product of these two rotations is easy if we use a matrix representation as below

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = 4_{xy}^{-1}4_{zt}^{+1}$$

Group (44-44).$\bar{1}$, isomorphism class (C4-C4).C2

For the group (44-44). $\bar{1}$ (order 16, space E^5) of family XXVIIIa we take for generators $4_{xy}^{+1}4_{zt}^{+1}$, $4_{xz}^{+1}4_{yt}^{-1}$ and the homothetie of order 2, $\bar{1}_{xyu}$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = \bar{1}_{\text{xyu}} .$$

The different operations of this group are obtained as previously (product of matrices).