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A Survey of the System *N* of Natural Numbers Assigned to Primary Teachers

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Abstract

From our standpoint, a school teacher should be acquainted, in a deeper way, with the content of teaching themes which are on school curricula. In case of primary teachers and their preparation for teaching mathematics, they should have a solid knowledge of the properties of the system *N* of natural numbers and an understanding of its position as being a basis upon which all other number systems are built. Up to some degree, these teachers should also be acquainted with further extensions of number systems going along the line as it is done in school: natural—positive rational—integer—rational—real numbers. These extensions are enlightened by Peacock's principle of invariance of the form—a rule derived for natural numbers, when expressed in general form (as a literal relation) continues to hold true in all extended systems. In Section 2 of this survey, a precise terminology is fixed which is needed for the study of the system *N* and in particular, for making a difference between syntactic and semantic concepts. The Cantor principle which expresses the dependence of conception of number on perception of set is also formulated and largely exploited in this paper. In Section 3, several rules are derived when different expressions denoting two different groupings of elements of a set are equated. Forgetting that the variables are bound to *N*, all these rules also express the properties of the extended number systems, as well as they are algebraic laws or their derivatives. At the end, discovering of rules of correspondence of sequences given by a number of their initial terms is considered as a type of exercises which help the development of the idea of variable. Some cases of finding formulae for sums of consecutive natural numbers are also included. This paper is intended to be a paradigmatic example how a mathematical content has to be elaborated to serve best the school teachers to deepen their knowledge of subject matter.

Subject Areas

Education

Keywords

Expressions and Relations in the System N, Distinction between Syntactic and Semantic Concepts, Derivation of the Rules in N and the Peacock's Principle of Invariance of Form, Formulae Expressing the Number of Dots (Circlets) of Regular Patterns

1. Introduction

The students at the institutions where primary school teachers are educated usually have on their curricula: a course of mathematics and a course of didactics of mathematics. The main aim of the latter of these courses is a clear presentation of the ways how teaching themes of primary school mathematics are elaborated to serve best the learner. But the ideas what should be the content of the former course vary considerably from country to country (and from institution to institution). Some of those ideas could be rather idealistic, for example, when the content contains elements of differential and integral calculus, elements of abstract algebra, etc. and when it is supposed that further and further learning of mathematics at upper levels deepens the knowledge of these students, but, in reality, it usually burdens them too much. We find that the content of this course has to be subordinate to the needs of the course of didactics of mathematics and one of its central themes should be "Repetition of School Mathematics".

Though the title of the above theme connotes a repetition, it does not mean that the same things (school math content) should be treated in exactly the same way as they are presented in the classroom. Actually, this content should be exposed in a deeper way, much beyond its didactical transformations found in school books. As the concepts synthesized in primary mathematics are developed further at the proximate stages of learning in the upper classes of elementary school, primary mathematics together with this further development is the school mathematics that is the subject of this repetition. To be more precise, the content that we have in mind should contain a clear and deeper exposition of extensions of number systems up along the way as it is done in school: N, (natural numbers); Q_+ (positive rationals); Z (integers); Q (rationals); R (reals). These extensions are enlightened by Peacock's principle of the permanence of form ([1] [2]). This principle says that a rule derived for the natural numbers, when expressed in general form (as a literal relation) continues to hold true in all extended systems. Therefore, we see that the starting point is the system N, which is also teaching and learning theme of primary mathematics. (A primary teacher is often directed wrongly to upper classes math, there to learn properties of arithmetic operations, though all such properties are carried over from the system N to its further extensions. Thus, a thorough acquaintance with the properties of operations in N is essential for both primary and all math teachers).

We also note that in this survey we will use some terms and concepts from secondary school math courses which these students have had, before they enrolled educational institutions, where they are trained. (That range of knowledge should, of course, be included in math course for which this survey is also sketched).

2. Expressions and Relations in the System *N* of Natural Numbers

The natural numbers are the basis from which all other number systems are built. On the one hand, there were mathematicians, called Naturalists, (among them the great classical mathematician Henri Poincaré, (1854-1912), the German mathematician Leopold Kronecker, (1823-1891), etc.) who considered the natural numbers to be a direct product of the human mind. A very well-known Cronecker's saying "God made the integers, all else is the work of man" expresses their point of view in a nice symbolic way. On the other hand, the so called Formalists considered the natural numbers to be constructions executed on the logical basis. We take here the system of natural numbers in the way it is established throughout learning in school, avoiding all formal constructions.

In the set theory, two sets have the same number of elements when there exists a 1 – 1 and onto mapping from one of these sets onto the other. As the collections of visible things in the surrounding world (together with their pictorial representations) consist the phenomenology upon which the ideas of natural numbers are formed, we will call this type of collections of objects, the *sets at the sensory level.* Operations with this type of sets are expressed by the use of natural language (without using the syntactic signs of set theory).

G. Cantor in his paper ([3]), expressed the set-number dependence, which we modify slightly to also express the dependence of conception of number on perception of set.

Cantor principle of invariance of number. Starting with perception of a set A of visible objects and abstracting (*forgetting*):

(i) *The nature of these objects*,

and

(ii) *Any kind of their organization.*

an abstract idea $\overline{\overline{A}}$ *of number results.*

Cantor uses two bars over the letter denoting a set to emphasize two above abstractions. Let us also add that under an organization of elements we understand any way how they can be ordered, arranged, grouped, etc.

Dealing with sets at sensory level and performing the activities of abstracting (governed by the Cantor principle), in the course of learning arithmetic in school, first the individual concepts of numbers 1, 2, ⋯, 10 are formed on the basis of counting as an ordered reciting of number names up to 10. Then, the block N_{20} of numbers up to 20 is established together with the operations of addition and subtraction and the relation "is less than". This block is, then, extended to the block N_{100} of numbers up to 100, within which the permanent meaning of multiplication and division is established. Further extensions are the block N_{1000}

of numbers up to 1000 and, ultimately, the system *N* of all natural numbers. Let us note that "establish" is the proper word for activities of forming concepts at the level of primary school, when they are learnt in the contact with series of examples (rather than by definitions).

2.1. Numerical Expressions

The simplest numerical expressions are decimal notations for natural numbers. For example, 2, 34, 258, 2017, etc., are such simple expressions. Next, the sums, as for example, 2 + 7, 21 + 8, 231 + 75, etc., the differences, as for example, 7 – 5, 18 – 13, 4825 – 794, etc., the products, as for example, 6 × 8, 31 × 52, 665 × 2337, etc., the quotients, as for example, 16:4, 125:25, 7020:90, etc. are also expressions. In general, expressions are sums, differences products and quotients of the already formed expressions which are put in the parentheses. For example when the components of the product are the sums 4 + 8 and 12 + 9, composing a new expression we write (4 + 8) × (12 + 9). When an expression is just decimal notation for a number, then it is not put in parentheses and also, as a matter of convention, products and quotients of two expressions are not put in parentheses. For example, we write. (9 – 5) × 7 instead of (9 – 5) × (7), (7 + 8) – (2 + 5) × (9 – 7) instead of (7 + 8) – ((2 + 5) × (9 – 7)), etc.

The value of an expression is the number which it stands for. Very different expressions can represent one and the same number. But the most informative of them all is the decimal notation of that number. That is why, when we say "find the value of an expression", we think of finding the decimal notation for that expression.

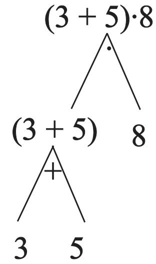
Parentheses are understood as the command "first do what is in parentheses". For example, when calculating the value of the expressions 20 – (16 – 7), 3 × 7 + 32:8, etc. we proceed in the following way: 20 – (16 – 7) = 20 – 9 = 11, 3 × 7 + 32:8 = 21 + 4 = 35, (in this example parentheses are omitted but the command is not forgotten), etc.

Let us now consider a number of examples:

(a) (3 + 5) × 8, (b) (7 – 4) × (8 + 3), (c) (7 + 17): 6, (d) 84: (5 + 7).

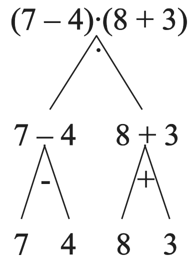
To express clearly the order in which operations are performed, expressions are decomposed in the form of a jotting called the *tree of an expression*.

(a) The expression (3 + 5) × 8 is seen as the product of expressions 3 + 5 and 8, while 3 + 5 is the sum of 3 and 5. This decomposition is represented by the following diagram.

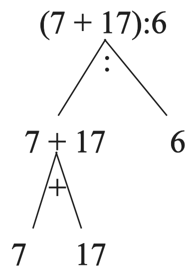


which is its tree.

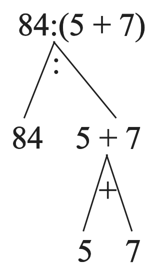
(b) The expression (7 − 4) × (8 + 3) is the product of the difference 7 − 4 and the sum 8 + 3 and its tree is:



(c) The expression (7 + 17):6 is the quotient of 7 + 17 and 6. Its tree is:



(d) Similarly, the tree of the expression 84:(5 + 7) is:



The expressions are just jottings and some of them do not have their value in *N*. For example, such are the expressions:

4 − 7, 13:5, 83 − 84, 411:412, 3548:5746, etc. (and let you yourself add more such examples). The reader should differentiate between syntactic concepts, as they are digits, decimal notations, sums, differences, products and quotients and expressions in general and semantic concepts, as they are numbers, values of expressions, etc.

2.2. Expressions with a Variable

It was François Viète (1540-1603), who introduced the concept of variable in mathematics. According to him a variable is a letter denoting not only one number but a whole species of numbers. In analogy with Grecian *logistica numerosa*, he called his new algebra *logistica speciosa*. Thus, a variable in a set is a letter denoting any element of that set and consequently, a *variable in the set N of all natural numbers* is a letter denoting any natural number. When a letter, say x, is added to the simplest expressions in N, combining expression with the operational signs, all jottings obtained in that way are called *expressions with a variable* (say with x).

Examples of expressions with the variable x are:

(a) $3x + 2$, (b) $7x - 23$, (c) $15 - x$, (d) $15: x$, etc.

(Let us recall that between a number and the letter x, the multiplication sign is not written. We write $7x$, $15x$, etc. understanding that 7, 15, etc. are multiplicands and x is the multiplier. We also include zero in the set N (as it is done in school arithmetic)).

Now let us consider the above examples.

(a) When x takes its values 0, 1, 2, 3, and so onwards the expression $3x + 2$ also takes its values $3 \times 0 + 2$, $3 \times 1 + 2$, $3 \times 2 + 2$, $3 \times 3 + 2$ and so onwards. This dependence of the value of the expression $3x + 2$ on the values of x is represented by the following table:

x	0	1	2	3	⋯
$3x + 2$	2	5	8	11	⋯

Ellipses in this table indicate that something is omitted, in the first row all natural numbers from 4 on are omitted and in the second row the values of the expression $2x + 3$ for those numbers are omitted.

(b) When x takes the values 0, 1, 2, and 3, the expression $7x - 23$ does not have its value in N. Dependence of the value of $7x - 23$ on the values 4, 5, 6, ... of the variable x is represented by the table:

x	4	5	6	7	⋯
$7x - 23$	5	12	19	26	⋯

(c) The expression $15 - x$ has its value in N only when x takes one of the following values 0, 1, 2, ⋯, 15. The table of this expression is

x	0	1	2	⋯	15
15 − x	15	14	13	⋯	0

(d) The expression $15:x$ has its value in N, for those values of x for which $15:x$ is a natural number. Those values are the factors of 15:1, 3, 5 and 15. In this case, the table is:

x	1	3	5	15
$15:x$	15	5	3	1

When an expression with the variable x in N is given, then each of the natural numbers is a value for x, but for some of these numbers the value of the expression is a natural number and for some others is not. The set of all natural numbers for which an expression has its value in N is called the *domain of definition* of that expression. For the above examples, their domains of definition are:

(a) N, (b) {4, 5, 6, ⋯}, (c) {0, 1, 2, ⋯, 15}, (d) {1, 3, 5, 15}.

2.3. Relations in the system *N*

When the values of two numerical expressions are equated, then the equals sign is used to relate them. For example, we write

$8 + 7 = 15$, $21 - 9 = 12$, $4 \times 8 = 30 + 2$, $63: 9 = 7$, etc.

or

$8 + 7 = 16$, $21 - 9 = 11$, $4 \times 8 = 30 + 4$, $63: 9 = 8$, etc.

The jottings when two numerical expressions are related by the equals sign are called *equalities*. Some equalities may be true, as those in the first row above, some others may be false, as those in the second row above.

When between two numerical expressions one of the two signs: <, (less than) or >, (greater than) is written, then such jottings are called *inequalities*. Examples of inequalities are:

$3 + 5 < 4 + 5$, $12 - 5 > 12 - 6$, $7 \times 8 > 7 \times 7$, $30: 10 < 30:5$, etc.

or

$3 + 8 < 1 + 8$, $12 - 5 > 12 - 4$, $7 \times 8 > 8 \times 8$, $30: 5 < 30:5$, etc.

Inequalities in the first row above are true and those in the second row above are false.

When a statement, as the examples of equalities and inequalities, is logically evaluated as being true or false, then it is called a *proposition*.

When one or both of expressions related by the equals sign contain a variable, then such a jotting is called an *equation*. For examples, the jottings:

(a) $x + 7 = 15$, (b) $3x = 15$, (c) $15 - x = 12$, (d) $20 - x = 21 - x$, (*e*) $20 - x = x - 21$, etc. are equations. The *domain of definition of an equation* is the set of values of the variable x for which the both sides of the equation are defined (have their values in N).

(a) The expression $x + 7$ has a value in N for all values of x and 15 is a constant not dependent on x. Hence, the domain of definition of this equation is the set N.

(b) The expression $3x$ has for its domain of definition the set N and 15 is a constant. Thus, this equation has the set N for its domain of definition.

(c) The domain of definition of $15 - x$ is the set {0, 1, 2, ⋯, 15} and therefore, this set is also the domain of definition of this equation.

(d) The domains of definition of $20 - x$ and $21 - x$ are the sets {0, 1, 2, ⋯, 20} and {0, 1, 2, ⋯, 21}, respectively. Thus, the domain of definition of this equation is {0, 1, 2, ⋯, 20}.

(*e*) The expression $20 - x$ is defined when x takes the values in the set {0, 1, 2, ⋯, 20} and the expression $x - 21$ when x takes the values in {21, 22, 23, ⋯}. We see that there is no value of x for which both these expressions are defined. Thus, the domain of definition of this equation is the empty set $\varnothing$, *i.e.* this equation is not defined in N. As a jotting this equation has a meaning, but it has no semantic meaning.

Solution of an equation is the value of the variable x for which the corresponding equality becomes true.

(a) When x takes the value 8, the equality $8 + 7 = 15$ is a true proposition. For all other values such a proposition is false. Hence, 8 is the only solution of this equation.

(b) The solution of the equation $3x = 15$ is the number 5.

(c) When x takes the values in the set $\{0, 1, 2, \cdots, 15\}$, the only true equality is $15 - 3 = 12$ and, therefore, 3 is the only solution of this equation.

(d) For all values of x in the set $\{0, 1, 2, \cdots, 20\}$, the corresponding equalities are false and this equation has no solution.

(*e*) This equation has no meaning. (It would be wrong to say that such an equation does not have solutions).

When at least one of the two expressions related by one of the signs "<" or ">" contains the variable x, then such a jotting is called an *inequality*. For examples, the jottings:

(a) $x < 7$, (b) $14 - x < 4$, (c) $1 - x > 2$, (d) $1 - x > x - 2$, etc.

are inequalities. The *domain of definition of an inequality* is the set which is the intersection of sets being the domains of definition of the expressions which make that inequality.

(a) $x < 7$ is defined in the whole set N.

(b) The domain of definition of the inequality $14 - x < 4$ is the set $\{0, 1, 2, \cdots, 14\}$.

(c)The inequality $1 - x > 2$ has for its domain of definition the set $\{0, 1\}$.

(d) The domain of definition of the expression $1 - x$ is the set $\{0, 1\}$ and of the expression $x - 2$ the set $\{2, 3, 4, \cdots\}$. The intersection of these two sets is the empty set $\varnothing$. Hence, this inequality has no meaning in N.

A value of the variable x for which an inequality becomes a true proposition is called the *solution of the inequality* and all its solutions make a set which is called the *set of solutions of the inequality*.

(a) Substituting for x the values $0, 1, 2, \cdots, 6$ the true propositions: $0 < 7$, $1 < 7$, $2 < 7$, $\cdots$, $6 < 7$ are obtained, while all others: $7 < 7$, $8 < 7$, $9 < 7$, $\cdots$ are false. The set of solutions of this inequality is $\{0, 1, 2, \cdots, 6\}$.

(b) The inequality $14 - x < 4$ becomes a true propositions when x takes the values: 14, 13, 12, 11, ($14 - 14 < 4$, $14 - 13 < 4$, $14 - 12 < 4$, $14 - 11 < 4$) and for all other values of x the corresponding propositions are false: $14 - 10 < 4$, $14 - 9 < 4$, $\cdots$, $14 - 0 < 4$. The set of solutions of this inequality is $\{11, 12, 13, 14\}$.

(c) When x takes the values 0 or 1, the corresponding inequalities: $1 - 0 > 2$, $1 - 1 > 2$ are false propositions. The set of solutions of this inequality is the empty set $\varnothing$, *i.e.* the inequality has no solution in N.

(d) The inequality $1 - x > x - 2$ has no meaning in N and we would go wrong saying that the set of its solutions is the empty set.

Let us add to the end of this section that the symbols "=", "<" and ">" are called *relational signs* and a jotting which contains one of this signs is called a *relation*. Thus "relation" is a collective name for equations, equalities and inequalities. Two extra relational signs that are also often used are "≤" (less than or equal to) and "≥" (greater than or equal to) and a jotting which contains one of these signs is also called an inequality or less specifically, a relation. Let us also add that for a fixed natural number n, the sets of solutions of the inequalities $x \leq n$ and $x \geq n$ are the sets $\{0, 1, 2, \cdots, n\}$ and $\{n, n+1, n+2, \cdots\}$, respectively.

Comments. Our attention here has been concentrated on establishing of a precise terminology, but we have not followed a very formal exposition and the way how we formulate some similar facts often varies intentionally in some details. The English word "jotting" is exploited here probably beyond its usual meaning. The mere usage of this word emphasizes the syntactic character of concepts it refers to. As an example, when the meaning of "jotting" approximates what is our intension of its usage could be Russian word "запис'" (see, for example, [4]).

The usage of letters in primary arithmetic has for its goal the development of the idea of variable. To achieve this goal of early algebra, the task of solving world problems by means of equations has been in focus. But we find that such a task is fully recognized only when equations are formally solved and when they are a mighty technique for solving word problems. This means that we find that this task should be postponed to upper classes of elementary school and in primary school, the simplest types of equations and inequalities suffice to "move" a letter to take all values from sets of numbers.

3. Derivation of Arithmetical Rules

The letter x is usually used to denote a variable which occurs in an equation or an inequality. But when the variables are free and when they denote any natural number, then they are usually denoted by letters k, l, m, n, p from the middle of the alphabet.

By doing long series of corresponding examples and as a result of learning, children in school form mental images and schemes which are basis upon which the concepts of numbers and arithmetic operations gain their permanent meaning. And since the psychologists warn us to not identify inner representations with any visual realities, we will use set theoretic language to describe precisely the type of examples which are experienced establishing permanent meaning of these operations. Thus an *additive scheme* is a pair of disjoint sets together with their union. Let m and n be the numbers of elements of these sets and s the number of elements of their union. When m and n are given and s is to be found, we say that an *addition task* follows this scheme. But when s and m, (resp. n) are given and n, (resp. m) is to be found, we say that a *subtraction task* follows this scheme. Similarly, a *multiplicative scheme* is a family of m disjoint sets each having n elements. Let p be the number of elements of the union of these sets. When m and n are given and p is to be found, we say that a *multiplication task* follows this scheme. But when p and m, (resp. n) are given and n, (resp. m) is to be found, we say that a *division task* follows this scheme.

Now we intend to derive several rules expressing properties of arithmetical operations in the system N, which, when carried over to the extended systems continue to hold true. When these rules are arranged and organized logically, selecting some of them as being basic and deducing from them all others, then these basic rules are called axioms of arithmetic. We are not going to do it and each of the rules that we will derive, we will consider being independent of all others. In order to make a distinction, here we use the verb "derive" and "deduce" is reserved for performing proofs.

As the number does not depend on the nature of elements of sets, proceeding further, we use some models of boxes with marbles instead of using the general terms of set theory.

(1) Suppose that a red box contains m marbles and a blue one n marbles. Altogether, it is $m + n$ marbles. Reversing the order, the blue box contains n marbles and the red one m marbles. Altogether, it is $n + m$ marbles. Equating two expressions denoting the same number of marbles, we get

$$m+n=n+m\,.$$

In didactics of mathematics this equality is called the *rule of interchange of the places of summands* and in mathematics; it is called the *commutative law for addition.*

(2) In a red, a blue and a green box there are k, m and n marbles, respectively. In the red and blue boxes there are $k + m$ marbles and in the green one n marbles. Altogether, it is $(k + m) + n$ marbles. In the red box there are k marbles and in the blue and green ones $m + n$ marbles. Altogether, it is $k + (m + n)$ marbles. Equating two expressions for the same number of marbles, we get

$$(k+m)+n=k+(m+n)$$

In didactics of mathematics, this relation is called the *rule of association of summands* and, in mathematics; it is called the *associative law for addition*.

As these two ways of association produce one and the same value, it is not necessary to indicate them and we can write simply $k + m + n$. This jotting is called the sum of three numbers. Changing the order of taking the three boxes and associating in both ways, the following sequence of sums is obtained: $(k+ m) + n$, $k + (m + n)$, $(k + n) + m$, $k + (n + m)$, $(m + k) + n$, $m + (k + n)$, $(m + n) + k$, $m + (n + k)$, $(n + k) + m$, $n + (k + m)$, $(n + m) + k$, $n + (m + k)$ and they all represent the total number of marbles and, hence, they, taken two by two, can be equated. Thus we derive the *general rule of association of summands*, summands can be associated in both ways and in an arbitrary order.

Having four boxes containing k, l, m and n marbles associating them, two by two, in all possible ways each of the following jottings denote the total number of marbles:

$$\big((k+l)+m\big)+n,\big(k+(l+m)\big)+n,(k+l)+(m+n),$$
$$k+\big((l+m)+n\big),k+\big(l+(m+n)\big)$$

As the number does not depend on the way how sets are grouped, all these jottings denote one and the same number and, hence, they can be equated two by two. Not indicating the way of association we can write simply $k + l + m + n$.

Proceeding inductively, given n natural numbers $k_1, k_2, \cdots, k_n$, their sum will not depend on the way how they are associated and, accordingly, it is denoted writing $k_1 + k_2 + \cdots + k_n$.

Let us note that an application of the commutative rule and the associative rule, the general rule of association is easily deduced (though here, we do not deduce but we derive). The reader should also recognize the rule of association as the way of reducing sums of three (or more) numbers to the sums of two numbers. Thereby, as a consequence, we need the addition table for two one-digit numbers, but we do not need such a table for three (or more) numbers.

(3) A box contains m red and blue marbles and n of them are blue. The number of red marbles is $m - n$. When k blue marbles are added, (removed) the total number of marbles is $m + k$, $(m - k)$ and $n + k$, $(n - k)$ of them are blue. This does not affect the number of red marbles which is $(m + k) - (n + k)$, $((m - k) - (n - k))$. Equating two expressions for the same number of red marbles, we get

$$m - n = (m + k) - (n + k),\ m - n = (m - k) - (n - k)$$

In order to ensure the meaning in N, in the case of the first of these equalities it has to be supposed that $m \geq n$ and, in addition, in the case of the second one, $n \geq k$ is supposed. These relations are called the *rules of preservation of the value of a difference.*

(4) A box contains k marbles which are red, blue or green. The number of blue and green marbles is m and n is the number of green ones.

$k - m$	$m - n$	n
red	blue	green

Now we can express the number of red and green marbles in two ways: (i) $k - (m - n)$ and (ii) $(k - m) + n$. Equating these two expressions, we obtain

$$k - (m - n) = (k - m) + n\ .$$

This relation is called the *rule of subtracting a difference from a number.*

(5) A box contains k red, blue and green marbles. The number of blue marbles is m and n is the number of green marbles.

$k - (m + n)$	m	n
red	blue	green

Expressing the number of red marbles in two ways: (i) $k - (m + n)$ and (ii) $(k - m) - n$ and equating these two expressions, we get

$$k - (m + n) = (k - m) - n$$

This relation is called the *rule of subtracting a sum from a number.*

(6) In a box there are m red and n blue marbles. When k red and l blue mar-

bles are removed, $m-k$ red and $n-l$ blue marbles remain. Altogether $(m-k)+(n-l)$ marbles remain. There were $m+n$ marbles and $k+l$ of them were removed. In the box, $(m+n)-(k+l)$ marbles remain. Equating two expressions for the same number of marbles that remain, we get

$$(m-k)+(n-l)=(m+n)-(k+l).$$

This relation is called the *rule of adding differences* or the *rule of subtracting sums.*

(7) The fact that the relation $m+n=s$ is true whenever one of the following two relations $m=s-n$, $n=s-m$ is true and vice versa is called the *rule of interdependence of addition and subtraction.*

The fact that $m=s-n$ is true whenever $n=s-m$ is true and vice versa is called the *rule of exchange of places of the subtrahend and the difference*, which is analogous to the rule of exchange of the places of summands in the case of addition.

(8) A box contains m marbles, n of them are blue or yellow and k of them are green or yellow and l of them are yellow. The rest of them are red.

n

red	n − l—blue	l—yellow	k − l—green

k

Equating two expressions for the number of red marbles, we get

$$(m-n)-(k-l)=(m-k)-(n-l).$$

This relation is called the *rule of subtracting differences.*

(9) A useful model for derivation of the properties of multiplication will be rectangular arrangements of small circles. The arrangement that follows is supposed to have m rows, in each of them n circlets.

O OO ··· O
O OO ··· O
O OO ··· O
··· ···
O OO ··· O

In each of m rows, there are n circlets. Altogether, it is mn circlets. In each of n columns, there are m circlets. Altogether, it is nm circlets. Equating two expressions for the total number of circlets, we obtain

$$mn=nm.$$

This is the *rule of the interchange of places of factors* or *the commutative law for multiplication* how it is called in mathematics.

Let us recall that the multiplication sign is omitted in front of a variable and an open parenthesis.

(10) In each of k rows of the following arrangement:

O O···OOO···O
O O···OOO···O
··· ···

O O···OOO···O

there are m red and n blue circlets. Altogether, there are $k(m + n)$ circlets. The number of red circlets is km and of the blue ones is kn. Therefore, it is $km + kn$ circlets. Equating two expressions for the same number of circlets, we get

$$k(m+n) = km + kn.$$

This relation is called the *rule of multiplication of a sum* and, in mathematics; it is called the *distributive law*.

Considering the same arrangement and supposing that there are m circlets in each of k rows, n of which are blue. Equating two expressions for the total number of red circlets, the following relation is obtained

$$k(m-n) = km - kn,$$

which is called the *rule of multiplication of differences.*

(11) In each of k packages there are m boxes, each box contains n marbles. In k packages there are m boxes, altogether, it is km boxes and the total number of marbles is $(km)n$. In each package there are m boxes and in each box n marbles. In a package there are mn marbles and the total number of marbles is $k(mn)$. Equating two expressions for the total number of marbles, the equality

$$(km)n = k(mn),$$

follows, which is called the *rule of association of factors* or, in mathematics, the *associative law for multiplication.*

Since the products involving three numbers do not depend on the way of association, they are simply denoted writing kmn and this expression is called the product of three numbers. Combining this rule with the rule of exchange of places of factors, it is easily seen that the products of three numbers do not depend on the order in which these numbers are taken.

(12) The fact that the relation $mn = p$ is true, whenever one of the two relations $p:m = n$, $p:n = m$ is true and *vice versa*, is called the *rule of interdependence of multiplication and division.*

The fact that $p:m = n$ is true whenever $p:n = m$ is true and *vice versa*, is called the *rule of interchange of places of the divisor and the quotient.*

(13) In a package of n boxes, there are m marbles, in each box the same number of marbles. In a box there is $m:n$ marbles. In k packages of such boxes there are km marbles and kn boxes. The number of marbles in a box is $(km):(kn)$. Equating two expressions for the same number of marbles in a box, the relation

$$m:n = (km):(kn)$$

is obtained, which is called the *rule of preservation of the quotient.*

Comments. We hope that the content of this section could direct the early algebra towards the derivation of the rules and their exploitation for the transformation of expressions in N. Some of the rules that we have derived can be deduced from some others and a sequence of new relations can also be deduced.

We suggest that the reader deduces, for example, the relation under (8) (abbreviating, we will refer to it, writing (8)), applying (7) and (6).

(Put $(k-l)-(m-n)=x-y$, then $k-l=(m-n)+(x-y)=(m+x)-(n+y)$. From $k=m+x$ and $l=n+y$, we get $x=k-m$ and $y=l-n$).

Putting $l=0$ in the relation (6), we get

(6') $(m+n)-k=(m-k)+n$.

Putting $k=I+j$ in the relation (4) and applying (6') and association of the summands, we get $(I+j)-(m-n)=((I+j)-m)+n=((I-m)+j)+n=(I-m)+(j+n)$. We have obtained

(14) $(I+j)-(m-n)=(I-m)+(j+n)$.

Let us now deduce the relation

(15) $(m-n)(m+n)=m^2-n^2$.

Applying (10) and then (6), (5) and (6'), we have

$$\begin{aligned}(m-n)(m+n)&=m(m-n)+n(m-n)=\left(m^2-mn\right)+\left(mn-n^2\right)\\&=\left(m^2+mn\right)-\left(mn+n^2\right)=\left(\left(mn+m^2\right)-mn\right)-n^2\\&=\left(\left(mn-mn\right)+m^2\right)-n^2.\end{aligned}$$

For centuries, mathematicians had been applying these rules as a matter of routine and it was French priest and mathematician F.-J. Servois the first to propose the terms "commutative" and "distributive", (in 1814). Let us remark that the acquaintance with the rules of arithmetic helps a primary teacher understand various arithmetical procedures (see, for example, [5]). Some of these rules and their variants adopted for classroom teaching should be the main subject of early algebra (multiple perspectives of which are discussed in [6], and particularly in [7]).

4. Discovering Rules of Correspondence

Whenever we have an expression with a variable: (a) $7-x$, (b) $x-4$, (c) $3n$, etc.

We have a function (a correspondence) defined, taking for its domain of definition the domain of definition of that expression and in this context the set N is always taken as its codomain. Three above expressions define the following three functions

(a) $x\to 7-x$, (b) $x\to x-4$, (c) $n\to 3n$

which have for their domains of definition the sets $\{0, 1, 2, \cdots, 7\}$, $\{4, 5, 6, \cdots\}$, N, respectively.

In some parts of television program, participants, when competing, solve puzzles like the following ones: Discover a certain law in the formation of the sequences:

(a) 2, 4, 6, 8, ⋯ (b) 1, 3, 5, 7, ⋯

and prolong them writing the term that follows. Such puzzles are too easy and they will be somewhat more demanding if, for example, their 99-th term is to be found. In the case of sequence under (a), the dependence of the place of a term and its value is: in the place 1, the term is 2 or 2×1, in the place 2, the term is 4 or 2 × 2, in the place 3, the term is 6 or 2 × 3 and in the place 4, the term is 8 or 2 × 4 and in the case of the sequence under (b), the terms are 1 less than the corresponding ones under (a). Hence, they can be written as: 2 × 1 − 1, 2 × 2 − 1, 2 × 3 − 1, 2 × 4 − 1, ⋯ Thus, the terms in the place 99 are: (a) 2 × 99, (= 198), (b) 2 × 99 − 1, (= 197).

A solution of these puzzles which would be more general and more useful is when we find that in the place n, the terms are: (a) $2n$, (b) $2n - 1$ and when the rules of correspondence are (a) $n \to 2n$, (b) $n \to 2n - 1$.

Discovering of the rules of correspondence are assignments where the variable n is used in the way which demonstrates its right meaning of being a represent of arbitrary natural numbers. This type of exercises is very instructive and they should not be turned into brainteasers. To make the discovering of rules easier, groups of similar exercises should always be given. For example:

(a) Discover the rule of correspondence of the following sequences:

(i) 5, 10, 15, 20, ⋯, (ii)3, 8, 13, 18, ⋯, (iii) 1, 6, 11, 16, ⋯,

(iv) 7, 12, 17, 22, ⋯, (v) 9, 14, 19, 24, ⋯.

((i) $n \to 5n$, (ii) $n \to 5n - 2$, (iii) $n \to 5n - 4$, (iv) $n \to 5n + 2$, (v) $n \to 5n + 4$).

(b) Discover the rule of correspondence of the sequence:

(i) 1, 4, 9, 16, ⋯, (ii) 0, 3, 8, 15, ⋯, (iii) 2, 5, 10, 15, ⋯

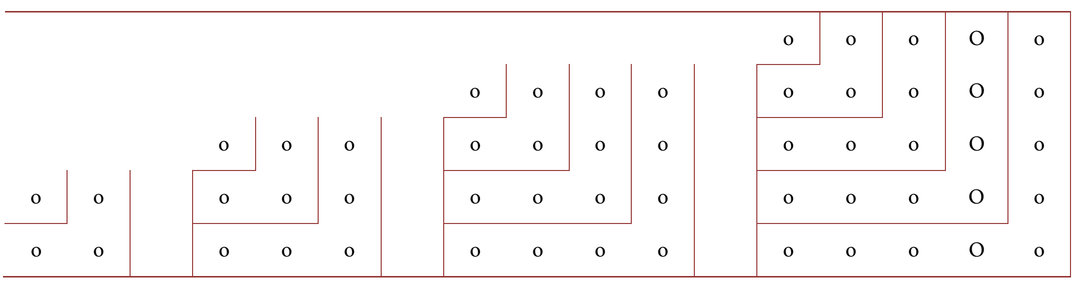
((i) $n \to n^2$, (ii) $n \to n^2 - 1$. (iii) $n \to n^2 + 1$).

(c) Given rule of correspondence $n \to n^2 - n$, write first five terms of the sequence and, then, discover the rule of correspondence of the sequences:

(i) 1, 3, 7, 13, 21, ⋯, (ii) 3, 5, 9, 15, 23, ⋯, (iii) 5, 7, 11, 17, 25, ⋯

(0, 2, 6, 12, 20, ⋯, (i) $n \to n^2 - n + 1$, (ii) $n \to n^2 - n + 3$, (iii) $n \to n^2 - n + 5$). Etc.

Looking at the following picture:

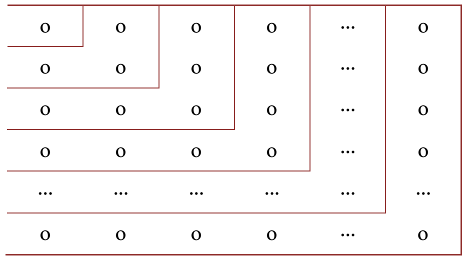


we see four square arrangements having 2 × 2, 3 × 3, 4 × 4, 5 × 5 circlets .When the circlets of these arrangements are grouped in subsets separated by "right angles", their numbers are 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7, 1 + 3 + 5 + 7 + 9. Equating expressions of circlets for each of these arrangements, we obtain the equalities

1 + 3 = 2 × 2, 1 + 3 + 5 = 3 × 3, 1 + 3 + 5 + 7 = 4 × 4, 1+ 3 + 5 + 7 + 9 = 5 × 5.

This procedure can be prolonged to obtain the equality for the sums of consecutive odd numbers starting with 1.

It was Pythagoras who first discovered these relations and we, using the variable n, are going to derive the general formula for this type of sums. Let us suppose that the following picture:



represents a square arrangement of n circlets in each of its rows. In the arrangement, there are $n \times n$ circlets. But looking at the circlets separated in subsets by "right angles", the last subset has n circlets arranged horizontally and $n - 1$ more in the vertical column. Altogether, it is $2n - 1$ circlets. Now the total number of circlets can also be written as the sum $1+3+5+\cdots+2n-1$. Equating this sum with $n \times n$, we obtain the formula for the sum of n consecutive odd numbers starting with 1

$$1+2+3+5+\cdots+2n-1=n^2 .$$

As the general form of an even number is 2n and of an odd one $2n - 1$, we can rewrite the above formula as follows

$$2\times 1-1+2\times 2-1+2\times 3-1+\cdots+2n-1=n^2$$

or

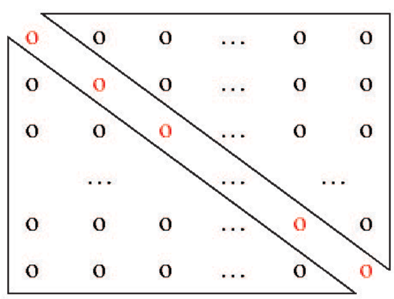
$$2(1+2+3+\cdots+n)-n=n^2 .$$

Thus, we have obtained the formula

$$1+2+3+\cdots+n=\left(n^2+n\right):2,$$

for the sum of n consecutive natural numbers starting with 1.

Now we will show how the above formula is derived directly, using the following square arrangement, having $n + 1$ rows (and $n + 1$ columns):



On the left side of the diagonal (red circlets) there are $1+2+3+\cdots+n$ black circlets and that the same number of black circlets is on the right side of the diagonal. Altogether, it is $2(1+2+3+\cdots+n)$ black circlets. Looking differently, in each of $n + 1$ rows, there are n black circlets. Altogether, it is $n(n + 1)$ black circlets. Equating two expressions for the total number of black circlets, we get

$$2(1+2+3+\cdots+n)=n(n+1)$$

or

$$1+2+3+\cdots+n=\left(n\left(n+1\right)\right):2.$$

We suggest to the reader to use this formula and to deduce:

(a) $2+4+6+\cdots+2n=n\left(n+1\right)$, $1+3+5+\cdots+2n-1=n^2$.

(b) The formula for the sum of n consecutive numbers divisible by k: $k + 2k + 3k + \cdots + nk$.

(c) The formula for the sum of all consecutive natural numbers from m to $m + k$.

Final Remarks. The meaning of natural numbers and operations with them is based on the phenomenology which consists of sets at the sensory level and manipulation with them. When extending this system of numbers the meaning has always to be in foreground. Thus, when the system of positive rational numbers is concerned, the measuring of quantities with a common unite of measure is a basis of the meaning. In the case of integers such a basis are the scales having series of marks above and below zero (thermometers, tables indicating the level of water in a river, etc.). In the case of real numbers the intuitive basis is Descartes' geometrical model, a line with a point fixed to be the origin and a segment fixed to be the unit segment and all other segments taken to be conveyors of meaning of those numbers. The rules derived in Section 3 of this paper serve for the verification of the rules in the extended systems. For example, when addition in Q_+ is defined, then

$$k/n+l/n=\left(k+l\right)/n,\ l/n+k/n=\left(l+k\right)/n$$

and being $k + l = l + k$ (in N), the commutative law for addition also holds true in Q_+ etc.

Children in primary school should develop up to some degree the skills of transforming (simple) numerical and literal expressions. Teachers should be aware of these tasks of early algebra and they should consider them more important than the calculating skills. We hope that the content of this paper could help them realize the significance of these teaching requirements.

5. Conclusion

At the end, let us say that we hope this paper shall be just the first step which can inspire the researchers to work on the formation of a genre of mathematical literature aimed at the improvement of teachers' education.

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