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On Functions of K-Balanced Matroids

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Abstract

In this paper, we prove an analogous to a result of Erdös and Rényi and of Kelly and Oxley. We also show that there are several properties of k-balanced matroids for which there exists a threshold function.

Keywords

K-Balanced, Matroid, Projective Geometry, Threshold Function

1. Introduction

We begin with some background material, which follows the terminology and notation in [1]. Let $M=(E,F)$ denote the matroid on the ground set E with flats F. All matroids considered in this paper are loopless. In particular, if M is a matroid on a set E and $X \subseteq E$, then $r(X)$ will denote the rank of X in M. We shall be considering projective geometries over a fixed finite field $GF(q)$, recalling that (see, for example [2]) the number $\begin{bmatrix} r \\ n \end{bmatrix}$ of rank-n subspaces of the projective geometry $PG(r-1, q)$ is

$$\frac{(q^r-1)(q^{r-1}-1)\cdots(q^{r-n+1}-1)}{(q^n-1)(q^{n-1}-1)\cdots(q-1)}.$$

The uniform matroid of rank r and size n is denoted by $U_{r,n}$ where $r=0,1,\cdots,n$. When $r = n$, the matroid $U_{r,r}$ is called free and when $r = n = 0$, the matroid $U_{0,0}$ is called the empty matroid. For more on matroid theory, the reader is referred to [1]-[15]. Let k be a nonnegative integer. The k-density of a matroid M with rank greater than k is given by $d_k(M)=\frac{|M|}{r(M)-k}$, where $|M|$ is the size of the ground set of M and $r(M)$ is the rank of the matroid M. A matroid M is **k-balanced** if $r(M) > \left(k(k+1)\right)/2$ and

$$d_k(M) \le d_k(M) \qquad (1)$$

for all non-empty submatroids $H \sqsubseteq M$ and **strictly k-balanced** if the inequality is strict for all such $H \neq M$. When $k = 0$, M is called **balanced** and when $k = 1$, M is called **strongly balanced**.

A **random** submatroid ω_r of the projective geometry $PG(r-1,q)$ is obtained from $PG(r-1,q)$ by deleting elements so that each element has, independently of all other elements, probability 1 − p of being deleted and probability 1 − p of being retained. In this paper, we take p to be a function $p(r)$ of r. Let A be a fixed property which a matroid may or may not possess and $P_{r,p}(A)$ denotes the probability that ω_r has property A. We shall show that there are several properties A of k-balanced matroids for which there exists a function $t(r)$ such that

$$\lim_{r\to\infty} P_{r,p}(A) = \begin{cases} 0, & \lim_{r\to\infty} \dfrac{P}{t(r)} = 0 \\ 1, & \lim_{r\to\infty} \dfrac{P}{t(r)} = \infty \end{cases}$$

If such a function exists, it is called a **threshold function** for the property A. For more on these notions, the reader is referred [16] [17].

2. K-Balanced Matroids

In this section, we prove the following main result which is analogous to Theorem 1 of Erdös and Rényi [16] and to Theorem 1.1 of Kelly and Oxley [17].

Theorem 1. *Let m and n be fixed positive integers with n ≤ m and suppose that* $B_{n,m}$ *denote a non-empty set of k-balanced simple matroids, each of which have m elements and rank n and is representable over GF(q). Then a threshold function for the property B that* ω_r *has a submatroid isomorphic to some member of* $B_{n,m}$ *is* $q^{\frac{-rn}{m}}$.

Proof. Let X and $B_{n,m}$ denote the number of submatroids of the matroid ω_r and $PG(n-1,q)$ respectively which are isomorphic to some member of $B_{n,m}$. Then

$$P_{r,p}(B) = P(X \neq 0) \leq EX$$

by definition of expectation. Therefore

$$P_{r,p}(B) \leq \begin{bmatrix} r \\ n \end{bmatrix} B_{n,m} p^m \leq B_{n,m} p^m q^{rn} \leq B_{n,m} \left(\frac{p}{q^{-\frac{rn}{m}}} \right)^m .$$

Thus, if $\lim_{r\to\infty} \frac{p}{q^{-\frac{rn}{m}}} = 0$, then $\lim_{r\to\infty} P_{r,p}(B) = 0$.

Now suppose that $\lim_{n\to\infty} \frac{p}{q^{-\frac{rn}{m}}} = \infty$. We need to show that, in this case, $\lim_{n\to\infty} P_{r,p}(B) = 1$. Let $D_{m,n}$ be the set of subsets A of $PG(r-1,q)$ for which the restriction $PG(r-1,q)\,|\,A$ of $PG(r-1,q)$ to A is isomorphic to some member of $B_{n,m}$. Then

$$EX^2 = \sum_{A_1 \in D_{m,n}} \sum_{A_2 \in D_{m,n}} p^{|A_1 \cup A_2|} = \sum_{i=0}^{m} p^{m+i} \propto_i \tag{2}$$

where $\propto_i$ equals the number of ordered pairs (A_1, A_2) such that $A_1, A_2 \in D_{m,n}$ and $|A_1 \cap A_2| = m - i$. Thus

$$EX^2 \le p^{2m} \left[\left(B_{m,n} \begin{bmatrix} r \\ n \end{bmatrix} \right)^2 + \sum_{i=0}^{m-1} p^{i-m} \propto_i \right].$$

We now want to obtain upper bounds on the numbers $\propto_0, \propto_1, \cdots, \propto_{m-1}$, so suppose that $A_1, A_2 \in D_{m,n}$ and $|A_1 \cap A_2| = m - i$ where $0 \le i \le m-1$. Then as $PG(r-1,q) | A$ is k-balanced,

$$\left(|A_1 \cap A_2|\right) / \left(r(A_1 \cap A_2) - k\right) \le m/(n-k)$$

and so $r(A_1 \cap A_2) \ge ((m-i)(n-k))/m + k$. It follows that

$$r(A_2) - r(A_1 \cap A_2) \le n - ((m-i)(n-k))/m - k$$
$$= (i(n-k))/m \le (in)/m$$

and hence $r(A_2) - r(A_1 \cap A_2) \le \lfloor (in)/m \rfloor$ where $\lfloor (in)/m \rfloor$ is the floor of $(in)/m$.

Now $\propto_i = \beta_i \gamma_i$ where β_i is the number of ways to choose A_1 and γ_i is the number of ways to choose A_2 so that $|A_1 \cap A_2| = m - i$, A_1 having already been chosen. Clearly $\beta_i = B_{m,n} \begin{bmatrix} r \\ n \end{bmatrix}$. Once A_1 has been chosen, there are at most $\binom{m}{m-i}$ choices for the subset $A_1 \cap A_2$ of A_1. Further, once $A_1 \cap A_2$ has been chosen, A_2 must be contained in some rank n subspace W of $PG(r\text{-}1,q)$ which contain the chosen set $A_1 \cap A_2$. The number δ of such subspaces W is bounded above by

$$\left((q^r - q^s)/(q-1)\right)\left((q^r - q^{s+1})/(q-1)\right) \cdots \left((q^r - q^{n-1})/(q-1)\right),$$

where $s = r(A_1 \cap A_2)$. Thus $\delta \le q^{r(n-1)}$. But it was shown above that $n - s \le \lfloor (in)/m \rfloor$; hence $\delta \le q^{r\lfloor in/m \rfloor}$. Once W has been chosen, there are at most $B_{m,n}$ choices for A_2. We conclude that

$$\gamma_i \le \binom{m}{m-i} q^{r\lfloor in/m \rfloor} B_{m,n}$$

and hence

$$\alpha_i \le \begin{bmatrix} r \\ n \end{bmatrix} B_{m,n}^2 \binom{m}{m-i} q^{r\lfloor in/m \rfloor}. \tag{3}$$

Now as $EX = \begin{bmatrix} r \\ n \end{bmatrix} B_{m,n} p^m$, we have by Equation (2), that

$$\frac{EX^2}{(EX)^2} \le 1 + \left(B_{m,n} \begin{bmatrix} r \\ n \end{bmatrix} \right)^{-2} + \sum_{i=0}^{m-1} p^{i-m} \propto_i .$$

Hence, by Equation (2),

$$\frac{EX^2}{(EX)^2} \le 1 + \left(B_{m,n} \begin{bmatrix} r \\ n \end{bmatrix} \right)^{-2} + \sum_{i=0}^{m-1} p^{i-m} \begin{bmatrix} r \\ n \end{bmatrix} B_{m,n}^2 \binom{m}{m-i} q^{r\left\lfloor \frac{in}{m} \right\rfloor} . .$$

Thus $\dfrac{EX^2}{(EX)^2} \le 1 + \sum_{i=0}^{m-1} p^{i-m} \binom{m}{m-i} \dfrac{q^{r\left\lfloor \frac{in}{m} \right\rfloor}}{\begin{bmatrix} r \\ n \end{bmatrix}} \le 1 + \sum_{i=0}^{m-1} p^{i-m} \binom{m}{m-i} \dfrac{q^{r\left\lfloor \frac{in}{m} \right\rfloor}}{q^{n(r-n)}}$

Since $\begin{bmatrix} r \\ n \end{bmatrix} \ge q^{n(r-n)}$. Thus

$$\frac{EX^2}{(EX)^2} \le 1 + \sum_{i=0}^{m-1} p^{i-m} q^{-rn + r\left\lfloor \frac{in}{m} \right\rfloor} \binom{m}{m-i} q^{n^2}. \tag{4}$$

Now consider $p^{i-m} q^{-rn + r\left\lfloor \frac{in}{m} \right\rfloor}$. We have

$$q^{-rn + r\left\lfloor \frac{in}{m} \right\rfloor} \le q^{-r\left(n - \frac{in}{m}\right)} = \left(q^{rn/m}\right)^{i-m}.$$

Thus $p^{i-m} q^{-rn + r\left\lfloor \frac{in}{m} \right\rfloor} \le \left(pq^{\frac{rn}{m}}\right)^{i-m}$. But $\lim_{r\to\infty} pq^{\frac{rn}{m}} = \infty$, hence $\lim_{r\to\infty} \left(pq^{rn/m}\right)^{i-m} = 0$ for $0 \le i \le m-1$. It follows from Equation (4) that $\lim_{r\to\infty} \sup \dfrac{EX^2}{(EX)^2} \le 1$; hence $\lim_{r\to\infty} \dfrac{EX^2}{(EX)^2} = 1$. Therefore, by Chebyshev's Inequality, $\lim_{r\to\infty} P(X \ne 0) = 1$. We conclude that $q^{\frac{-rn}{m}}$ is indeed a threshold function for the property *B*.

Corollary 1 *If n is a fixed positive integer, then a threshold function for the property that* ω_r *has an n-element independent set is* q^{-r}.

Corollary 2 *If m is a fixed positive integer exceeding two, then a threshold function for the property that* ω_r *has an m-element circuit is* $q^{\frac{-r(m-1)}{m}}$.

Corollary 3 *If n is a fixed positive integer, then a threshold function for the property that* ω_r *contains a submatroid isomorphic to* $PG(n-1,q)$ *is* $q^{\frac{-rn(q-1)}{q^n-1}}$.

To show that the preceding three results are valid, we are required to check that the appropriate submatroids are *k*-balanced. For example, in Corollary 1, the *n*-element independent set must be *k*-balanced; this is the free matroid $U_{n,n}$. Corollary 2 requires one to verify that an *m*-element circuit is *k*-balanced; this is precisely the uniform matroid $U_{m-1,m}$, while in Corollary 3, the projective geometry $PG(n-1,q)$ needs to be *k*-balanced. For a more thorough discussion of this material, the reader is referred to Proposition 2 and Theorem 5 in [2].

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