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Competition Numbers of a Kind of Pseudo-Halin Graphs

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Abstract

For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and characterizing a graph by its competition number has been one of important research problems in the study of competition graphs. A 2-connected planar graph G with minimum degree at least 3 is a pseudo-Halin graph if deleting the edges on the boundary of a single face f_0 yields a tree. It is a Halin graph if the vertices of f_0 all have degree 3 in G. In this paper, we compute the competition numbers of a kind of pseudo-Halin graphs.

Keywords

Competition Graph, Competition Number, Halin Graph, Generalized Halin Graph, Pseudo-Halin Graph

1. Introduction and Preliminary

Let $G=(V,E)$ be a graph in which V is the vertex set and E the edge set. We always use $|V|$ and $|E|$ to denote the vertex number and the edge number of G, respectively. The notion of competition graph was introduced by Cohen [1] in connection with a problem in ecology. Let $D=(V,A)$ be a digraph in which V is the vertex set and A the set of directed arcs. The *competition graph* $C(D)$ of D is the undirected graph G with the same vertex set as D and with an edge $uv\in E(G)$ if and only if there exists some vertex $x\in V(D)$ such that $(u,x),(v,x)\in A(D)$. We say that a graph G is a competition graph if there exists a digraph D such that $C(D)=G$.

Roberts [2] observed that every graph together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. The *competition number*

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$k(G)$ of a graph G is defined to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph. It is difficult to compute the competition number of a graph in general as Opsut [3] has shown that the computation of the competition number of a graph is an NP-hard problem. But it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number. Recently, many papers related to graphs' competition numbers appear. Kim *et al.* [4] studied the competition numbers of connected graphs with exactly one or two triangles. Sano [5] studied the competition numbers of regular polyhedra. Kim *et al.* [6] studied the competition numbers of Johnson graphs. Park and Sano [7] [8] studied the competition numbers of some kind of hamming graphs. Kim *et al.* [9] studied the competition numbers of the complement of a cycle. Furthermore, there are some papers (see [10] [11] [12] [13] [14]) focused on the competition numbers of the complete multipartite graphs, and some papers (see [15]-[21]) concentrated on the relationship between the competition number and the number of holes of a graph. A cycle of length at least 4 of a graph as an induced subgraph is called a hole of the graph. We use I_r to denote the graph consisting only of r isolated vertices, and $G \cup I_r$ the disjoint union of G and I_r. The induced subgraph $G[S]$ of G is a subgraph of G whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S.

All graphs considered in this paper are simple and connected. For a vertex v in a graph *G*, let the open neighborhood of v be denoted by $N_G(v) = \{u \mid u \text{ is adjacent to } v\}$. For any set S of vertices in G, we define the neighborhood of S in G to be the set of all vertices adjacent to vertices in S, this set is denoted by $N_G(S)$. An in-neighbor of a vertex v in digraph D is a vertex u such that $(u,v) \in A(D)$; an out-neighbor of a vertex v is a vertex w such that $(v,w) \in A(D)$. We denote the sets of in-neighbors and out-neighbors of v in D by $N_D^-(v)$ and $N_D^+(v)$, respectively.

A subset S of the vertex set of a graph G is called a *clique* of G if $G[S]$ is a complete graph. For a clique S of a graph G and an edge e of G, we say e is covered by S if both of the endpoints of e are contained in S. An edge clique cover of a graph G is a family of cliques such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_e(G)$ of a graph G is the minimum size of an edge clique cover of G. A vertex clique cover of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The *vertex clique cover number* $\theta_v(G)$ of a graph G is the minimum size of a vertex clique cover of G.

A generalized Halin graph $G = T \cup C$ is a plane graph consisting of a plane embedding of a tree T and a cycle C connecting the leaves (vertices of degree 1) of T such that C is the boundary of the exterior face. The tree T and the cycle C are called the characteristic tree and the adjoint cycle of G, respectively. For each $v \in V(C)$, if $N_T(v) \cap N_T(x) = \varnothing$ for any vertex $x \in N_C(v)$, then we called v a simple leaf of T, otherwise, a compound leaf of T. Denote all simple leaves of T by $V_1(C)$ and all compound leaves of T by $V_2(C)$, respectively. It is obvious that $V(C) = V_1(C) \cup V_2(C)$, and $V_1(C) \cap V_2(C) = \varnothing$. Let $V_1' = N_T(V_1(C))$,

$V_2' = N_T(V_2(C))$. A generalized Halin graph G is a Halin graph when each interior vertex of G has degree at least 3.

A 2-connected planar graph G with minimum degree at least 3 is a pseudo-Halin graph if deleting the edges on the boundary of a single face f_0 yields a tree. It is a Halin graph if the vertices of f_0 all have degree 3 in G. The face f_0 is the exterior face; the others are interior faces. Vertices of f_0 are exterior vertices; the others are interior vertices. Vertices of f_0 with degree 3 in G are regular vertices; the others are irregular vertices. Let $R(f_0)$ and $I(f_0)$ denote the sets of regular and irregular vertices in f_0, respectively.

The main theme of this paper is to study the competition numbers of a kind of pseudo-Halin graphs with $|I(f_0)|=1$, and gets the exact value or the upper bound of the competition number of this kind of pseudo-Halin graphs.

2. Lemmas

We first introduce two useful Lemmas.

Lemma 1 (Harary *et al.* [22]). *Let $D=(V,A)$ be a digraph. Then D is acyclic if and only if there exists an ordering of vertices, $\sigma=[v_1,v_2,\cdots,v_n]$, such that one of the following two conditions holds*:

1) For all $i,j\in\{1,2,\cdots,n\}$, $(v_i,v_j)\in A$ implies that $i<j$;

2) For all $i,j\in\{1,2,\cdots,n\}$, $(v_i,v_j)\in A$ implies that $i>j$.

By this lemma, if D is an acyclic digraph, we can find a vertex labelling $\sigma: V\to\{1,2,\cdots,|V|\}$ so that whenever $(x,y)\in A$, $\sigma(y)<\sigma(x)$. We call σ an acyclic labelling of D. Conversely, if D is a digraph with an acyclic labelling, then D is acyclic.

Lemma 2 (Kim and Roberts [4]). *For a tree T and a vertex v of T, there is an acyclic digraph D so that $T\cup\{v_0\}$ is the competition graph of D for v_0 not in T and so that v has only outgoing arcs in D.*

Kim and Roberts [4] proved Lemma 2 by the following algorithm.

Let $T_1=T$, $V(D_1)=V(T)$, and $A(D_1)=\varnothing$. Choose a vertex v_1 of degree 1 from T_1. If v^1 is adjacent to v_1 in T_1, let $T_2=T-v_1$, $V(D_2)=V(D_1)\cup\{v_0\}$ for some vertex v_0 not in T, and $A(D_2)=\{(v_1,v_0),(v^1,v_0)\}$. Having defined T_i and D_i, choose a vertex v_i of degree 1 from T_i. If v^i is adjacent to v_i in T_i, then let $T_{i+1}=T_i-v_i$, $V(D_{i+1})=V(D_i)$, and $A(D_{i+1})=A(D_i)\cup\{(v_i,v_{i-1}),(v^i,v_{i-1})\}$. Repeat this last step until D_n has been defined. Let $D=(V(D_n),A(D_n))$. In the procedure, we may avoid selecting v until we select all other vertices since there are at least two vertices of degree 1 in a tree with more than one vertex.

In fact, this algorithm provides an acyclic labelling $\sigma=[v_0,v_1,v_2,\cdots,v_n]$ of D such that $v_n=v^{n-1}$, $v_{n-1}v_n\in E(T)$, and v_{n-1} and v_n have only outgoing arcs in D, where $|V(T)|=n$.

Opsut [3] gave the following two lower bounds for the competition number of a graph.

Theorem 1 (Opsut [3]). *For any graph G*, $k(G)\ge\theta_e(G)-|V(G)|+2$.

Theorem 2 (Opsut [3]). *For any graph G*, $k(G)\ge\min\{\theta_v(N_G(v))|v\in V(G)\}$.

Lemma 3. *For any generalized Halin graph $G=T\cup C$*,

$$k(G)\le\begin{cases}2, & V_1(C)=\varnothing;\\ |V_1(C)|+1, & V_1(C)\ne\varnothing.\end{cases}$$

Proof. Let $G=T\cup C$ be a generalized Halin graph, where T and C are the characteristic tree and the adjoint cycle of G, respectively. Suppose that along cycle C by clockwise order we can partition $V_2(C)$ (if $V_2(C)\ne\varnothing$) into $k\ge 1$ subsets $V_2^i(C)$ such that $V_2(C)=\cup_{i=1}^k V_2^i(C)$ and the vertices in each $V_2^i(C)$ are consecutive on C, where $1\le i\le k$. Let u^i be the common neighbor of the vertices in $V_2^i(C)$, where $1\le i\le k$. We assume that the vertices in $V_2^i(C)$ by clockwise order are $v_1^i,v_2^i,\cdots,v_{\alpha_i}^i$, where $\alpha_i\ge 2$ and $1\le i\le k$. Denote all vertices on C between $v_{\alpha_i}^i$ and v_1^{i+1} by $V_1^i(C)$, where $1\le i\le k$ and $v_1^{k+1}=v_1^1$. We assume that the vertices in $V_1^i(C)$ (if $V_1^i(C)\ne\varnothing$) by clockwise order are $x_1^i,x_2^i,\cdots,x_{\beta_i}^i$, where $\beta_i\ge 1$ and $i\in\{1,2,\cdots,k\}$. Note that $V_1(C)=\cup_{i=1}^k V_1^i(C)$, and if $V_1(C)\ne\varnothing$ then we always let $V_1^k(C)\ne\varnothing$. If $V_2(C)=\varnothing$, then let $V_1^1(C)=V(C)$ and arbitrarily select a vertex in $V(C)$ as x_1^1.

By Lemma 1 and the algorithm in the proof of Lemma 2, we can construct an acyclic digraph D so that $T\cup\{v_0\}$ is the competition graph of D for v_0 not in T, and get an acyclic labelling

$$\sigma:V\cup\{v_0\}\to\{1,2,\cdots,|V|+1\}$$

of D so that

$$\sigma(v_0)=1;$$

$$\sigma(v_j^i)=1+\sum_{k=1}^{i-1}(\alpha_k+\beta_k)+j,\ \text{ where } 1\le i\le k \text{ and } 1\le j\le\alpha_i;$$

$$\sigma(x_j^i)=1+\sum_{k=1}^{i}\alpha_k+\sum_{k=1}^{i-1}\beta_k+j,\ \text{ where } 1\le i\le k \text{ and } 1\le j\le\beta_i.$$

Note that, if $V_2(C)=\varnothing$, then let $\alpha_1=0$, if $V_1^i=\varnothing$, then let $\beta_i=0$, where $i\in\{1,2,\cdots,k\}$, and we always have $\sum_{k=1}^0(\alpha_k+\beta_k)=0$ and $\sum_{k=1}^0\beta_k=0$. Label the other vertices of T arbitrarily.

Case 1. $V_1(C)=\varnothing$.

Let D_1 be a digraph whose vertex set is

$$V(G)\cup\{v_0,v_1\}$$

and whose arc set is

$$A(D)\cup\bigcup_{i=1}^{k}\bigcup_{j=2}^{\alpha_i}\left\{\left(v_j^i,v_{j-2}^i\right)\right\}\cup\bigcup_{i=1}^{k-1}\left\{\left(v_1^{i+1},v_{\alpha_i-1}^i\right)\right\}$$
$$\cup\left\{\left(v_1^1,v_1\right),\left(v_{\alpha_k}^k,v_1\right)\right\}-\bigcup_{i=1}^{k}\left\{\left(u^i,v_{\alpha_i-1}^i\right)\right\},$$

where $v_0^1=v_0$, $v_0^i=v_{\alpha_{i-1}}^{i-1}$ for each $i\in\{2,3,\cdots,k\}$, and v_0,v_1 are new added vertices.

Case 2. $V_2(C)=\varnothing$.

Let D_2 be a digraph whose vertex set is

$$V(G)\cup\{v_0\}\cup\left\{y_1^1,y_2^2,\cdots,y_{\beta_1}^1\right\}$$

and whose arc set is

$$A(D)\cup\bigcup_{j=1}^{\beta_1}\left\{\left(x_j^1,y_j^1\right),\left(x_{j+1}^1,y_j^1\right)\right\},$$

where $x_{\beta_1+1}^1=x_1^1$, and all y_j^1 $\left(1\le j\le\beta_1\right)$ are new added vertices.

Case 3. $V_1(C)\ne\varnothing$ and $V_2(C)\ne\varnothing$.

Let D_3 be a digraph whose vertex set is

$$V(G)\cup\{v_0\}\cup\bigcup_{1\le i\le k,\beta_i>0}\left\{y_1^i,y_2^i,\cdots,y_{\beta_i}^i\right\}$$

and whose arc set is

$$A(D)\cup\bigcup_{i=1}^{k}\bigcup_{j=2}^{\alpha_i}\left\{\left(v_j^i,v_{j-2}^i\right)\right\}\cup\bigcup_{1\le i\le k,\beta_i>0}\left\{\left(x_1^i,v_{\alpha_i-1}^i\right)\right\}\cup\bigcup_{1\le i\le k,\beta_i=0}\left\{\left(v_1^{i+1},v_{\alpha_i-1}^i\right)\right\}$$
$$\cup\bigcup_{1\le i\le k,\beta_i>0}\bigcup_{j=1}^{\beta_i}\left\{\left(x_j^i,y_j^i\right),\left(x_{j+1}^i,y_j^i\right)\right\}-\bigcup_{i=1}^{k}\left\{\left(u^i,v_{\alpha_i-1}^i\right)\right\},$$

where $v_0^1=v_0$ and $v_0^i=x_{\beta_{i-1}}^{i-1}$ (if $\beta_{i-1}>0$) or $v_{\alpha_{i-1}}^{i-1}$ (if $\beta_{i-1}=0$) for any $i\in\{2,3,\cdots,k\}$, $x_{\beta_k+1}^k=v_1^1$ and $x_{\beta_i+1}^i=v_1^{i+1}$ for any $i\in\{1,2,\cdots,k-1\}$ such that $\beta_i>0$. All y_j^i $\left(1\le i\le k,1\le j\le\beta_i\right)$ are new added vertices.

We note that D_1, D_2 and D_3 are acyclic. This is because every arc added here goes either from a big label vertex to a small label vertex or from a vertex in $V(G)$ to a new added vertex not in $V(G)\cup\{v_0\}$. It is not difficult to check that

$$C(D_1)=G\cup\{v_0,v_1\},$$

$$C(D_2)=V(G)\cup\{v_0\}\cup\left\{y_1^1,y_2^2,\cdots,y_{\beta_1}^1\right\}\text{ and}$$

$$C(D_3)=G\cup\{v_0\}\cup\bigcup_{1\le i\le k,\beta_i>0}\left\{y_1^i,y_2^i,\cdots,y_{\beta_i}^i\right\}.$$

And we know that $V_1(C)=\left\{x_1^1,x_2^2,\cdots,x_{\beta_1}^1\right\}$ if $V_2(C)=\varnothing$, and $V_1(C)=\bigcup_{1\le i\le k,\beta_i>0}\left\{x_1^i,x_2^i,\cdots,x_{\beta_i}^i\right\}$ if $V_1(C)\ne\varnothing$ and $V_2(C)\ne\varnothing$. So the result follows.

□

Lemma 4. *For any not* K_4 *generalized Halin graph* $G=T\cup C$,

$$\left|V_1(C)\right|=\theta_e(G)-\left|V(G)\right|+1.$$

Proof. Let $G=T\cup C$ be a not K_4 generalized Halin graph, where T and C are the characteristic tree and the adjoint cycle of G, respectively. Since $G\left[V(G)\setminus V(C)\right]$ is a tree, then $\theta_e\left(G\left[V(G)\setminus V(C)\right]\right)=\left|V(G)\right|-\left|V(C)\right|-1$. Note that $G\left[V(C)\cup V_2'\right]$ just includes all triangles in G and the edges in C, so $\theta_e\left(G\left[V(C)\cup V_2'\right]\right)=\left|V(C)\right|$. It is easy to check that $\theta_e\left(G\left[V_1(C)\cup V_1'\right]\right)=\left|V_1(C)\right|$. Furthermore, each pair of graphs $G\left[V(G)\setminus V(C)\right]$, $G\left[V(C)\cup V_2'\right]$ and $G\left[V_1(C)\cup V_1'\right]$ has not any common edges. So we have

$$\begin{aligned}\theta_e(G)&=\theta_e\left(G\left[V(G)\setminus V(C)\right]\right)+\theta_e\left(G\left[V(C)\cup V_2'\right]\right)+\theta_e\left(G\left[V_1(C)\cup V_1'\right]\right)\\&=\left(\left|V(G)\right|-\left|V(C)\right|-1\right)+\left|V(C)\right|+\left|V_1(C)\right|\\&=\left|V(G)\right||+\left|V_1(C)\right|-1\end{aligned}$$

or

$$|V_1(C)| = \theta_e(G) - |V(G)| + 1. \qquad \square$$

3. Pseudo-Halin Graph

Now we consider a pseudo-Halin graph G with the exterior face f_0 and $I(f_0) = \{x\}$. Let u and v be the neighbors of x on the boundary of f_0. Without lose of generality, we may always assume that u is on the left of x and v on the right of x. Let $G' = G - \{xu, xv\} + uv$. Note that G' is a generalized Halin graph since $d_{G'}(x) \geq 2$ is accepted. See Figure 1. Let $G' = T \cup C'$, where T and C' are the characteristic tree and the adjoint cycle of G', respectively. Then it is easy to see that C' is got from the boundary of f_0 by deleting the edges xu and xv, and adding the edge uv. So we have $V(C') = R(f_0)$. Let $b \neq x$ be another neighbor of u on cycle C'. The characteristic tree T of G' is just the tree got from G by deleting the edges on the boundary of the face f_0. So T may also be called the characteristic tree of G. Let u' be the neighbor of u in T and v' the neighbor of v in T, respectively.

We construct a graph G'' from G' by replacing the edge uv by a path $ux'v$, and joining x with x'. It is not difficult to see that G'' is a Halin graph. Since every Halin graph contains a triangle, and x' is not a vertex of any triangle in G'', then G' also contains a triangle. Therefore $V_2(C') \neq \varnothing$. So we just need to consider the following cases.

Theorem 3. *Let* G *be a pseudo-Halin graph with* $I(f_0) = \{x\}$, *and* C' *the adjoint cycle of graph* $G' = G - \{xu, xv\} + uv$. *If* $V_1(C') = \varnothing$, *then* $k(G) = 2$.

Proof. Suppose that G is a pseudo-Halin graph with $I(f_0) = \{x\}$ and $V_1(C') = \varnothing$, where C' is the adjoint cycle of graph $G' = G - \{xu, xv\} + uv$. Denote and labelling the vertices of G' in a similar way as used in Lemma 3. By Lemma 3, $k(G') \leq 2$. Let $v = v_1^1$ and by the similar way used in the proof of Lemma 3, there is a digraph D' such that

$$C(D') = G' \cup \{v_0, v_1\},$$

and

$$\{(v, v_0), (v', v_0), (u, v_1), (v, v_1), (u, b)\} \subset A(D')$$

but

$$(u', b) \notin A(D'),$$

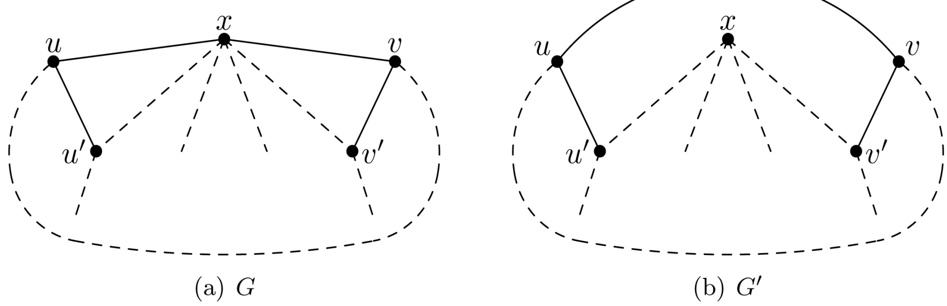

Figure 1. G and G'.

where v_0 and v_1 are two isolated vertices not in G'. Note that $N^{-}_{D'}(b)=\{u\}$. Let

$$V(D)=V(D')$$

and

$$A(D)=A(D')\cup\{(x,b),(x,v_1)\}\setminus\{(u,v_1)\}.$$

See Figure 2. Note that D is acyclic since every arc added here goes from a big label vertex to a small label vertex. It is easy to see that $C(D)=G\cup\{v_0,v_1\}$. So we have $k(G)\le 2$. On the other hand, since for each vertex $v\in V(G)$, $d_G(v)\ge 3$, and note that the maximal clique in G is a triangle, so we have $\theta_v(N_G(v))\ge 2$. By Theorem 2, $k(G)\ge \min\{\theta_v(N_G(v))\,|\,v\in V(G)\}\ge 2$. Therefore $k(G)=2$. □

Lemma 5. *Let* G *be a pseudo-Halin graph with* $I(f_0)=\{x\}$, *and* C' *the adjoint cycle of graph* $G'=G-\{xu,xv\}+uv$. *If* $V_1(C')\ne\varnothing$, *then we have* $k(G)\le |V_1(C')|+2$.

Proof. Suppose that G is a pseudo-Halin graph with $I(f_0)=\{x\}$ and $V_1(C')\ne\varnothing$, where C' is the adjoint cycle of graph $G'=G-\{xu,xv\}+uv$. Denote and labelling the vertices of G' in a similar way as used in Lemma 3. By Lemma 3, $k(G')\le |V_1(C')|+1$, and there is a digraph D' such that $C(D')=G'\cup I_{|V_1(C')|+1}$, where $I_{|V_1(C')|+1}$ are $|V_1(C')|+1$ isolated vertices not in G'. By the similar way used in the proof of Lemma 3, there exists a vertex $y\in I_{|V_1(C')|+1}$ or $y\in V(C')$ such that $uy,vy\in A(D')$. Let

$$V(D)=V(D')\cup\{w\}$$

and

$$A(D)=A(D')\cup\{(x,w),(u,w),(x,y)\}\setminus\{(u,y)\},$$

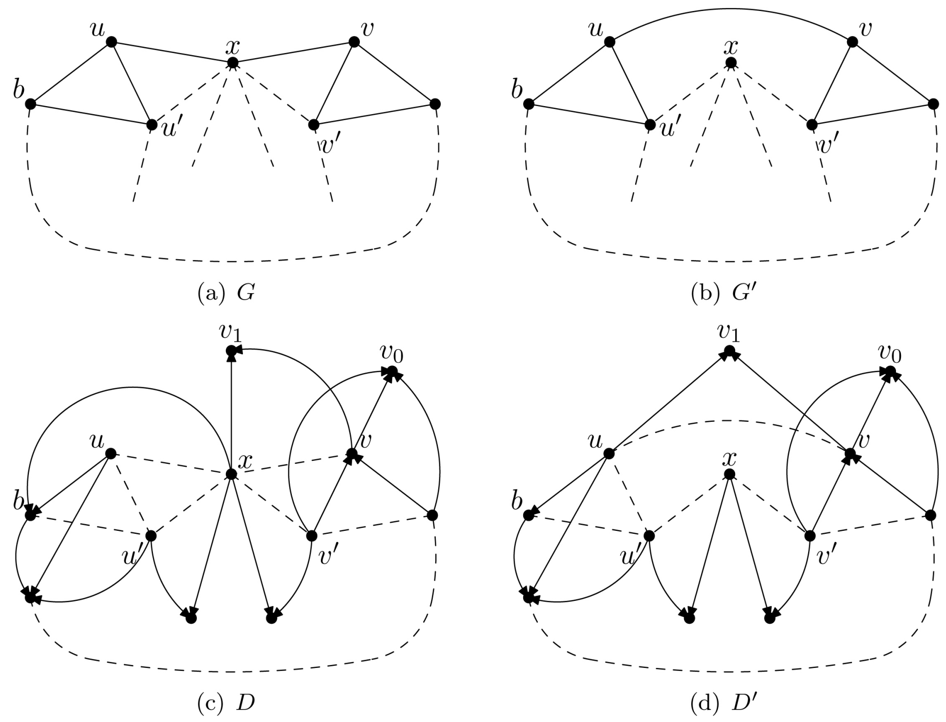

Figure 2. G and D $(V_1=\varnothing)$.

where w is a new added vertex to D'. Note that D is acyclic since every arc added here goes from a big label vertex to a small label vertex or to a new added vertex. It is easy to see that $C(D) = G \cup I_{|V_1(C')|+1} \cup \{w\}$. So we have $k(G) \le |V_1(C')| + 2$. □

Lemma 6. *Let* G *be a pseudo-Halin graph with* $I(f_0) = \{x\}$, *and* C' *the adjoint cycle of graph* $G' = G - \{xu, xv\} + uv$.

1) If $xu', xv' \notin E(G)$, then $\theta_e(G') = \theta_e(G) - 1$,

2) If $xu', xv' \in E(G)$, then $\theta_e(G') = \begin{cases} \theta_e(G)+3, & u,v \in V_1(C'); \\ \theta_e(G)+1, & u,v \in V_2(C'); \\ \theta_e(G)+2, & \text{otherwise}, \end{cases}$

3) If $xp' \in E(G)$, but $xq' \notin E(G)$, then $\theta_e(G') = \begin{cases} \theta_e(G)+1, & p \in V_1(C'); \\ \theta_e(G), & p \in V_2(C'), \end{cases}$ where $\{p, q\} = \{u, v\}$.

Proof. 1) $xu', xv' \notin E(G)$.

Obviously, $\{x, u\}$ and $\{x, v\}$ are two maximal cliques of G. Since $\{u, v\}$ is a maximal clique of G', then $\theta_e(G') = \theta_e(G) - 1$.

2) $xu', xv' \in E(G)$.

It is easy to see that $\{x, u, u'\}$ and $\{x, v, v'\}$ are two maximal cliques of G. Note that uv is a maximal clique of G'. If $u, v \in V_1(C')$, then $\{x, u'\}$, $\{u, u'\}$, $\{x, v'\}$ and $\{v, v'\}$ are all maximal cliques of graph G'. So $\theta_e(G') = \theta_e(G) + 3$; If $u, v \in V_2(C')$, then $\{x, u'\}$ and $\{x, v'\}$ are two maximal cliques of graph G', but $\{u, u'\}$ and $\{v, v'\}$ not. Hence $\theta_e(G') = \theta_e(G) + 1$; Otherwise, say $u \in V_1(C')$ and $v \in V_2(C')$. Then $\{x, u'\}$, $\{u, u'\}$ and $\{x, v'\}$ are all maximal cliques of graph G', but $\{v, v'\}$ not. Therefore $\theta_e(G') = \theta_e(G) + 2$.

3) $xu' \in E(G)$ but $xv' \notin E(G)$ (or $xv' \in E(G)$ but $xu' \notin E(G)$).

Without lose of generality, we just need to consider the case $xu' \in E$ but $xv' \notin E$. By the proof of Case (1), $\theta_e(G - xu + uv) = \theta_e(G)$. If $v \in V_1(C')$, then $\{x, v'\}$ and $\{v, v'\}$ are all maximal cliques of graph G'. If $v \in V_2(C')$, then $\{x, v'\}$ is a maximal clique of graph G', but $\{v, v'\}$ not. So the result follows. □

For a pseudo-Halin graph G, suppose that f_0 be the exterior face of G and $I(f_0) = \{x\}$. Note that G' can not be K_4, so by Lemmas 4 we have
$|V_1(C')| = \theta_e(G') - |V(G)| + 1$, by Lemma 5 we have
$k(G) \le |V_1(C')| + 2 \le \theta_e(G') - |V(G)| + 3$. On the other hand, by Theorem 1 we have $k(G) \ge \theta_e(G) - |V(G)| + 2$. So by lemmas 6, we have the following result.

Theorem 4. *Let* G *be a pseudo-Halin graph with* $I(f_0) = \{x\}$ *and* $V_1(C') \ne \varnothing$, *where* C' *is the adjoint cycle of graph* $G' = G - \{xu, xv\} + uv$.

1) If $xu', xv' \notin E$, then $k(G) = \theta_e(G) - |V(G)| + 2$,

2) If $xu', xv' \in E$, then

$$\theta_e(G) - |V(G)| + 2 \le k(G) \le \begin{cases} \theta_e(G) - |V(G)| + 6, & u,v \in V_1(C'); \\ \theta_e(G) - |V(G)| + 4, & u,v \in V_2(C'); \\ \theta_e(G) - |V(G)| + 5, & \text{otherwise}, \end{cases}$$

3) If $xp' \in E$, and $xq' \notin E$, then

$$\theta_e(G)-|V(G)|+2\le k(G)\le\begin{cases}\theta_e(G)-|V(G)|+4, & p\in V_1(C');\\ \theta_e(G)-|V(G)|+3, & p\in V_2(C'),\end{cases}$$

where $\{p,q\}=\{u,v\}$.

4. Concluding Remarks

In this paper, we study the competition numbers of a kind of pseudo-Halin graphs with just one irregular vertex.

For a pseudo-Halin graph G with the exterior face f_0 and $|I(f_0)|=1$, we show that if all leaves of the characteristic tree of G are compound leaves, then $k(G)=2$, otherwise, $\theta_e(G)-|V(G)|+2\le k(G)\le\theta_e(G)-|V(G)|+6$. Even we proved that $k(G)=\theta_e(G)-|V(G)|+2$ for some cases, but we can not provide the accurate value of the competition number of G for other cases. So it would be valuable to get the accurate value of the competition number of the pseudo-Halin graph with just one irregular vertex, and it may be interesting to study the competition numbers of general pseudo-Halin graphs.

For a digraph $D=(V,A)$, if we partition V into k types, then we may construct a undirected graph $C^k(D)=(V,E)$ of D as follows

1) $uv\in E$ if and only if there exists some vertex $x\in V$ such that $(u,x),(v,x)\in A$ and u,v are of same type, or

2) $uv\in E$ if and only if there exists some vertex $x\in V$ such that $(u,x),(v,x)\in A$ and u,v are of different types.

It is easy to see that $C^1(D)=C(D)$ for a given digraph D, and we note that multitype graphs can be used to study the multi-species in ecology and have been deeply studied, see [23] [24]. So these generalizations of competition graphs may be more realistic and more interesting.

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