

On the Quantum Mechanical Treatment of the Bateman-Morse-Feshbach Damped Oscillator with Variable Mass

Akira Suzuki¹, Hiroki Majima²

¹Department of Physics, Faculty of Science, Tokyo University of Science, Tokyo, Japan

²Department of Physics, General Education Division, Salesian Polytechnic, Tokyo, Japan

Email: asuzuki@rs.kagu.tus.ac.jp, majima@salesio-sp.ac.jp

How to cite this paper: Suzuki, A. and Majima, H. (2016) On the Quantum Mechanical Treatment of the Bateman-Morse-Feshbach Damped Oscillator with Variable Mass. *Journal of Modern Physics*, 7, 2329-2340. <http://dx.doi.org/10.4236/jmp.2016.716201>

Received: October 17, 2016

Accepted: December 12, 2016

Published: December 15, 2016

Copyright © 2016 by authors and Scientific Research Publishing Inc.

This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

The harmonic oscillator with time-dependent (indefinite and variable) mass subject to the force proportional to velocity is studied by extending Bateman's dual Lagrangian and Hamiltonian formalism. To study the quantum analog of such a dissipative system, the Batemann-Morse-Feshbach classical Hamiltonian of the damped harmonic oscillator with varying (time-dependent) mass is canonically quantized. In order to discuss the stability of the quantum dissipative system due to the influence of varying mass and the dissipative force, we derived a formula for the vacuum state of the dissipative system with the help of quantum field theoretical framework. It is shown that the formula based on this simple model could be used to study the influence of dissipation such as the instability due to the dissipative force and/or the variable mass. It is understood that the change in the oscillator mass corresponds to a control parameter in quantum dissipative systems.

Keywords

Canonical Quantization, Dissipative System, Dumped Harmonic Oscillator, Variable Mass, Control Parameter

1. Introduction

The quantum damped oscillator has been studied by many researchers to understand dissipation in quantum theory since the damped harmonic oscillator is one of the simplest systems revealing the dissipation of energy. It is well known that quantum damped harmonic oscillator is studied within two representations of the model system. One representation is the Bateman-Feshbach-Tikochinsky (BFT) oscillator (often called

the Bateman oscillator) as a closed system with two degrees of freedom [1] [2]. The other representation is the Caldirola-Kanai (CK) oscillator as an open system with one degree of freedom [3] [4] [5].

The damped harmonic oscillator (DHO) is described by $x(t)$ subject to the 2nd-order linear differential equation with constant coefficients, where coefficient $\gamma(>0)$ in the first derivative term is called a damping coefficient and k is the harmonic coefficient while m is the mass coefficient (constant):

$$m\ddot{x} + \gamma\dot{x} + kx = 0, \quad (1)$$

where the overdot denotes the derivative with respect to t . Depending on the relation between damping and harmonic coefficients we have three different cases and the general solutions of Equation (1) are:

(a) The over-damping case: $\Omega_o^2 \equiv \gamma^2/4m^2 - k/m > 0$,

$$x(t) = (C_1 e^{+\Omega_o t} + C_2 e^{-\Omega_o t}) e^{-\gamma t/2m}. \quad (2)$$

(b) The critical-damping case: $\Omega_c^2 = k/m - \gamma^2/4m^2 = 0$,

$$x(t) = (C_3 + C_4 t) e^{-\gamma t/2m}. \quad (3)$$

(c) The under-damping case: $\Omega_u^2 = k/m - \gamma^2/4m^2 > 0$,

$$x(t) = (C_5 e^{+i\Omega_u t} + C_6 e^{-i\Omega_u t}) e^{-\gamma t/2m}. \quad (4)$$

The last case is the most interesting case and Celeghini *et al.* [6] rigorously studied classical and quantum damped harmonic oscillator with a constant mass. In this paper we will study the case where the oscillator's mass changes with time. When $\gamma = 0$, Equation (1) is reduced to the standard harmonic oscillator equation of motion. Throughout the paper we consider the $\gamma \neq 0$ case along with a time-dependent mass $m = m(t)$, *i.e.*, we consider dissipation by using the simple model. The harmonic oscillator described by Equation (1) represents a dissipative system of which energy is not conserved although the γ is time-independent. In order to establish the canonical formalism for the dissipative system we have to construct a Lagrangian-Hamiltonian form in any case. Bateman's formulation [1] resolves this problem of dissipation, where the dynamics of the system is described by Equation (1) in the classical theory, see below.

The BFT (Bateman) damped oscillator [1] [2] is regarded as an *open* system in which energy is dissipated by interacting with a heat bath. Bateman's formulation [1] for the DHO resolves this problem of dissipation albeit the dynamics of the system is described by Equation (1). Bateman has shown that in order to apply the standard canonical formalism¹ of classical mechanics to dissipative systems, one can double the numbers of degrees of freedom. The new degrees of freedom are assumed to represent a reservoir, also called heat bath. Applying this idea to the damped harmonic oscillator one obtains a pair of damped oscillators, so-called *Bateman's dual* or *mirror image system* [1], represented by

¹In order to establish the canonical formalism for the dissipative systems we have to construct a Lagrangian-Hamiltonian form in any cases.

$$m\ddot{x} + \gamma\dot{x} + kx = 0, \quad m\ddot{y} - \gamma\dot{y} + ky = 0. \quad (5)$$

This closed system includes a primary one expressed by x -variable and its time reversed image by y -variable. According to this, the energy dissipated by the oscillator is completely absorbed at the same time by the mirror image oscillator, and thus the energy of the total system is conserved. Actually these equations can be derived from the Lagrangian:

$$\mathcal{L} = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) - kxy. \quad (6)$$

It should be noted that this Lagrangian does not depend on time explicitly. By Legendre transforming Equation (6), Bateman obtained the Hamiltonian:

$$\mathcal{H} = \dot{x}p_x + \dot{y}p_y - \mathcal{L} = \frac{p_x p_y}{m} + \frac{\gamma}{2m}(yp_y - xp_x) + \left(k - \frac{\gamma^2}{4m}\right)xy, \quad (7)$$

where we used $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{y} - \gamma y/2$ and $p_y = \partial\mathcal{L}/\partial\dot{y} = m\dot{x} + \gamma x/2$. The Bateman classical dual-Hamiltonian (7) has been rediscovered by Morse and Feshbach [7] and its detailed quantum mechanical analysis was performed by Feshbach and Tikochinski [2]. The quantum Bateman system for a DHO has been analyzed by many workers since it can be regarded as a simple dissipative model system. For review and references, see Ref. [8] [9] [10] [11]. We shall further exploit quantum mechanical treatments of the DHO by extending to the case where the oscillator mass is time-dependent. It is interesting to note that the time-dependent mass $m = m(t)$ plays the same role of the control parameter for damping as the damping factor γ in the DHO does in the dissipative system (see Equation (10), Section 2).

In this paper, we treat the Hamiltonian formulation and quantization of the DHO where the oscillator mass is time-dependent and study the effect of these control parameters $m(t)$ and γ on dissipation in quantum dissipative systems by examining the stability of vacuum state by using the simple model represented by the DHO with varying (time-dependent) mass by employing the theoretical scheme of Majima and Suzuki [11] and study dissipation in quantum dissipative systems in order to understand the dissipation in quantum dissipative systems.

2. Classical Theory

Let us consider the case where the oscillator's mass is time-dependent: $m = m(t)$ [12]. The *kinetic* momentum of the oscillator is then defined by

$$p \equiv m(t)\dot{x}. \quad (8)$$

Now we differentiate Equation (8) with respect to t , we obtain

$$\frac{d}{dt}p = \dot{m}(t)\dot{x} + m(t)\ddot{x}. \quad (9)$$

When the oscillator with variable mass is subject to the external force $F_{\text{ex}} = -\gamma\dot{x} - kx$, the equation of motion for the damped harmonic oscillator with variable mass $m(t)$ is given by

$$m(t)\ddot{x} + \dot{m}(t)\dot{x} + \gamma\dot{x} + kx = 0. \tag{10}$$

Thus, dynamics of the damped harmonic oscillator with variable mass $m(t)$ is governed by this equation of motion. We note that the second term in Equation (10) arises due to the oscillator mass being time-dependent. Damping occurs from the two sources $-\gamma\dot{x}$ and $-\dot{m}(t)\dot{x}$ and thus the varying mass plays the same role as the damping coefficient γ , that could be a control parameter for the damping.

By applying Bateman’s dual oscillator formulation, the equations of motion for the dual system of the damped harmonic oscillator (10) may be expressed by the following equations of motion:

$$m(t)\ddot{x} + \dot{m}(t)\dot{x} + \gamma\dot{x} + kx = 0, \quad m(t)\ddot{y} + \dot{m}(t)\dot{y} - \gamma\dot{y} + ky = 0. \tag{11}$$

If we do not employ an explicit time-dependent dissipative function, the Lagrangian leading to Equation (11) can be expressed by

$$\mathcal{L} = m(t)\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) - kxy. \tag{12}$$

It is interesting to note that the form of Equation (12) is similar to Equation (6) but the mass m is time-dependent: $m = m(t)$.

Lagrange equations of motion for the Lagrangian (12) reproduce correctly the dual equations of motion (11): the first equation represents a damped harmonic oscillator with variable mass, while the second one can be considered as its time-reversed image.

Let us define the canonical momenta (p_x, p_y) for our dual oscillator system by using the Lagrangian (12):

$$p_x \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(t)\dot{y} - \frac{\gamma}{2}y, \quad p_y \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}} = m(t)\dot{x} + \frac{\gamma}{2}x. \tag{13}$$

It should be noted that these canonical momenta defined in Equation (13) are different from the kinetic momenta defined by Equation (8). In order to obtain the Hamiltonian of this dual system, we apply Legendre transformation to the Lagrangian function (12) in a following way:

$$\mathcal{H} = \dot{x}p_x + \dot{y}p_y - \mathcal{L} = \dot{x}p_x + \dot{y}p_y - m(t)\dot{x}\dot{y} - \frac{\gamma}{2}(x\dot{y} - y\dot{x}) + kxy. \tag{14}$$

Expressing Equation (14) in terms of the canonical momenta (p_x, p_y) with the use of Equation (13), we can obtain the Hamiltonian function of the dual system for the damped harmonic oscillators with variable mass; by this transformation the velocities (\dot{x}, \dot{y}) are transformed into the new variables of momenta (p_x, p_y) . The Hamiltonian function $\mathcal{H} = \mathcal{H}(x, y, p_x, p_y, t)$ of the system can then be expressed by

$$\mathcal{H} = \frac{p_x p_y}{m(t)} + \frac{\gamma}{2m(t)}(y p_y - x p_x) + \left(k - \frac{\gamma^2}{4m(t)} \right) xy. \tag{15}$$

This is the extended Bateman dual-Hamiltonian for which Hamilton’s equations of motion reproduce correctly the doubled system. Since the energy of the total system is constant, the system of damped harmonic oscillator and its time-reversed image is a closed system described by the Hamiltonian function (15). We can write the canonical

equations of Hamilton as follows:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_y}{m(t)} - \frac{\gamma}{2m(t)} x, \quad \dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_x}{m(t)} + \frac{\gamma}{2m(t)} y, \quad (16)$$

$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = \frac{\gamma}{2m(t)} p_x - \left(k - \frac{\gamma^2}{4m(t)} \right) y, \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = -\frac{\gamma}{2m(t)} p_y - \left(k - \frac{\gamma^2}{4m(t)} \right) x. \quad (17)$$

The Hamilton equations of motion reproduce correctly the classical damped harmonic oscillator systems.

The Poisson brackets of the dual system are

$$\{x, p_x\} = \{y, p_y\} = 1, \quad \{x, p_y\} = \{y, p_x\} = \{x, y\} = \{p_x, p_y\} = 0. \quad (18)$$

The Poisson bracket formulation of Hamilton's equations is given by

$$\dot{x} = \{x, \mathcal{H}\}, \quad \dot{p}_x = \{p_x, \mathcal{H}\}; \quad \dot{y} = \{y, \mathcal{H}\}, \quad \dot{p}_y = \{p_y, \mathcal{H}\}. \quad (19)$$

It should be noted that the Hamiltonian (15) is a constant of the motion since $\{\mathcal{H}, \mathcal{H}\} = \partial \mathcal{H} / \partial t = 0$. It is thus concluded that the energy dissipated by the original oscillator is completely absorbed by the dual of the system.

3. Quantum Theory

Let us consider the quantal case. Canonical quantization for the dual Hamiltonian \mathcal{H} in Equation (15) can be done by applying the standard quantization rules:

$$[\hat{x}, \hat{p}_x] = i\hbar = [\hat{y}, \hat{p}_y]; \quad [\hat{x}, \hat{y}] = 0 = [\hat{p}_x, \hat{p}_y], \quad (20)$$

where position and momentum operators are denoted respectively by \hat{x} , \hat{p}_x , etc. The quantized Hamiltonian $\hat{\mathcal{H}}$ is then expressed by

$$\hat{\mathcal{H}} = \frac{\hat{p}_x \hat{p}_y}{m(t)} + m(t) \Omega(t)^2 \hat{x} \hat{y} + \frac{\gamma}{2m(t)} [\hat{y} \hat{p}_y - \hat{x} \hat{p}_x], \quad (21)$$

where

$$\Omega(t) \equiv \left[\frac{1}{m(t)} \left(k - \frac{\gamma^2}{4m(t)} \right) \right]^{1/2}. \quad (22)$$

Note that the mass variable m depends on time t , so that the common frequency of the two damped oscillators Ω defined by Equation (23) also depends on time. Here and hereafter we drop hat ($\hat{\quad}$) from the operators for the sake of simplicity of the notations.

Now we introduce the pairs of the annihilation and creation operators (a_j, a_j^\dagger) ($j = 1, 2$):

$$a_1 \equiv \frac{1}{\sqrt{2\hbar\Omega(t)}} \left[\frac{p_x}{\sqrt{m(t)}} - i\sqrt{m(t)}\Omega(t)x \right], \quad (23)$$

$$a_1^\dagger \equiv \frac{1}{\sqrt{2\hbar\Omega(t)}} \left[\frac{p_x}{\sqrt{m(t)}} + i\sqrt{m(t)}\Omega(t)x \right], \quad (24)$$

$$a_2 \equiv \frac{1}{\sqrt{2\hbar\Omega(t)}} \left[\frac{p_y}{\sqrt{m(t)}} - i\sqrt{m(t)}\Omega(t)y \right], \tag{25}$$

$$a_2^\dagger \equiv \frac{1}{\sqrt{2\hbar\Omega(t)}} \left[\frac{p_y}{\sqrt{m(t)}} + i\sqrt{m(t)}\Omega(t)y \right]. \tag{26}$$

The creation operator a_j^\dagger is the Hermitian conjugate of the annihilation operator a_j . These operators a_j and a_j^\dagger ($j=1,2$) satisfy the following commutation rules:

$$[a_1, a_1^\dagger] = 1 = [a_2, a_2^\dagger], [a_1, a_2] = 0 = [a_1, a_2^\dagger]. \tag{27}$$

The Hamiltonian (21) can then be expressed in terms of these creation and annihilation operators:

$$\mathcal{H} = \hbar\Omega(t)(a_1^\dagger a_2 + a_1 a_2^\dagger) - \frac{i\hbar\gamma}{4m(t)} \left[(a_1^2 - a_1^{\dagger 2}) - (a_2^2 - a_2^{\dagger 2}) \right], \tag{28}$$

where $\Omega(t)$ is given by Equation (23).

The second-quantized Hamiltonian (28) is not a simple form and it is difficult to clarify the physical meaning of each term in the particle picture. We perform the following linear canonical transformation by introducing new operators \mathcal{A} and \mathcal{B} , which define the canonical transformations [9]:

$$\mathcal{A} \equiv \frac{1}{\sqrt{2}}(a_1 + a_2), \quad \mathcal{B} \equiv \frac{1}{\sqrt{2}}(a_1 - a_2), \tag{29}$$

and their conjugates, which resort to Equations (23)-(26). These new operators $(\mathcal{A}, \mathcal{B})$ obey the same algebra as in Equation (27), that is, the following canonical commutation rules hold for the new operators $(\mathcal{A}, \mathcal{A}^\dagger, \mathcal{B}, \mathcal{B}^\dagger)$:

$$[\mathcal{A}, \mathcal{A}^\dagger] = 1 = [\mathcal{B}, \mathcal{B}^\dagger], [\mathcal{A}, \mathcal{B}] = 0 = [\mathcal{A}, \mathcal{B}^\dagger]. \tag{30}$$

Thus these operators construct a *dual* Hilbert (Fock) space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $(\mathcal{A}, \mathcal{A}^\dagger)$ on \mathcal{H}_A and $(\mathcal{B}, \mathcal{B}^\dagger)$ on \mathcal{H}_B .

The Hamiltonian (21) in the Schrödinger picture (SP) can be expressed in terms of the new operators $(\mathcal{A}, \mathcal{A}^\dagger, \mathcal{B}, \mathcal{B}^\dagger)$ in a simple form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1; \tag{31}$$

$$\mathcal{H}_0 \equiv \hbar\Omega(t)(\mathcal{A}^\dagger \mathcal{A} - \mathcal{B}^\dagger \mathcal{B}), \tag{32}$$

$$\mathcal{H}_1 \equiv i\hbar\Gamma(t)(\mathcal{A}^\dagger \mathcal{B}^\dagger - \mathcal{A}\mathcal{B}), \tag{33}$$

where $\Gamma(t) \equiv \gamma/2m(t)$. We note that the Hamiltonians \mathcal{H}_0 and \mathcal{H}_1 in SP are both time-dependent through Ω and Γ since they depend on the time dependent mass $m(t)$.

In order to see the effect of varying mass, let us define the vacuum states, $|0\rangle_A$ for the system (A) spanned by the operator \mathcal{A} and $|0\rangle_B$ for the system (B) spanned by the operator \mathcal{B} :

$$\mathcal{A}|0\rangle_A = 0 = \mathcal{B}|0\rangle_B. \tag{34}$$

Then the vacuum state of the Hamiltonian \mathcal{H} on the dual Hilbert space can be described by the direct product of $|0\rangle_{\mathcal{A}}$ and $|0\rangle_{\mathcal{B}}$:

$$|0\rangle\rangle \equiv |0,0\rangle \equiv |0\rangle_{\mathcal{A}} \otimes |0\rangle_{\mathcal{B}} \quad (35)$$

since any operators on $\mathcal{H}_{\mathcal{A}}$ commutes with any operators on $\mathcal{H}_{\mathcal{B}}$.

The SP evolution operator $\mathcal{U}(t, t_0)$ is generally expressed by

$$\mathcal{U}(t, t_0) = T \exp \left[-\frac{i}{\hbar} \int_{t_0}^t \mathcal{H} dt \right], \quad (36)$$

where the symbol T designates the time-ordering operator. Then we can define a vacuum state at a time t for a dissipative system as

$$|0(t)\rangle\rangle \equiv e^{-i\mathcal{H}t/\hbar} |0\rangle\rangle. \quad (37)$$

By using the Hamiltonian \mathcal{H} expressed in terms of the operators $(\mathcal{A}, \mathcal{B})$ and their conjugate operators [see Equations (31)-(33)] and the relations (30) and (34), the vacuum state (37) can be evaluated straightforwardly. The vacuum state $|0(t)\rangle\rangle$ at time t is then explicitly given by

$$|0(t)\rangle\rangle = \frac{1}{\cosh(\Gamma t)} \exp \left[\tanh(\Gamma t) \mathcal{A}^\dagger \mathcal{B}^\dagger \right] |0\rangle\rangle. \quad (38)$$

Recalling $\Gamma = \gamma/2m(t)$, Equation (38) can be expressed by

$$|0(t)\rangle\rangle = \frac{1}{\cosh[\gamma t/2m(t)]} \exp \left[\tanh(\gamma t/2m(t)) \mathcal{A}^\dagger \mathcal{B}^\dagger \right] |0\rangle\rangle. \quad (39)$$

This equation forms the basis for further evaluation of the vacuum state of the system associated with oscillator's variable mass and other parameters characterizing the system. In the following we consider the effect of variable mass $m(t)$ on the dissipated system.

4. Effect of Variable Mass

Let us study the effect of variable mass/dissipative force on the present dissipative system by looking at the vacuum states with the use of Equation (39) since the vacuum state sensitively reflects the stability (dissipation) of the system. Here we consider the following cases: (i) $m(t) = m_0$, (ii) $m(t) = m_0 t$, (iii) $m(t) = m_0 t^{-1}$, and (iv) $m(t) = m_0 \exp(\gamma t)$, where m_0 is the mass value (constant) at $t = 0$.

We first consider the case (i) for a constant mass, *i.e.*, $m(t) = m_0$. The vacuum state, Equation (39), is then given by that of the damped harmonic oscillator obtained in Ref. [11]:

$$|0(t)\rangle\rangle = \frac{1}{\cosh(\gamma t/2m_0)} \exp \left[\tanh(\gamma t/2m_0) \mathcal{A}^\dagger \mathcal{B}^\dagger \right] |0\rangle\rangle. \quad (40)$$

Figure 1 shows the time development of the vacuum state $|0\rangle\rangle$ in $t \in [-\infty, \infty]$. Here we see that the physical vacuum $|0\rangle\rangle$ (solid line) increases with time, reaches its maximum value and then decreases. This asymmetric shape of the $|0\rangle\rangle$ vs. t plot can be explained as follows: As seen from Equation (38), the vacuum state $|0\rangle\rangle$ is the

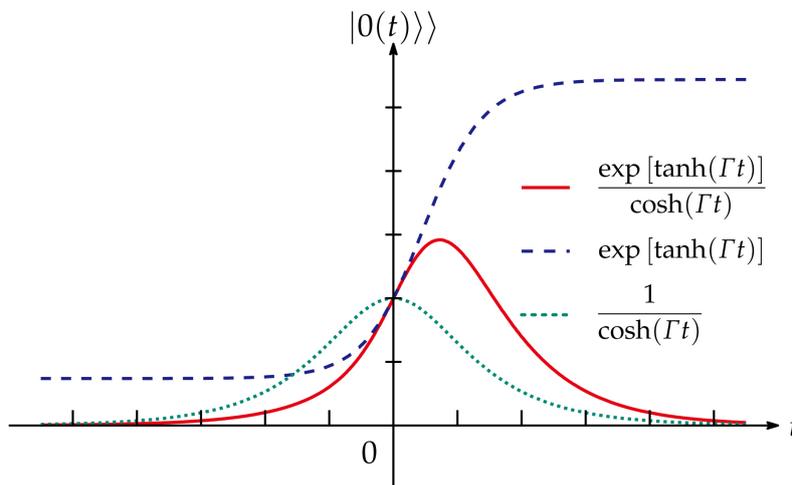


Figure 1. (Color online) The vacuum state $|0(t)\rangle$ vs. t for a TDM with $\Gamma \equiv \gamma/2m(t)$, where $m(t) = m_0$.

product of $1/\cosh(\Gamma t)$ and $\exp[\tanh(\Gamma t)]$. The function $1/\cosh(\Gamma t)$ (dotted line) is symmetric about $t = 0$, which converges 0 at $t = \pm\infty$, while the function $\tanh(\Gamma t)$ increases monotonically with time for $t \in [0, \infty]$, where $\tanh(\Gamma t)$ takes the values between 0.1 and 1 for $t = 1/\Gamma$. Accordingly, the monotonically increasing function $\exp[\tanh(\Gamma t)]$ converges about 2.5 for $t = 2/\Gamma$. From these results, we can say qualitatively that $|0(t)\rangle$, which is the product of $1/\cosh(\Gamma t)$ and $\exp[\tanh(\Gamma t)]$, has a peak at $t = 2/\Gamma$, and then decreases monotonically and converges 0 as t goes infinity. This asymmetry of the vacuum state seen in **Figure 1** is due essentially to the presence of dissipative (resistive) force. In other words, this collapse of an initial state (instability of vacuum state) is characteristic of the dissipative system considered here. We may say that in our dissipative model system, the time reversal symmetry of the vacuum state breaks down essentially due to the presence of the time reversed resistive (damping) force in the present model system.

Next we consider the case (ii). The vacuum state, Equation (39), is then given by

$$|0(t)\rangle = \frac{1}{\cosh(\gamma/2m_0)} \exp[\tanh(\gamma/2m_0) \mathcal{A}^\dagger \mathcal{B}^\dagger] |0\rangle. \tag{41}$$

It is interesting to note that the vacuum state does *not* change with time when the mass changes linearly with time t . In such a case, the vacuum state remains in a static state. See **Figure 2**.

Next we consider the case (iii), where the mass decreases with time: $m(t) = m_0 t^{-1}$. The time development of the vacuum state $|0\rangle$ in $t \in [-\infty, \infty]$ is then shown in **Figure 3**. In this case, the vacuum state decays abruptly in a short period of time faster than the case for a constant mass $m = m_0$ (cp **Figure 1**).

Finally we consider the case (iv), where the mass increases exponentially with time: $m(t) = m_0 \exp(\gamma t)$ [see **Figure 4**]. The Kanai Hamiltonian [3] represents a particle of

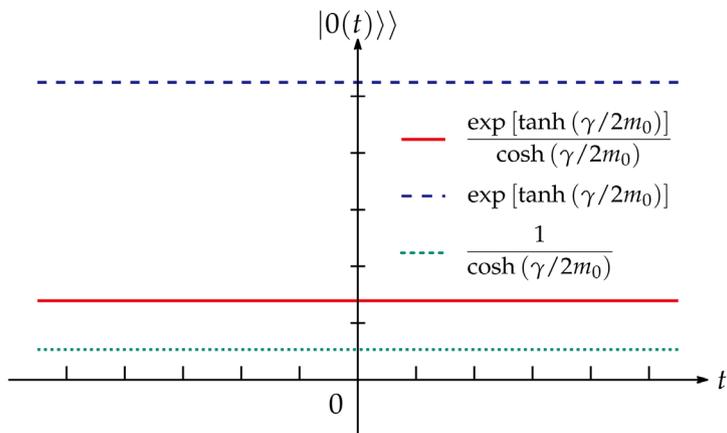


Figure 2. (Color online) The vacuum state $|0(t)\rangle$ vs. t for a TDM with $\Gamma \equiv \gamma/2m(t)$, where $m(t) = m_0 t$.

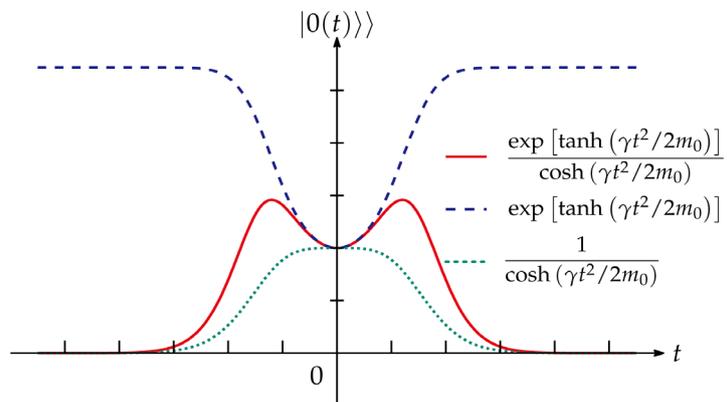


Figure 3. (Color online) The vacuum state $|0(t)\rangle$ vs. t for a TDM with $\Gamma \equiv \gamma/2m(t)$, where $m(t) = m_0 t^{-1}$.

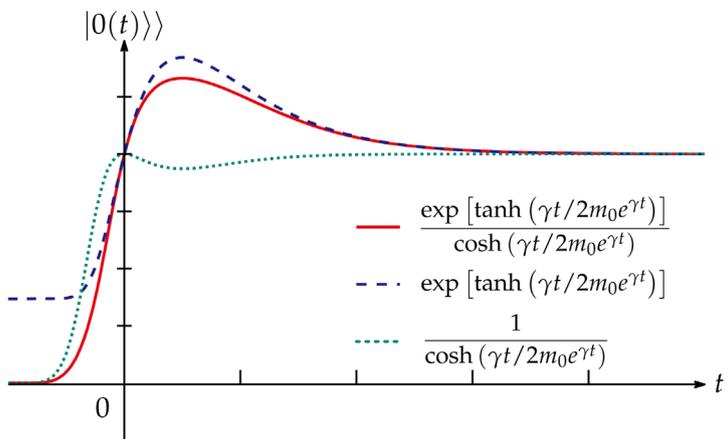


Figure 4. (Color online) The vacuum state $|0(t)\rangle$ vs. t for a TDM with $\Gamma \equiv \gamma/2m(t)$, where $m(t) = m_0 \exp(\gamma t)$.

varying mass, $m(t) = m_0 \exp(\gamma t)$. Indeed this type of mass change explains the peculiar quantum mechanical features of the system represented by the Kanai Hamiltonian, when misinterpreted as representing a particle of fixed mass subject to a damping force [11]. In our formulation, Equation (39) for $m(t) = m_0 \exp(\gamma t)$ is given by

$$|0(t)\rangle\rangle = \frac{1}{\cosh(\gamma t e^{-\gamma t}/2m_0)} \exp\left[\tanh(\gamma t e^{-\gamma t}/2m_0) \mathcal{A}^\dagger \mathcal{B}^\dagger\right] |0\rangle\rangle. \tag{42}$$

In order to study the effects of the dissipative Hamiltonian \mathcal{H}_t [cf. Equation (33)] on the vacuum state, one could directly compute $\exp(-i\mathcal{H}t/\hbar)|0\rangle\rangle$ by making use of Equation (39) along with the Hamiltonian \mathcal{H} [Equation (31)]. The vacuum-to-vacuum (V-to-V) transition amplitude (viz., the vacuum survival probability amplitude) subject to \mathcal{H}_0 [Equation (32)] can be calculated straightforwardly. We generally obtain the V-to-V transition amplitude in the limit $t \rightarrow \infty$:

$$\langle\langle 0|0(t)\rangle\rangle = \exp(-\ln \cosh \Gamma t) \xrightarrow{t \rightarrow \infty} 0. \tag{43}$$

These are of dissipative nature, their time evolution being controlled by $e^{-\Gamma t}$ for large t , as Equation (43) shows. This is a general expression for the V-to-V transition amplitude in the presence of external force field, from which we can study the effects of the external and the dissipative force fields through \mathcal{H}_0 and \mathcal{H}_t on the dissipative system expressed by the Hamiltonian \mathcal{H} [see Equations (31)-(33)].

It has been shown that the proper way to perform the canonical quantization of the damped harmonic oscillator is to work in the framework of Quantum Field Theory (QFT) [13]. In our formulation for many degrees of freedom, $|0(t)\rangle\rangle$ is formally (at finite volume) expressed by the free vacuum state $|0(t)\rangle\rangle$ at time t as $|0(t)\rangle\rangle \equiv e^{-i\mathcal{H}_0 t/\hbar} |0\rangle\rangle$, which can be calculated straightforwardly by using the explicit form of the system Hamiltonian \mathcal{H} [see Equations (31)-(33)]:

$$\begin{aligned} |0(t)\rangle\rangle &\equiv \exp(-i\mathcal{H}t/\hbar)|0\rangle\rangle \\ &= \exp \sum_k \Gamma_k t (\mathcal{A}_k^\dagger \mathcal{B}_k^\dagger - \mathcal{A}_k \mathcal{B}_k) |0\rangle\rangle = \prod_k \frac{1}{\cosh \Gamma_k t} \exp\left[\tanh(\Gamma_k t) \mathcal{A}_k^\dagger \mathcal{B}_k^\dagger\right] |0\rangle\rangle. \end{aligned} \tag{44}$$

In QFT we have to consider infinitely many degrees of freedom. Thus, by using the continuous limit relation $\sum_k \mapsto V/(2\pi)^3 \int d^3k$, we obtain

$$\langle\langle 0(t')|0(t)\rangle\rangle \xrightarrow{V \rightarrow \infty} 0 \quad \forall t, t' \text{ with } t \neq t', \tag{45}$$

provided that $\int d^3k \ln \cosh \Gamma_k t$ is finite and positive. Equation (45) means that the representation at a given time t is unitarily inequivalent to the representation at any different time $t' \neq t$ in the infinite volume limit: the system spans a whole set of unitarily inequivalent representations as time evolves (n.b. each of them is labeled by different values of t).

5. Summary and Conclusions

To sum up, we have studied the DHO with a variable mass as a simple model for a dissipative system, following the theoretical scheme of Majima and Suzuki [11]. In

Section 2, we developed the classical theory for the DHO with a variable (time-dependent) mass $m(t)$. By introducing the *kinetic momentum* $p \equiv m(t)\dot{x}$, the equation of motion obtained for the DHO with a time-dependent mass, Equation (10), is different from the DHO with a constant mass; as seen in Equation (10), the new force term $-\dot{m}(t)\dot{x}$ appears in addition to the damping force $-\gamma\dot{x}$. This suggests the form of $m(t)$ plays the same role as γ in the damping force term. It is worth to mention that the replacement of m by $m(t)$ in the equation of motion (1) for the DHO is *not* allowed when we consider the dynamics for the DHO with a time-dependent mass (since the kinetic momentum \neq the canonical momentum). The mass $m(t)$ plays the same role as the damping constant γ . In other words, the time-dependent mass $m(t)$ could be regarded as the *control parameter* of damping (dissipation). Depending on the form of the time dependency of the mass, we expect that the oscillator mass $m(t)$ plays a passive or active role for the stability of the oscillator state and the dissipation.

Introducing the *canonical momenta* (13) by using the obtained Lagrangian (12) for the time-dependent mass $m(t)$, we obtained the classical Hamiltonian (15) in terms of the canonical momenta for the dissipative oscillator and its mirror image oscillator by applying the classical theory of Bateman's dual oscillator formulation.

In Section 3, we extended the theory developed in Section 2 to the quantum case, where we showed and discussed in detail how to derive the second quantized form of the Hamiltonian in terms of creation and annihilation operators (a^\dagger, a) for this dual system. The resultant Hamiltonian (28) is not a simple form and it is hard to clarify the physical meaning of each term in the particle picture. We showed that the second-quantized Hamiltonian (28) can be expressed in a simple form by introducing new creation and annihilation operators $(\mathcal{A}^\dagger, \mathcal{A}, \mathcal{B}^\dagger, \mathcal{B})$, which construct a dual Hilbert (Fock) space $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, $(\mathcal{A}, \mathcal{A}^\dagger)$ on $\mathcal{H}_{\mathcal{A}}$ and $(\mathcal{B}, \mathcal{B}^\dagger)$ on $\mathcal{H}_{\mathcal{B}}$. Then the Hamiltonian (28) can be expressed in a simple form (see Equations (31)-(33)).

In order to discuss the stability of the system arising from the change of oscillator mass in time, we focus on the change of the vacuum state of the system due to the change in the mass causing dissipation/stability of the system. We derived the general formula of the vacuum state $|0\rangle\rangle$ of the system (38), which forms the basis of the present study of stability of the dissipated system. We have shown that the stability of the system due to the mass change and/or other parameters characterizing the system can be studied by looking at the behaviour of a vacuum state since the vacuum state contains useful information on the system.

In Section 4, we studied the effect of variable mass/dissipative force on the quantum dissipative system by using the formula (40). We considered the following cases: (i) $m(t) = m_0$, (ii) $m(t) = m_0 t$, (iii) $m(t) = m_0 t^{-1}$, and (iv) $m(t) = m_0 \exp(\gamma t)$. Characteristic features of the time development of vacuum state $|0(t)\rangle\rangle$ for a TDM listed above are shown **Figures 1-4**, respectively.

Noticing that the time-dependent mass could be a *control parameter* for dissipation/damping of the DHO with varying (time-dependent) mass, we developed the theory to

investigate dissipation (damping) in the quantum theory and quantum dissipated system by employing the DHO as the simple model system. The time-dependent oscillator mass $m(t)$ as well as the damping factor γ plays an important role for damping (dissipation). Controlling these parameter, we can study the effect of dissipation in quantum dissipative systems.

References

- [1] Bateman, H. (1931) *Physical Review*, **38**, 815. <https://doi.org/10.1103/PhysRev.38.815>
- [2] Fechbach, H. and Tikochinsky, Y. (1977) *The New York Academy of Sciences*, **38**, 44.
- [3] Kanai, E. (1948) *Progress of Theoretical Physics*, **3**, 440. <https://doi.org/10.1143/ptp/3.4.440>
- [4] Caldilola, P. (1941) *Nouovo Cimento*, **18**, 393. <https://doi.org/10.1007/BF02960144>
- [5] Greenberger, D.M. (1979) *Journal of Mathematical Physics*, **20**, 762. <https://doi.org/10.1063/1.524148>
- [6] Celeghini, E., Rasetti, M. and Vitiello, G. (1992) *Annals of Physics (New York)*, **215**, 156. [https://doi.org/10.1016/0003-4916\(92\)90302-3](https://doi.org/10.1016/0003-4916(92)90302-3)
- [7] Morse, P.M. and Feshbach, H. (1953) *Methods of Theoretical Physics*. Vol. 1, McGraw-Hill, New York.
- [8] Dekker, H. (1977) *Physical Review A*, **16**, 2126. <https://doi.org/10.1103/PhysRevA.16.2126>
- [9] Dekker, H. (1981) *Physical Reports*, **80**, 1. [https://doi.org/10.1016/0370-1573\(81\)90033-8](https://doi.org/10.1016/0370-1573(81)90033-8)
- [10] Um, C., Yeon, K. and George, T.F. (2002) *Physics Reports*, **362**, 63. [https://doi.org/10.1016/S0370-1573\(01\)00077-1](https://doi.org/10.1016/S0370-1573(01)00077-1)
- [11] Majima, H. and Suzuki, A. (2011) *Annals of Physics*, **326**, 3000. <https://doi.org/10.1016/j.aop.2011.08.002>
- [12] Greenberger, D.M. (1979) *Journal of Mathematical Physics*, **20**, 771. <https://doi.org/10.1063/1.524149>
- [13] Blasone, M., Jizba, P. and Vitiello, G. (2011) *Quantum Field Theory and Its Macroscopic Manifestations: Boson Condensation, Ordered Patterns and Topological Defects*. Imperial Colledge Press, London. <https://doi.org/10.1142/p592>



Submit or recommend next manuscript to SCIRP and we will provide best service for you:

Accepting pre-submission inquiries through Email, Facebook, LinkedIn, Twitter, etc.
A wide selection of journals (inclusive of 9 subjects, more than 200 journals)
Providing 24-hour high-quality service
User-friendly online submission system
Fair and swift peer-review system
Efficient typesetting and proofreading procedure
Display of the result of downloads and visits, as well as the number of cited articles
Maximum dissemination of your research work

Submit your manuscript at: <http://papersubmission.scirp.org/>

Or contact jmp@scirp.org