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General Solution and Stability of Quattuordecic Functional Equation in Quasi β-Normed Spaces

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Abstract

In this paper, we introduce the following quattuordecic functional equation

$$\begin{aligned}&f(x+7y)-14f(x+6y)+91f(x+5y)-364f(x+4y)+1001f(x+3y)\\&-2002f(x+2y)+3003f(x+y)-3432f(x)+3003f(x-y)-2002f(x-2y)\\&+1001f(x-3y)-364f(x-4y)+91f(x-5y)-14f(x-6y)+f(x-7y)\\&=14!\,f(y),\end{aligned}$$

investigate the general solution and prove the stability of this quattuordecic functional equation in quasi β-normed spaces by using the fixed point method.

Keywords

Quattuordecic Functional Equation, Fixed Point Method, Hyers-Ulam Rassias Stability, Quasi-β-Normed Space

1. Introduction

The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. He stated that if $(G_1,.)$ is a group and let $(G_2,*)$ be a metric group with metric $d(.,.)$: Given $\epsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)*h(y))<\epsilon$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(G(x),H(x))<\epsilon$ for all $x\in G_1$?

The case of approximately additive functions was solved by D. H. Hyers [2] under the assumption that both E_1 and E_2 are Banach spaces. He stated that for $\epsilon>0$ and

$f: E_1 \to E_2$ such that $\| f(x+y) - f(x) - f(y) \| \le \varepsilon$ for all $x, y \in E_1$, then there exists a unique additive mapping $T: E_1 \to E_2$ such that $\| f(x) - T(x) \| \le \varepsilon$ for all $x \in E_1$. This result is called Hyers-Ulam stability.

Hyers Theorem was generalized by Th. M. Rassias [3] for linear mappings by considering an unbounded Cauchy difference. The stability problem of several functional equations has been extensively investigated by a number of authors, and there are many interesting results concerning this problem [4]-[17].

Very recently the general solution and the stability of the quintic and sextic functional equation in quasi-β-normed spaces via fixed point method were discussed by [18]. The general solution, the stability of the septic and Octic functional equations, *viz.*

$$\begin{aligned}&f(x+4y) - 7f(x+3y) + 21f(x+2y) - 35f(x+y) - 35f(x)\\&-21f(x-y) + 7f(x-2y) - f(x-3y) = 5040f(y)\end{aligned}$$

and

$$\begin{aligned}&f(x+4y) - 8f(x+3y) + 28f(x+2y) - 56f(x+y) - 70f(x) - 56f(x-y)\\&+28f(x-2y) - 8f(x-3y) + f(x-4y) = 40320f(y)\end{aligned}$$

in quasi-β-normed spaces were investigated by T. Z. Xu *et al.* [18].

J. M. Rassias and Mohamed Eslamian discussed the general solution of a Nonic functional equation

$$\begin{aligned}&f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) - 126f(x+y) - 126f(x)\\&+84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!\,f(y)\end{aligned}$$

and proved the stability of nonic functional equation [19] in quasi-β-normed spaces by applying the fixed point method.

A fixed point approach for the stability of Decic functional equation

$$\begin{aligned}&f(x+5y) - 10f(x+4y) + 45f(x+3y) - 120f(x+2y) - 210f(x+y) - 252f(x)\\&+210f(x-y) - 120f(x-2y) + 45f(x-3y) - 10f(x-4y) + f(x-5y) = 10!\,f(y)\end{aligned}$$

in quasi-β-normed spaces was investigated by K. Ravi *et al.* [20].

Very recently, K. Ravi and Senthil Kumar discussed the undecic and duodecic functional equation and its stability in quasi-β-normed spaces.

In this paper, the authors are interested in finding the general solution and stability of Quattuordecic functional equation

$$\begin{aligned}&f(x+7y) - 14f(x+6y) + 91f(x+5y) - 364f(x+4y) + 1001f(x+3y)\\&-2002f(x+2y) + 3003f(x+y) - 3432f(x) + 3003f(x-y) - 2002f(x-2y)\\&+1001f(x-3y) - 364f(x-4y) + 91f(x-5y) - 14f(x-6y) + f(x-7y)\\&= 14!\,f(y),\end{aligned} \tag{1}$$

where $14! = 87178291200$ in quasi-β-normed spaces by using fixed point method.

The functional Equation (1) is called Quattourdecic functional equation because the function $f(x) = Kx^{14}$ satisfies the Equation (1).

In Section 2, we have given necessary definitions. In Section 3, we discuss the general solution of the functional Equation (1). In Section 4, we investigate the stability of Quattuordecic functional Equation (1) in quasi-β-normed spaces and we provide a counter example to show that the functional Equation (1) is not stable.

2. Preliminaries

We recall some basic concepts concerning quasi-β-normed spaces introduced by J. M. Rassias and H. M. Kim [14] in 2009. Let β be a fixed real number with $0<\beta\le 1$, and let K denote either R or C. Let X be linear space over K. A quasi-β-norm $\|.\|$ is a real valued function on X satisfying the following three conditions:

1) $\|x\|\ge 0$, for all $x\in X$; and $\|x\|=0$ iff $x=0$,

2) $\|\lambda x\|=|\lambda|^{\beta}\|x\|$ for all $\lambda\in K$, and all $x\in X$,

3) there is a constant $k\ge 1$ such that $\|x+y\|\le K(\|x\|+\|y\|)$.

For all $x,y\in X$. A quasi-β-normed space is a pair $(X,\|.\|)$, where $\|.\|$ is a quasi-β on X. The smallest possible K is called the modules of concavity of $\|.\|$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $\|.\|$ is called a (β,p)-norm $0<\beta\le 1$ if $\|x+y\|^{p}\le\|x\|^{p}+\|y\|^{p}$ $\forall\, x,y\in X$.

In this space a quasi-β-Banach space is called a (β,p)-Banach space. We can refer to [18] for the concept of quasi-normed spaces and p-Banach spaces. Given a p-norm, the formula $d(x,y)\le\|x-y\|^{p}$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem, each quasi-norm is equal to some p-norm. Since it is much easier to work with p-norms then quasi-norms, we restrict our attention mainly to p-norms.

Using fixed point theorem, Xu *et al.* [18] proved the following impotent lemma.

Lemma 1. *Let* $i\in -1,1$ *be fixed,* $a,s\in N$ *with* $a\ge 2$, *and* $\varphi:X\to[0,\infty)$ *be a function such that there exists an* $L<1$ *with* $\varphi(a^{i}x)\le a^{is\beta}L\varphi(x)$ *for all* $x\in X$. *Let* $f:X\to Y$ *be a mapping satisfying*

$$\left\|f(ax)-a^{s}f(x)\right\|_{Y}\le\varphi(x)\quad\forall x,y\in X. \tag{2}$$

Then there exists a uniquely determined mapping $F:X\to Y$ such that

$$\left\|f(x)-F(x)\right\|_{Y}\le\frac{1}{a^{s\beta}\left|1-L^{i}\right|}\varphi(x)\quad\forall x\in X. \tag{3}$$

3. General Solution of Functional Equation

In this section, let X and Y be vector spaces. In the following Theorem, we investigate the general solution of the functional Equation (1).

Theorem 1. *A function* $f:X\to Y$ *is a solution of the Quattuordecic functional Equation* (1) *if and only if f is of the form* $f(x)=A^{14}(x)$ *for all* $x\in X$, *where* A^{14} *is the diagonal of the* 14*-additive symmetric mapping* $A_Q:X^{14}\to Y$.

Proof. Assume that f satisfies the functional Equation (1). Replacing (x,y) by $(0,0)$ in (1), we have $f(0)=0$. Replacing (x,y) by (x,x) in (1), we get

$$\begin{aligned} &f(8x)-14f(7x)+91f(6x)-364f(5x)+1001f(4x)-2002f(3x)\\ &+3003f(2x)-3432f(x)+3003f(0)-2002f(-x)+1001f(-2x)\\ &-364f(-3x)+91f(-4x)-14f(-5x)+f(-6x)=14!f(x). \end{aligned} \tag{4}$$

Substituting (x,y) by $(x,-x)$ in (1), we obtain

$$\begin{aligned} &f(-6x)-14f(-5x)+91f(-4x)-364f(-3x)+1001f(-2x)-2002f(-x)\\ &+3003f(0)-3432f(x)+3003f(0)-2002f(3x)+1001f(4x)-364f(5x)\\ &+91f(6x)-14f(7x)+f(8x)=14!f(-x). \end{aligned} \tag{5}$$

Subtracting Equations (5) and (4), we get

$$f(-x)=f(x). \tag{6}$$

Replacing (x,y) with $(0,2x)$ in (1), one gets

$$\begin{aligned} &2f(14x)-28f(12x)+182f(10x)-728f(8x)+2002f(6x)-4004f(4x)\\ &+6006f(2x)=14!f(2x) \end{aligned}$$

and

$$\begin{aligned} &f(14x)-14f(12x)+91f(10x)-364f(8x)+1001f(6x)-2002f(4x)\\ &+43589142597f(2x)=0. \end{aligned} \tag{7}$$

Replacing (x,y) with $(7x,x)$ in (1), one gets

$$\begin{aligned} &f(14x)-14f(13x)+91f(12x)-364f(11x)+1001f(10x)-2002f(9x)\\ &+3003f(8x)-3432f(7x)+3003f(6x)-2002f(5x)+1001f(4x)\\ &-364f(3x)+91f(2x)=87178291214f(x). \end{aligned} \tag{8}$$

Subtracting the Equations (7) and (8), we obtain

$$\begin{aligned} &14f(13x)-105f(12x)+364f(11x)-910f(10x)+2002f(9x)\\ &-3367f(8x)+3432f(7x)-2002f(6x)+2002f(5x)-3003f(4x)\\ &+364f(3x)-43589142688f(2x)+87178291214f(x)=0. \end{aligned} \tag{9}$$

Replacing (x,y) with $(6x,x)$ in (1) and multiplying by 14, we have

$$\begin{aligned} &14f(13x)-196f(12x)+1274f(11x)-5096f(10x)+14014f(9x)\\ &-28028f(8x)+42042f(7x)-48048f(6x)+42042f(5x)-28028f(4x)\\ &+14014f(3x)-5096f(2x)-1220496075512f(x)=0. \end{aligned} \tag{10}$$

Subtracting Equations (9) and (10), we obtain

$$\begin{aligned} &91f(12x)+910f(11x)+4186f(10x)-12012f(9x)+24661f(8x)\\ &-38610f(7x)+46046f(6x)-40040f(5x)+25025f(4x)-13650f(3x)\\ &-43589137592f(2x)+1307674366726f(x)=0. \end{aligned} \tag{11}$$

Replacing (x,y) with $(5x,x)$ in (1) and multiplying by 91, we have

$$\begin{aligned}&91f(12x)-1274f(11x)+8281f(10x)-33124f(9x)+91091f(8x)\\&-182182f(7x)+273273f(6x)-312312f(5x)+273273f(4x)\\&-182182f(3x)+91182f(2x)-7933224533598f(x)=0.\end{aligned}\tag{12}$$

Subtracting Equations (11) and (12), we have

$$\begin{aligned}&364f(11x)-4095f(10x)+21112f(9x)-66430f(8x)+143572f(7x)\\&-227227f(6x)+272272f(5x)-248248f(4x)+168532f(3x)\\&-43589228774f(2x)+9240898900324f(x)=0.\end{aligned}\tag{13}$$

Replacing (x,y) with $(4x,x)$ in (1) and multiplying by 364, we have

$$\begin{aligned}&364f(11x)-5096f(10x)+33124f(9x)-132496f(8x)+364364f(7x)\\&-728728f(6x)+1093092f(5x)-1249248f(4x)+1093456f(3x)\\&-733824f(2x)-317328975599312f(x)=0.\end{aligned}\tag{14}$$

Subtracting Equations (13) and (14), we obtain

$$\begin{aligned}&1001f(10x)-12012f(9x)+66066f(8x)-220792f(7x)\\&+501501f(6x)-820820f(5x)+1001000f(4x)-924924f(3x)\\&+43588494950f(2x)-40973796499636f(x)=0.\end{aligned}\tag{15}$$

Replacing (x,y) with $(3x,x)$ in (1) and multiplying by 1001, we obtain

$$\begin{aligned}&1001f(10x)-14014f(9x)+91091f(8x)-364364f(7x)\\&+1002001f(6x)-2004002f(5x))+3007004f(4x)-3449446f(3x)\\&+3097094f(2x)-87265471859566f(x)=0.\end{aligned}\tag{16}$$

Subtracting Equations (15) and (16), one gets

$$\begin{aligned}&2002f(9x)-25025f(8x)+143572f(7x)-500500f(6x)\\&+1183182f(5x)-2006004f(4x)+2524522f(3x)\\&-43591592044f(2x)+128239268359202f(x)=0.\end{aligned}\tag{17}$$

Replacing (x,y) with $(2x,x)$ in (1) and multiplying by 2002, we have

$$\begin{aligned}&2002f(9x)-28028f(8x)+182182f(7x)-728728f(6x)\\&+2006004f(5x)-4036032f(4x)+6194188f(3x)\\&-7599592f(2x)-174530930966392f(x)=0.\end{aligned}\tag{18}$$

Subtracting Equations (17) and (18), we obtain

$$\begin{aligned}&3003f(8x)-38610f(7x)+228228f(6x)-822822f(5x)+2030028f(4x)\\&-3669666f(3x)-43583992452f(2x)+302770199325594f(x)=0.\end{aligned}\tag{19}$$

Replacing (x,y) with (x,x) in (1) and multiply by 3003, we have

$$\begin{aligned}&3003f(8x)-42042f(7x)+276276f(6x)-1135134f(5x)+3279276f(4x)\\&-7105098f(3x)+12024012f(2x)-261796424791902f(x)=0.\end{aligned}\tag{20}$$

Subtracting Equations (19) and (20), one gets

$$\begin{aligned}&3432 f(7x)-48048 f(6x)+312312 f(5x)-1249248 f(4x)\\&+3435432 f(3x))-43596016464 f(2x)+564566624117496 f(x)=0.\end{aligned} \tag{21}$$

Replacing (x, y) with $(0, x)$ in (1) and multiplying by 1716, we have

$$\begin{aligned}&3432 f(7x)-48048 f(6x)+312312 f(5x)-1249248 f(4x)\\&+3435432 f(3x)-6870864 f(2x))+14959793739204 f(x)=0.\end{aligned} \tag{22}$$

Subtracting Equations (20) and (21), we have

$$-43589145600 f(2x)+714164561510400 f(x)=0$$

or

$$f(2x)=2^{14} x. \tag{23}$$

On the other hand, one can rewrite the functional Equation (1) in the form

$$\begin{aligned}&f(x)-\frac{1}{3432} f(x+7y)+\frac{7}{1716} f(x+6y)-\frac{7}{264} f(x+5y)+\frac{7}{66} f(x+4y)\\&-\frac{7}{24} f(x+3y)+\frac{7}{12} f(x+2y)-\frac{7}{8} f(x+y)-\frac{7}{8} f(x-y)+\frac{7}{12} f(x-2y)\\&-\frac{7}{24} f(x-3y)+\frac{7}{66} f(x-4y)-\frac{7}{264} f(x-5y)+\frac{7}{1716} f(x-6y)\\&-\frac{1}{3432} f(x-7y)+25401600 f(y)=0\end{aligned} \tag{24}$$

for all $x \in X$. By ([17], Theorems 3.5 and 3.6), *f* is a generalized polynomial function of degree at most 14, that is, *f* is of the form

$$\begin{aligned}f(x)=&A^{14}x+A^{13}x+A^{12}x+A^{11}x+A^{10}x+A^{9}x+A^{8}x+A^{7}x\\&+A^{6}x+A^{5}x+A^{4}x+A^{3}x+A^{2}x+A^{1}x+A^{0}x\end{aligned} \tag{25}$$

for all $x \in X$.

Here, $A^0x = A^0$ is an arbitrary element of *y* and A^ix is the diagonal of the *i*-additive symmetric map $A_i : X^i \to Y$ ($i = 1, 2, 3, \cdots, 14$) by $f(0) = 0$ and $f(-x) = f(x)$, for all $x \in X$, we get $A^0x = A^0 = 0$ and the function *f* is even. Thus, we have

$$A^{13}x+A^{11}x+A^{9}x+A^{7}x+A^{5}x+A^{3}x+A^{1}x=0$$

it follows that

$$f(x)=A^{14}x+A^{12}x+A^{10}x+A^{8}x+A^{6}x+A^{4}x+A^{2}x.$$

Using Equations (25) and $A^n(rx) = r^n A^n(x)$, we obtain

$$\begin{aligned}&2^{14}\left(A^{14}x+A^{12}x+A^{10}x+A^{8}x+A^{6}x+A^{4}x+A^{2}x\right)\\&=2^{14}A^{14}x+2^{12}A^{12}x+2^{10}A^{10}x+2^{8}A^{8}x+2^{6}A^{6}x+2^{4}A^{4}x+2^{2}A^{2}x.\end{aligned}$$

for all $x \in X$ and $r \in Q$. It follows that

$$A^{12}x+A^{10}x+A^{8}x+A^{6}x+A^{4}x+A^{2}x=0$$

for all $x \in X$. Hence $f(x) = A^{14}x$.

Conversely, assume that $f(x) = A^{14}x$ for all $x \in X$, where $A^{14}x$ is the diagonal of the 14-additive symmetric map $A_{14}: X^{14} \to Y$ from

$$\begin{aligned} A^{14}(x+y) = {} & A^{14}x + A^{14}y + 14A^{13.1}(x,y) + 91A^{12.2}(x,y) + 364A^{11.3}(x,y) \\ & + 1001A^{10.4}(x,y) + 2002A^{9.5}(x,y) + 3003A^{8.6}(x,y) + 3432A^{7.7}(x,y) \\ & + 3003A^{6.8}(x,y) + 2002A^{5.9}(x,y) + 1001A^{4.10}(x,y) + 364A^{3.11}(x,y) \\ & + 91A^{2.12}(x,y) + 14A^{1.13}(x,y), \end{aligned} \tag{26}$$

and

$$A^{14}(rx) = r^{14}\ A^{14}x,\ A^{13}(x,ry) = r \cdot A^{13.1}(x,y),\ A^{12.2}(x,ry) = r^2 A^{12.2}(x,y),$$

$$A^{11.3}(x,ry) = r^3 A^{11.3}(x,y),\ A^{10.4}(x,ry) = r^4 A^{10.4}(x,y),$$

$$A^{9.5}(x,ry) = r^5 A^{9.5}(x,y),\ A^{8.6}(x,ry) = r^6 A^{8.6}(x,y),$$

$$A^{7.7}(x,ry) = r^7 A^{7.7}(x,y),\ A^{6.8}(x,ry) = r^8 A^{6.8}(x,y),$$

$$A^{5.9}(x,ry) = r^9 A^{5.9}(x,y),\ A^{4.10}(x,ry) = r^{10} A^{4.10}(x,y),$$

$$A^{3.11}(x,ry) = r^{11} A^{3.11}(x,y),\ A^{2.12}(x,ry) = r^{12} A^{2.12}(x,y),$$

$$A^{1.13}(x,ry) = r^{13} A^{1.13}(x,y),$$

for all $x, y \in X$ and $r \in Q$. We see that f satisfies the Equation (1). This completes the proof of the Theorem.

4. Stability of Quattuordecic Functional Equation

Throughout this section, we assume that X is a linear space, Y is a (β, p) Banach space with (β, p)-norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$. We establish the following stability for the Quarttuordecic functional equation in quasi β-normed spaces. For a given mapping $f: X \to Y$, we define the difference operator

$$\begin{aligned} Df(x,y) = {} & f(x+7y) - 14f(x+6y) + 91f(x+5y) - 364f(x+4y) \\ & + 1001f(x+3y) - 2002f(x+2y) + 3003f(x+y) - 3432f(x) \\ & + 3003f(x-y) - 2002f(x-2y) + 1001f(x-3y) - 364f(x-4y) \\ & + 91f(x-5y) - 14f(x-6y) + f(x-7y) - 87178291200f(y). \end{aligned} \tag{27}$$

Theorem 2. *Let* $i \in \{-1, 1\}$ *be fixed and* $\varphi: X \times X \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with* $\varphi(2^i x, 2^i y) \le 16384^{i\beta} L\varphi(x,y)$ *for all* $x, y \in X$. *Let* $f: X \to Y$ *be a mapping satisfying*

$$\|Df(x,y)\|_Y \le \varphi(x,y), \tag{28}$$

for all $x, y \in X$. Then there exists a unique Quattuordecic mapping $Q: X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{1}{16384^{\beta}\|1 - L^j\|}\varphi_Q(x,y), \tag{29}$$

for all $x, y \in X$, where

$$\begin{aligned}
\varphi_Q(x,y) = {} & K\varphi(7x,x) + \frac{K^3}{2^\beta}\varphi(0,2x) + \frac{K^4}{50803200}\varphi(0,0) + \frac{K^{11}}{58060800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) \\
& + \frac{K^{12}}{87091200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big) + \frac{K^{13}}{174182400}\big(\varphi(6x,6x)+\varphi(6x,-6x)\big) \\
& + \frac{K^{14}}{479001600}\big(\varphi(8x,8x)+\varphi(8x,-8x)\big) + \frac{K^{15}}{1916006400}\big(\varphi(10x,10x)+\varphi(10x,-10x)\big) \\
& + \frac{K^{16}}{12454041600}\big(\varphi(12x,12x)+\varphi(12x,-12x)\big) + \frac{14^\beta K^8}{6227020800}\varphi(0,0) \\
& + \frac{K^{17}}{174356582400}\big(\varphi(14x,14x)+\varphi(14x,-14x)\big) + 14^\beta K^8\varphi(6x,x) \\
& + \frac{14^\beta K^8}{87178291200}\big(\varphi(x,x)+\varphi(x,-x)\big) + 91^\beta K^7\varphi(5x,x) + \frac{91^\beta K^8\varphi(0,0)}{958003200} \\
& + \frac{91^\beta K^9}{6227020800}\big(\varphi(x,x)+\varphi(x,-x)\big) + \frac{91^\beta K^{10}}{87178291200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) \\
& + 364^\beta K^6\varphi(4x,x) + \frac{364^\beta K^7\varphi(0,0)}{239500800} + \frac{364^\beta K^8}{958003200}\big(\varphi(x,x)+\varphi(x,-x)\big) \\
& + \frac{364^\beta K^9}{6227020800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) + \frac{364^\beta K^{10}}{87178291200}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big) \\
& + 1001^\beta K^5\varphi(3x,x) + \frac{1001^\beta K^6\varphi(0,0)}{87091200} + \frac{1001^\beta K^7}{239500800}\big(\varphi(x,x)+\varphi(x,-x)\big) \\
& + \frac{1001^\beta K^8}{958003200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) + \frac{1001^\beta K^9}{6227020800}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big) \\
& + \frac{1001^\beta K^{10}}{87178291200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big) + 2002^\beta K^4\varphi(2x,x) + \frac{2002^\beta K^5\varphi(0,0)}{43545600} \\
& + \frac{2002^\beta K^6}{87091200}\big(\varphi(x,x)+\varphi(x,-x)\big) + \frac{2002^\beta K^7}{239500800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) \\
& + \frac{2002^\beta K^8}{958003200}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big) + \frac{2002^\beta K^9}{6227020800}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big) \\
& + \frac{2002^\beta K^{10}}{87178291200}\big(\varphi(5x,5x)+\varphi(5x,-5x)\big) + 3003^\beta K^3\varphi(x,x) + \frac{3003^\beta K^4\varphi(0,0)}{29030400} \\
& + \frac{3003^\beta K^5}{43545600}\big(\varphi(x,x)+\varphi(x,-x)\big) + \frac{3003^\beta K^6}{87091200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) \\
& + \frac{3003^\beta K^7}{239500800}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big) + \frac{3003^\beta K^8}{958003200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big) \\
& + \frac{3003^\beta K^9}{6227020800}\big(\varphi(5x,5x)+\varphi(5x,-5x)\big) + \frac{3003^\beta K^{10}}{87178291200}\big(\varphi(6x,6x)+\varphi(6x,-6x)\big) \\
& + 1716^\beta K^2\varphi(0,x) + \frac{1716^\beta K^3\varphi(0,0)}{25401600} + \frac{1716^\beta K^4}{29030400}\big(\varphi(x,x)+\varphi(x,-x)\big) \\
& + \frac{1716^\beta K^5}{43545600}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big) + \frac{1716^\beta K^6}{87091200}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big) \\
& + \frac{1716^\beta K^7}{239500800}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big) + \frac{1716^\beta K^8}{958003200}\big(\varphi(5x,5x)+\varphi(5x,-5x)\big) \\
& + \frac{1716^\beta K^9}{6227020800}\big(\varphi(6x,6x)+\varphi(6x,-6x)\big) + \frac{1716^\beta K^{10}}{87178291200}\big(\varphi(7x,7x)+\varphi(7x,-7x)\big).
\end{aligned} \tag{30}$$

Proof. Replacing $x = 0, y = 0$ in (28), we get

$$\|f(0)\|_Y \le \frac{1}{87178291200}\varphi(0,0). \tag{31}$$

Replacing (x, y) by (x, x) in (28), we arrive that

$$\begin{aligned}&\| f(8x) - 14f(7x) + 91f(6x) - 364f(5x) + 1001f(4x) - 2002f(3x)\\ &+3003f(2x) - 3432f(x) + 3003f(0) - 2002f(-x) + 1001f(-2x)\\ &-364f(-3x) + 91f(-4x) - 14f(-5x) + f(-6x) - 87178291200f(x)\|_Y\\ &\le \varphi(x,x).\end{aligned} \tag{32}$$

Replacing (x, y) by $(x, -x)$ in (28), we have

$$\begin{aligned}&\| f(-6x) - 14f(-5x) + 91f(-4x) - 364f(-3x) + 1001f(-2x) - 2002f(-x)\\ &+3003f(0) - 3432f(x) + 3003f(2x) - 2002f(3x) + 1001f(4x) - 364f(5x)\\ &+91f(6x) - 14f(7x) + f(8x) - 87178291200f(-x)\|_Y \le \varphi(x,-x).\end{aligned} \tag{33}$$

From Equations (32) and (33), we obtain

$$\|f(x) - f(-x)\|_Y \le \frac{K}{87178291200}\left(\varphi(x,x) + \varphi(x,-x)\right). \tag{34}$$

Replacing (x, y) with $(0, 2x)$ in (28), we arrive that

$$\begin{aligned}&\| f(14x) - 14f(12x) + 91f(10x) - 364f(8x) + 1001f(6x) - 2002f(4x)\\ &+3003f(2x) - 3432f(0) + 3003f(-2x) - 2002f(-4x) + 1001f(-6x)\\ &-364f(-8x) + 91f(-10x) - 14f(-12x) + f(-14x) - 87178291200f(2x)\|_Y\\ &\le \varphi(0,2x),\end{aligned} \tag{35}$$

for all $x \in X$. By (31), (34) and (35), we have

$$\begin{aligned}&\| f(14x) - 14f(12x) + 91f(10x) - 364f(8x) + 1001f(16x) - 2002f(4x) - 87178285194\|_Y\\ &\le \frac{K}{2^\beta}\varphi(0,2x) + \frac{K^2}{50803200}\varphi(0,0) + \frac{K^3}{58060800}\left(\varphi(2x,2x) + \varphi(2x,-2x)\right)\\ &+\frac{K^4}{87091200}\left(\varphi(4x,4x) + \varphi(4x,-4x)\right) + \frac{K^5}{174182400}\left(\varphi(6x,6x) + \varphi(6x,-6x)\right)\\ &+\frac{K^6}{479001600}\left(\varphi(8x,8x) + \varphi(8x,-8x)\right) + \frac{K^7}{1916006400}\left(\varphi(10x,10x) + \varphi(10x,-10x)\right)\\ &+\frac{K^8}{12454041600}\left(\varphi(12x,12x) + \varphi(12x,-12x)\right)\\ &+\frac{K^9}{174356582400}\left(\varphi(14x,14x) + \varphi(14x,-14x)\right).\end{aligned} \tag{36}$$

Replacing (x, y) with $(7x, x)$ in (28), we have

$$\begin{aligned}&\| f(14x) - 14f(13x) + 91f(12x) - 364f(11x) + 1001f(10x)\\ &-2002f(9x) + 3003f(8x) - 3432f(7x) + 3003f(6x) - 2002f(5x)\\ &+1001f(4x) - 364f(3x) + 91f(2x) + 87178291214f(x)\|_Y \le K\varphi(7x,x).\end{aligned} \tag{37}$$

From (36) and (37), we arrive that

$$
\begin{aligned}
&\| 14f(13x) - 105f(12x) + 364f(11x) - 910f(10x) + 2002f(9x) - 3367f(8x) \\
&+ 3432f(7x) - 2002f(6x) + 2002f(5x) - 3003f(4x) + 364f(3x) \\
&- 43589142688f(2x) + 87178291214f(x) \|_Y \\
&\le K\varphi(7x,x) + \frac{K^2}{2^\beta}\varphi(0,2x) + \frac{K^3}{50803200}\varphi(0,0) \\
&\quad + \frac{K^4}{58060800}\left(\varphi(2x,2x) + \varphi(2x,-2x)\right) \\
&\quad + \frac{K^5}{87091200}\left(\varphi(4x,4x) + \varphi(4x,-4x)\right) \\
&\quad + \frac{K^6}{174182400}\left(\varphi(6x,6x) + \varphi(6x,-6x)\right) \\
&\quad + \frac{K^7}{479001600}\left(\varphi(8x,8x) + \varphi(8x,-8x)\right) \\
&\quad + \frac{K^8}{1916006400}\left(\varphi(10x,10x) + \varphi(10x,-10x)\right) \\
&\quad + \frac{K^9}{12454041600}\left(\varphi(12x,12x) + \varphi(12x,-12x)\right) \\
&\quad + \frac{K^{10}}{174356582400}\left(\varphi(14x,14x) + \varphi(14x,-14x)\right).
\end{aligned}
\tag{38}
$$

Replacing (x,y) with $(6x,x)$ in (28), one finds that

$$
\begin{aligned}
&\| 14f(13x) - 196f(12x) + 1274f(11x) - 5096f(10x) + 14014f((9x) \\
&- 28028f(8x) + 42042f(7x) - 48048f(6x) + 42042f(5x) - 28028f(4x) \\
&+ 14014f(3x) - 5096f(2x) - 1220496075512f(x) \|_Y \\
&\le 14^\beta K\varphi(6x,x) + \frac{14^\beta K^2}{6227020800}\varphi(0,0) + \frac{14^\beta K^3}{87178291200}\left(\varphi(x,x) + \varphi(x,-x)\right).
\end{aligned}
\tag{39}
$$

Utilizing (38) and (39), we find that

$$
\begin{aligned}
&\| 91f(12x) + 910f(11x) + 4186f(10x) - 12012f(19x) + 24661f(8x) \\
&- 38610f(7x) + 46046f(6x) - 40040f(5x) + 25025f(4x) - 13650f(3x) \\
&- 43589137592f(2x) + 1307674366726f(x) \|_Y \\
&\le K^2\varphi(7x,x) + \frac{K^3}{2^\beta}\varphi(0,2x) + \frac{K^4}{50803200}\varphi(0,0) + \frac{K^5}{58060800}\left(\varphi(2x,2x) + \varphi(2x,-2x)\right) \\
&\quad + \frac{K^6}{87091200}\left(\varphi(4x,4x) + \varphi(4x,-4x)\right) + \frac{K^7}{174182400}\left(\varphi(6x,6x) + \varphi(6x,-6x)\right) \\
&\quad + \frac{K^8}{479001600}\left(\varphi(8x,8x) + \varphi(8x,-8x)\right) + \frac{K^9}{1916006400}\left(\varphi(10x,10x) + \varphi(10x,-10x)\right) \\
&\quad + \frac{K^{10}}{12454041600}\left(\varphi(12x,12x) + \varphi(12x,-12x)\right) + 14^\beta K^2\varphi(6x,x) + \frac{14^\beta K^2}{6227020800}\varphi(0,0) \\
&\quad + \frac{K^{11}}{174356582400}\left(\varphi(14x,14x) + \varphi(14x,-14x)\right) + \frac{14^\beta K^2}{87178291200}\left(\varphi(x,x) + \varphi(x,-x)\right).
\end{aligned}
\tag{40}
$$

Replacing (x,y) with $(5x,x)$ in (28), we obtain

$$\begin{aligned}
&\| 91 f(12x) - 1274 f(11x) + 8281 f(10x) - 33124 f(9x) + 91091 f(8x) \\
&- 182182 f(7x) + 273273 f(6x) + 312312 f(5x) + 273273 f(4x) \\
&- 182182 f(3x) + 91182 f(2x) - 7933224533598 f(x) \|_Y \\
&\le 91^{\beta} K \varphi(5x, x) + \frac{91^{\beta} K^2 \varphi(0,0)}{958003200} + \frac{91^{\beta} K^3}{6227020800} \left(\varphi(x,x) + \varphi(x,-x) \right) \\
&\quad + \frac{91^{\beta} K^4}{87178291200} \left(\varphi(2x,2x) + \varphi(2x,-2x) \right).
\end{aligned} \tag{41}$$

From (40) and (41), we arrive at

$$\begin{aligned}
&\| 364 f(11x) - 4095 f(10x) + 21112 f(9x) - 66430 f(8x) + 143572 f(7x) \\
&- 227227 f(6x) + 272272 f(5x) - 248248 f(4x) + 168532 f(3x) \\
&- 43589228774 f(2x) + 9240898900324 f(x) \|_Y \\
&\le K^3 \varphi(7x, x) + \frac{K^3}{2^{\beta}} \varphi(0, 2x) + \frac{K^4}{50803200} \varphi(0,0) \\
&\quad + \frac{K^6}{58060800} \left(\varphi(2x,2x) + \varphi(2x,-2x) \right) \\
&\quad + \frac{K^7}{87091200} \left(\varphi(4x,4x) + \varphi(4x,-4x) \right) + \frac{K^8}{174182400} \left(\varphi(6x,6x) + \varphi(6x,-6x) \right) \\
&\quad + \frac{K^9}{479001600} \left(\varphi(8x,8x) + \varphi(8x,-8x) \right) \\
&\quad + \frac{K^{10}}{1916006400} \left(\varphi(10x,10x) + \varphi(10x,-10x) \right) \\
&\quad + \frac{K^{11}}{12454041600} \left(\varphi(12x,12x) + \varphi(12x,-12x) \right) + 14^{\beta} K^3 \varphi(6x, x) \\
&\quad + \frac{K^{12}}{174356582400} \left(\varphi(14x,14x) + \varphi(14x,-14x) \right) + \frac{14^{\beta} K^3}{6227020800} \varphi(0,0) \\
&\quad + \frac{14^{\beta} K^3}{87178291200} \left(\varphi(x,x) + \varphi(x,-x) \right) + 91^{\beta} K^2 \varphi(5x, x) + \frac{91^{\beta} K^3 \varphi(0,0)}{958003200} \\
&\quad + \frac{91^{\beta} K^4}{6227020800} \left(\varphi(x,x) + \varphi(x,-x) \right) + \frac{91^{\beta} K^5}{87178291200} \left(\varphi(2x,2x) + \varphi(2x,-2x) \right).
\end{aligned} \tag{42}$$

Replacing (x, y) with $(4x, x)$ in (28), we obtain

$$\begin{aligned}
&\| 364 f(11x) - 5096 f(10x) + 33124 f(9x) - 132496 f(8x) + 364364 f(7x) \\
&- 728728 f(6x) + 1093092 f(5x) - 1249248 f(4x) + 1093456 f(3x) \\
&- 733824 f(2x) - 317328975599312 f(x) \|_Y \\
&\le 364^{\beta} K \varphi(4x, x) + \frac{364^{\beta} K^2 \varphi(0,0)}{239500800} + \frac{364^{\beta} K^3}{958003200} \left(\varphi(x,x) + \varphi(x,-x) \right) \\
&\quad + \frac{364^{\beta} K^4}{6227020800} \left(\varphi(2x,2x) + \varphi(2x,-2x) \right) \\
&\quad + \frac{364^{\beta} K^5}{87178291200} \left(\varphi(3x,3x) + \varphi(3x,-3x) \right).
\end{aligned} \tag{43}$$

Using Equations (42) and (43), we get

$$\begin{aligned}&\big\|1001f(10x)-12012f(9x)+66066f(8x)-220792f(7x)+501501f(6x)\\&-820820f(5x)+1001000f(4x)-924924f(3x)+43588494950f(2x)\\&-40973796499636f(x)\big\|_Y\\&\le K\varphi(7x,x)+\frac{K^3}{2^\beta}\varphi(0,2x)+\frac{K^4}{50803200}\varphi(0,0)\\&\quad+\frac{K^7}{58060800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\&\quad+\frac{K^8}{87091200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big)+\frac{K^9}{174182400}\big(\varphi(6x,6x)+\varphi(6x,-6x)\big)\\&\quad+\frac{K^{10}}{479001600}\big(\varphi(8x,8x)+\varphi(8x,-8x)\big)\\&\quad+\frac{K^{11}}{1916006400}\big(\varphi(10x,10x)+\varphi(10x,-10x)\big)\\&\quad+\frac{K^{12}}{12454041600}\big(\varphi(12x,12x)+\varphi(12x,-12x)\big)+14^\beta K^4\varphi(6x,x)\\&\quad+\frac{K^{13}}{174356582400}\big(\varphi(14x,14x)+\varphi(14x,-14x)\big)+\frac{14^\beta K^4}{6227020800}\varphi(0,0)\\&\quad+\frac{14^\beta K^4}{87178291200}\big(\varphi(x,x)+\varphi(x,-x)\big)+91^\beta K^3\varphi(5x,x)+\frac{91^\beta K^4\varphi(0,0)}{958003200}\\&\quad+\frac{91^\beta K^5}{6227020800}\big(\varphi(x,x)+\varphi(x,-x)\big)+\frac{91^\beta K^6}{87178291200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\&\quad+364^\beta K^2\varphi(4x,x)+\frac{364^\beta K^3\varphi(0,0)}{239500800}+\frac{364^\beta K^4}{958003200}\big(\varphi(x,x)+\varphi(x,-x)\big)\\&\quad+\frac{364^\beta K^5}{6227020800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\&\quad+\frac{364^\beta K^6}{87178291200}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big).\end{aligned}\tag{44}$$

Replacing (x,y) with $(3x,x)$ in (28), one finds that

$$\begin{aligned}&\big\|1001f(10x)-14014f(9x)+91091f(8x)-364364f(7x)\\&+1002001f(6x)-2004002f(5x)+3007004f(4x)-3449446f(3x)\\&+3097094f(2x)-87265471859566f(x)\big\|_Y\\&\le 1001^\beta K\varphi(3x,x)+\frac{1001^\beta K^2\varphi(0,0)}{87091200}+\frac{1001^\beta K^3}{239500800}\big(\varphi(x,x)+\varphi(x,-x)\big)\\&\quad+\frac{1001^\beta K^4}{958003200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\&\quad+\frac{1001^\beta K^5}{6227020800}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big)\\&\quad+\frac{1001^\beta K^6}{87178291200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big).\end{aligned}\tag{45}$$

From (44) and (45), we arrive at

$$
\begin{aligned}
&\| 2002 f(9x) - 25025 f(8x) + 143572 f(7x) - 500500 f(6x) \\
&+ 1183182 f(5x) - 2006004 f(4x) + 2524522 f(3x) \\
&- 43591592044 f(2x) + 128239268359202 f(x) \|_Y \\
&\le K^5 \varphi(7x,x) + \frac{K^3}{2^\beta}\varphi(0,2x) + \frac{K^4}{50803200}\varphi(0,0) \\
&\quad + \frac{K^8}{58060800}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right) \\
&\quad + \frac{K^9}{87091200}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right) + \frac{K^{10}}{174182400}\left(\varphi(6x,6x)+\varphi(6x,-6x)\right) \\
&\quad + \frac{K^{11}}{479001600}\left(\varphi(8x,8x)+\varphi(8x,-8x)\right) \\
&\quad + \frac{K^{12}}{1916006400}\left(\varphi(10x,10x)+\varphi(10x,-10x)\right) \\
&\quad + \frac{K^{13}}{12454041600}\left(\varphi(12x,12x)+\varphi(12x,-12x)\right) + 14^\beta K^5 \varphi(6x,x) \\
&\quad + \frac{K^{14}}{174356582400}\left(\varphi(14x,14x)+\varphi(14x,-14x)\right) + \frac{14^\beta K^5}{6227020800}\varphi(0,0) \\
&\quad + \frac{14^\beta K^5}{87178291200}\left(\varphi(x,x)+\varphi(x,-x)\right) + 91^\beta K^4 \varphi(5x,x) + \frac{91^\beta K^5 \varphi(0,0)}{958003200} \\
&\quad + \frac{91^\beta K^6}{6227020800}\left(\varphi(x,x)+\varphi(x,-x)\right) + \frac{91^\beta K^7}{87178291200}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right) \\
&\quad + 364^\beta K^3 \varphi(4x,x) + \frac{364^\beta K^4 \varphi(0,0)}{239500800} + \frac{364^\beta K^5}{958003200}\left(\varphi(x,x)+\varphi(x,-x)\right) \\
&\quad + \frac{364^\beta K^6}{6227020800}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right) + \frac{364^\beta K^7}{87178291200}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right) \\
&\quad + 1001^\beta K^2 \varphi(3x,x) + \frac{1001^\beta K^3 \varphi(0,0)}{87091200} + \frac{1001^\beta K^4}{239500800}\left(\varphi(x,x)+\varphi(x,-x)\right) \\
&\quad + \frac{1001^\beta K^5}{958003200}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right) + \frac{1001^\beta K^6}{6227020800}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right) \\
&\quad + \frac{1001^\beta K^7}{87178291200}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right).
\end{aligned} \tag{46}
$$

Replacing (x,y) with $(2x,x)$ in (28), we obtain

$$
\begin{aligned}
&\| 2002 f(9x) - 28028 f(8x) + 182182 f(7x) - 728728 f(6x) + 2006004 f(5x) \\
&- 4036032 f(4x) + 6194188 f(3x) - 7599592 f(2x) - 174530930966392 f(x) \|_Y \\
&\le 2002^\beta K^1 \varphi(2x,x) + \frac{2002^\beta K^2 \varphi(0,0)}{43545600} + \frac{2002^\beta K^3}{87091200}\left(\varphi(x,x)+\varphi(x,-x)\right) \\
&\quad + \frac{2002^\beta K^4}{239500800}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right) + \frac{2002^\beta K^5}{958003200}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right) \\
&\quad + \frac{2002^\beta K^6}{6227020800}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right) \\
&\quad + \frac{2002^\beta K^7}{87178291200}\left(\varphi(5x,5x)+\varphi(5x,-5x)\right).
\end{aligned} \tag{47}
$$

Using Equations (46) and (47), one gets that

$$
\begin{aligned}
&\big\|3003f(8x)-38610f(7x)+228228f(6x)-822822f(5x)+2030028f(4x)\\
&-3669666f(3x)-43583992452f(2x)+302770199325594f(x)\big\|_Y\\
&\le K^6\varphi(7x,x)+\frac{K^3}{2^\beta}\varphi(0,2x)+\frac{K^4}{50803200}\varphi(0,0)+\frac{K^9}{58060800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\
&\quad+\frac{K^{10}}{87091200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big)+\frac{K^{11}}{174182400}\big(\varphi(6x,6x)+\varphi(6x,-6x)\big)\\
&\quad+\frac{K^{12}}{479001600}\big(\varphi(8x,8x)+\varphi(8x,-8x)\big)+\frac{K^{13}}{1916006400}\big(\varphi(10x,10x)+\varphi(10x,-10x)\big)\\
&\quad+\frac{K^{14}}{12454041600}\big(\varphi(12x,12x)+\varphi(12x,-12x)\big)+14^\beta K^6\varphi(6x,x)\\
&\quad+\frac{K^{15}}{174356582400}\big(\varphi(14x,14x)+\varphi(14x,-14x)\big)+\frac{14^\beta K^6}{6227020800}\varphi(0,0)\\
&\quad+\frac{14^\beta K^6}{87178291200}\big(\varphi(x,x)+\varphi(x,-x)\big)+91^\beta K^5\varphi(5x,x)+\frac{91^\beta K^6\varphi(0,0)}{958003200}\\
&\quad+\frac{91^\beta K^7}{6227020800}\big(\varphi(x,x)+\varphi(x,-x)\big)+\frac{91^\beta K^8}{87178291200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\
&\quad+364^\beta K^4\varphi(4x,x)+\frac{364^\beta K^5\varphi(0,0)}{239500800}+\frac{364^\beta K^6}{958003200}\big(\varphi(x,x)+\varphi(x,-x)\big)\\
&\quad+\frac{364^\beta K^7}{6227020800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)+\frac{364^\beta K^8}{87178291200}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big)\\
&\quad+1001^\beta K^3\varphi(3x,x)+\frac{1001^\beta K^4\varphi(0,0)}{87091200}+\frac{1001^\beta K^5}{239500800}\big(\varphi(x,x)+\varphi(x,-x)\big)\\
&\quad+\frac{1001^\beta K^6}{958003200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)+\frac{1001^\beta K^7}{6227020800}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big)\\
&\quad+\frac{1001^\beta K^8}{87178291200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big)+2002^\beta K^2\varphi(2x,x)+\frac{2002^\beta K^3\varphi(0,0)}{43545600}\\
&\quad+\frac{2002^\beta K^4}{87091200}\big(\varphi(x,x)+\varphi(x,-x)\big)+\frac{2002^\beta K^5}{239500800}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)\\
&\quad+\frac{2002^\beta K^6}{958003200}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big)+\frac{2002^\beta K^7}{6227020800}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big)\\
&\quad+\frac{2002^\beta K^8}{87178291200}\big(\varphi(5x,5x)+\varphi(5x,-5x)\big).
\end{aligned}
\tag{48}
$$

Replacing (x,y) with (x,x) in (28), we have

$$
\begin{aligned}
&\big\|3003f(8x)-42042f(7x)+276276f(6x)-1135134f(5x)\\
&+3279276f(4x)-7105098f(3x)+12024012f(2x)-261796424791902f(x)\big\|_Y\\
&\le 3003^\beta K^1\varphi(x,x)+\frac{3003^\beta K^2\varphi(0,0)}{29030400}+\frac{3003^\beta K^3}{43545600}\big(\varphi(x,x)+\varphi(x,-x)\big)\\
&\quad+\frac{3003^\beta K^4}{87091200}\big(\varphi(2x,2x)+\varphi(2x,-2x)\big)+\frac{3003^\beta K^5}{239500800}\big(\varphi(3x,3x)+\varphi(3x,-3x)\big)\\
&\quad+\frac{3003^\beta K^6}{958003200}\big(\varphi(4x,4x)+\varphi(4x,-4x)\big)+\frac{3003^\beta K^7}{6227020800}\big(\varphi(5x,5x)+\varphi(5x,-5x)\big)\\
&\quad+\frac{3003^\beta K^8}{87178291200}\big(\varphi(6x,6x)+\varphi(6x,-6x)\big).
\end{aligned}
\tag{49}
$$

Using Equation (48) and (49), we obtain

$$\begin{aligned}
&\| 3432 f(7x) - 48048 f(6x) + 312312 f(5x) - 1249248 f(4x) \\
&+ 3435432 f(3x)) - 43596016464 f(2x) + 564566624117496 f(x) \|_Y \\
&\le K^7 \varphi(7x, x) + \frac{K^3}{2^\beta} \varphi(0, 2x) + \frac{K^4}{50803200} \varphi(0,0) + \frac{K^{10}}{58060800} (\varphi(2x, 2x) + \varphi(2x, -2x)) \\
&+ \frac{K^{11}}{87091200} (\varphi(4x, 4x) + \varphi(4x, -4x)) + \frac{K^{12}}{174182400} (\varphi(6x, 6x) + \varphi(6x, -6x)) \\
&+ \frac{K^{13}}{479001600} (\varphi(8x, 8x) + \varphi(8x, -8x)) + \frac{K^{14}}{1916006400} (\varphi(10x, 10x) + \varphi(10x, -10x)) \\
&+ \frac{K^{15}}{12454041600} (\varphi(12x, 12x) + \varphi(12x, -12x)) + 14^\beta K^7 \varphi(6x, x) \\
&+ \frac{K^{16}}{174356582400} (\varphi(14x, 14x) + \varphi(14x, -14x)) + \frac{14^\beta K^7}{6227020800} \varphi(0,0) \\
&+ \frac{14^\beta K^7}{87178291200} (\varphi(x, x) + \varphi(x, -x)) + 91^\beta K^6 \varphi(5x, x) + \frac{91^\beta K^7 \varphi(0,0)}{958003200} \\
&+ \frac{91^\beta K^8}{6227020800} (\varphi(x, x) + \varphi(x, -x)) + \frac{91^\beta K^9}{87178291200} (\varphi(2x, 2x) + \varphi(2x, -2x)) \\
&+ 364^\beta K^5 \varphi(4x, x) + \frac{364^\beta K^6 \varphi(0,0)}{239500800} + \frac{364^\beta K^7}{958003200} (\varphi(x, x) + \varphi(x, -x)) \qquad (50) \\
&+ \frac{364^\beta K^8}{6227020800} (\varphi(2x, 2x) + \varphi(2x, -2x)) + \frac{364^\beta K^9}{87178291200} (\varphi(3x, 3x) + \varphi(3x, -3x)) \\
&+ 1001^\beta K^4 \varphi(3x, x) + \frac{1001^\beta K^5 \varphi(0,0)}{87091200} + \frac{1001^\beta K^6}{239500800} (\varphi(x, x) + \varphi(x, -x)) \\
&+ \frac{1001^\beta K^7}{958003200} (\varphi(2x, 2x) + \varphi(2x, -2x)) + \frac{1001^\beta K^8}{6227020800} (\varphi(3x, 3x) + \varphi(3x, -3x)) \\
&+ \frac{1001^\beta K^9}{87178291200} (\varphi(4x, 4x) + \varphi(4x, -4x)) + 2002^\beta K^3 \varphi(2x, x) + \frac{2002^\beta K^4 \varphi(0,0)}{43545600} \\
&+ \frac{2002^\beta K^5}{87091200} (\varphi(x, x) + \varphi(x, -x)) + \frac{2002^\beta K^6}{239500800} (\varphi(2x, 2x) + \varphi(2x, -2x)) \\
&+ \frac{2002^\beta K^7}{958003200} (\varphi(3x, 3x) + \varphi(3x, -3x)) + \frac{2002^\beta K^8}{6227020800} (\varphi(4x, 4x) + \varphi(4x, -4x)) \\
&+ \frac{2002^\beta K^9}{87178291200} (\varphi(5x, 5x) + \varphi(5x, -5x)) + 3003^\beta K^2 \varphi(x, x) + \frac{3003^\beta K^3 \varphi(0,0)}{29030400} \\
&+ \frac{3003^\beta K^4}{43545600} (\varphi(x, x) + \varphi(x, -x)) + \frac{3003^\beta K^5}{87091200} (\varphi(2x, 2x) + \varphi(2x, -2x)) \\
&+ \frac{3003^\beta K^6}{239500800} (\varphi(3x, 3x) + \varphi(3x, -3x)) + \frac{3003^\beta K^7}{958003200} (\varphi(4x, 4x) + \varphi(4x, -4x)) \\
&+ \frac{3003^\beta K^8}{6227020800} (\varphi(5x, 5x) + \varphi(5x, -5x)) + \frac{3003^\beta K^9}{87178291200} (\varphi(6x, 6x) + \varphi(6x, -6x)).
\end{aligned}$$

Replacing (x, y) with $(0, x)$ in (28), we obtain

$$
\begin{aligned}
&\left\|3432 f(7x)-48048 f(6x)+312312 f(5x)-1249248 f(4x)\right.\\
&\left.+3435432 f(3x)-6870864 f(2x)+14959793739204 f(x)\right\|_Y\\
&\le 1716^{\beta}K^{1}\varphi(0,x)+\frac{1716^{\beta}K^{2}\varphi(0,0)}{25401600}+\frac{1716^{\beta}K^{3}}{29030400}\left(\varphi(x,x)+\varphi(x,-x)\right)\\
&\quad+\frac{1716^{\beta}K^{4}}{43545600}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{1716^{\beta}K^{5}}{87091200}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right)\\
&\quad+\frac{1716^{\beta}K^{6}}{239500800}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right)+\frac{1716^{\beta}K^{7}}{958003200}\left(\varphi(5x,5x)+\varphi(5x,-5x)\right)\\
&\quad+\frac{1716^{\beta}K^{8}}{6227020800}\left(\varphi(6x,6x)+\varphi(6x,-6x)\right)+\frac{1716^{\beta}K^{9}}{87178291200}\left(\varphi(7x,7x)+\varphi(7x,-7x)\right).
\end{aligned}
\tag{51}
$$

From (50) and (51), we arrive at

$$
\begin{aligned}
&\left\|-43589145600 f(2x)+714164561510400 f(x)\right\|_Y = K^{8}\varphi(7x,x)+\frac{K^{3}}{2^{\beta}}\varphi(0,2x)+\frac{K^{4}}{50803200}\varphi(0,0)\\
&+\frac{K^{11}}{58060800}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{K^{12}}{87091200}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right)+\frac{K^{13}}{174182400}\left(\varphi(6x,6x)+\varphi(6x,-6x)\right)\\
&+\frac{K^{14}}{479001600}\left(\varphi(8x,8x)+\varphi(8x,-8x)\right)+\frac{K^{15}}{1916006400}\left(\varphi(10x,10x)+\varphi(10x,-10x)\right)\\
&+\frac{K^{16}}{12454041600}\left(\varphi(12x,12x)+\varphi(12x,-12x)\right)+\frac{14^{\beta}K^{8}}{6227020800}\varphi(0,0)+\frac{K^{17}}{174356582400}\left(\varphi(14x,14x)+\varphi(14x,-14x)\right)\\
&+14^{\beta}K^{8}\varphi(6x,x)+\frac{14^{\beta}K^{8}}{87178291200}\left(\varphi(x,x)+\varphi(x,-x)\right)+91^{\beta}K^{7}\varphi(5x,x)+\frac{91^{\beta}K^{8}\varphi(0,0)}{958003200}+\frac{91^{\beta}K^{9}}{6227020800}\left(\varphi(x,x)+\varphi(x,-x)\right)\\
&+\frac{91^{\beta}K^{10}}{87178291200}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+364^{\beta}K^{6}\varphi(4x,x)+\frac{364^{\beta}K^{7}\varphi(0,0)}{239500800}+\frac{364^{\beta}K^{8}}{958003200}\left(\varphi(x,x)+\varphi(x,-x)\right)\\
&+\frac{364^{\beta}K^{9}}{6227020800}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{364^{\beta}K^{10}}{87178291200}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right)+1001^{\beta}K^{5}\varphi(3x,x)+\frac{1001^{\beta}K^{6}\varphi(0,0)}{87091200}\\
&+\frac{1001^{\beta}K^{7}}{239500800}\left(\varphi(x,x)+\varphi(x,-x)\right)+\frac{1001^{\beta}K^{8}}{958003200}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{1001^{\beta}K^{9}}{6227020800}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right)\\
&+\frac{1001^{\beta}K^{10}}{87178291200}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right)+2002^{\beta}K^{4}\varphi(2x,x)+\frac{2002^{\beta}K^{5}\varphi(0,0)}{43545600}+\frac{2002^{\beta}K^{6}}{87091200}\left(\varphi(x,x)+\varphi(x,-x)\right)\\
&+\frac{2002^{\beta}K^{7}}{239500800}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{2002^{\beta}K^{8}}{958003200}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right)+\frac{2002^{\beta}K^{9}}{6227020800}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right)\\
&+\frac{2002^{\beta}K^{10}}{87178291200}\left(\varphi(5x,5x)+\varphi(5x,-5x)\right)+3003^{\beta}K^{3}\varphi(x,x)+\frac{3003^{\beta}K^{4}\varphi(0,0)}{29030400}+\frac{3003^{\beta}K^{5}}{43545600}\left(\varphi(x,x)+\varphi(x,-x)\right)\\
&+\frac{3003^{\beta}K^{6}}{87091200}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{3003^{\beta}K^{7}}{239500800}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right)+\frac{3003^{\beta}K^{8}}{958003200}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right)\\
&+\frac{3003^{\beta}K^{9}}{6227020800}\left(\varphi(5x,5x)+\varphi(5x,-5x)\right)+\frac{3003^{\beta}K^{10}}{87178291200}\left(\varphi(6x,6x)+\varphi(6x,-6x)\right)+1716^{\beta}K^{2}\varphi(0,x)+\frac{1716^{\beta}K^{3}\varphi(0,0)}{25401600}\\
&+\frac{1716^{\beta}K^{4}}{29030400}\left(\varphi(x,x)+\varphi(x,-x)\right)+\frac{1716^{\beta}K^{5}}{43545600}\left(\varphi(2x,2x)+\varphi(2x,-2x)\right)+\frac{1716^{\beta}K^{6}}{87091200}\left(\varphi(3x,3x)+\varphi(3x,-3x)\right)\\
&+\frac{1716^{\beta}K^{7}}{239500800}\left(\varphi(4x,4x)+\varphi(4x,-4x)\right)+\frac{1716^{\beta}K^{8}}{958003200}\left(\varphi(5x,5x)+\varphi(5x,-5x)\right)+\frac{1716^{\beta}K^{9}}{6227020800}\left(\varphi(6x,6x)+\varphi(6x,-6x)\right)\\
&+\frac{1716^{\beta}K^{10}}{87178291200}\left(\varphi(7x,7x)+\varphi(7x,-7x)\right).
\end{aligned}
\tag{52}
$$

Therefore,

$$\left\| f(2x) - 2^{14} f(x) \right\| \le \varphi_{14}(x)$$

for all $x \in X$. By Lemma 2.1, there exists a unique mapping $Q : X \to Y$ such that

$$Q(2x) = 2^{14} Q(x)$$

and

$$\left\| f(x) - Q(x) \right\|_y \le \frac{1}{16384^{\beta} \left| 1 - L^i \right|} \varphi_{14}(x) \tag{53}$$

for all $x \in X$. It remains to show that Q is a Quattuordecic mapping. From (28), we have

$$\begin{aligned} \left\| \frac{Df\left(2^{im}(x), 2^{im}(y)\right)}{16384^{im}} \right\|_y &\le 16384^{im\beta} \varphi\left(2^{im}(x), 2^{im}(y)\right) \\ &\le 16384^{im\beta} \left(16384^{i\beta} L\right)^m \varphi(x, y) \\ &= L^m \varphi(x, y) \end{aligned} \tag{54}$$

for all $x, y \in X$ and $m \in N$. Here $\left\| DQ(x, y) \right\|_Y = 0$, for all $x, y \in X$.

Therefore, the mapping $Q : X \to Y$ is a Quattuordecic mapping. The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of Quattuordecic functional Equation (1).

Corollary 1. *Let X be a quasi α-normed space with quasi α-norm $\left\| \cdot \right\|_X$, and let Y be a (β, p) Banach Space with (β, p)-norm $\left\| \cdot \right\|_Y$. Let δ, λ be a positive number with $\lambda \ne \frac{14\beta}{\alpha}$ and let $f : X \to Y$ be a mapping satisfying*

$$\left\| Df(x, y) \right\|_Y \le \delta\left(\left\| x \right\|_X^Y + \left\| y \right\|_X^Y \right)$$

for all $x, y \in X$. Then there exists a unique quattuordecic mapping $Q : X \to Y$ such that

$$\left\| f(x) - D(x) \right\|_Y = \begin{cases} \dfrac{\delta \epsilon_\lambda}{16384^{\beta} - 2^{\alpha\lambda}} \left\| x \right\|_X^{\lambda}, & \lambda \in \left(0, \dfrac{14\beta}{\alpha}\right) \\ \dfrac{2^{\lambda\alpha} \delta \epsilon_\lambda}{16384^{\beta}\left(2^{\alpha\lambda} - 16384^{\beta}\right)} \left\| x \right\|_X^{\lambda}, & \lambda \in \left(\dfrac{14\beta}{\alpha}, \infty\right) \end{cases} \tag{55}$$

where

$$\begin{aligned}\epsilon_\lambda &= \frac{K^3}{2^\beta} + K^2 1716^\beta + 2K^3 3003^\beta + 2002^\beta K^4\left(2^{\alpha\lambda}+1\right) + 1001^\beta K^5\left(3^{\alpha\lambda}+1\right)\\
&\quad + 364^\beta K^6\left(4^{\alpha\lambda}+1\right) + 91^\beta K^7\left(5^{\alpha\lambda}+1\right) + 14^\beta K^8\left(6^{\alpha\lambda}+1\right) + K^8\left(7^{\alpha\lambda}+1\right)\\
&\quad + 4\left(\frac{14^\beta K^8}{87178291200} + \frac{91^\beta K^9}{6227020800} + \frac{364^\beta K^8}{958003200} + \frac{1001^\beta K^7}{239500800} + \frac{2002^\beta K^6}{87091200}\right.\\
&\quad \left. + \frac{3003^\beta K^5}{43545600} + \frac{1716^\beta K^4}{29030400}\right) + 4\left(2^{\alpha\lambda}\right)\left(\frac{91^\beta K^{10}}{87178291200} + \frac{364^\beta K^9}{6227020800}\right.\\
&\quad \left. + \frac{1001^\beta K^8}{958003200} + \frac{2002^\beta K^7}{239500800} + \frac{3003^\beta K^6}{87091200} + \frac{1716^\beta K^5}{43545600} + \frac{K^{11}}{58060800}\right)\\
&\quad + 4\left(3^{\alpha\lambda}\right)\left(\frac{364^\beta K^{10}}{87178291200} + \frac{1001^\beta K^9}{6227020800} + \frac{2002^\beta K^8}{958003200} + \frac{3003^\beta K^7}{239500800} + \frac{1716^\beta K^6}{87091200}\right)\\
&\quad + 4\left(4^{\alpha\lambda}\right)\left(\frac{1001^\beta K^{10}}{87178291200} + \frac{2002^\beta K^9}{6227020800} + \frac{3003^\beta K^8}{958003200} + \frac{1716^\beta K^7}{239500800} + \frac{K^{12}}{87091200}\right)\\
&\quad + 4\left(5^{\alpha\lambda}\right)\left(\frac{2002^\beta K^{10}}{87178291200} + \frac{3003^\beta K^9}{6227020800} + \frac{1716^\beta K^8}{958003200}\right)\\
&\quad + 4\left(6^{\alpha\lambda}\right)\left(\frac{3003^\beta K^{10}}{87178291200} + \frac{1716^\beta K^9}{6227020800} + \frac{K^{13}}{174182400}\right) + \frac{4\left(7^{\alpha\lambda}\right)1716^\beta K^{10}}{87178291200}\\
&\quad + \frac{4\left(8^{\alpha\lambda}\right)K^{14}}{479001600} + \frac{4\left(10^{\alpha\lambda}\right)K^{15}}{1916006400} + \frac{4\left(12^{\alpha\lambda}\right)K^{16}}{12454041600} + \frac{4\left(14^{\alpha\lambda}\right)K^{17}}{174356582400}\end{aligned}$$

The following example shows that the assumption $\lambda \neq \frac{14\beta}{\alpha}$ cannot be omitted in Corollary 4.2. This example is a modification of well known example of Gajda [6] for the additive functional inequality.

Example 1. *Let* $\phi : R \to R$ *be defined by*

$$\phi(x) = \begin{cases} x^{14}, & \text{for } |x| < 1 \\ 1, & \text{for } |x| \geq 1. \end{cases} \tag{56}$$

consider the function $f : R \to R$ to be defined by

$$f(x) = \sum_{n=0}^{\infty} 4^{14n}\phi\left(4^n x\right) \qquad \forall x \in \mathbb{R}.$$

Then f satisfies the following functional inequality

$$\left|Df(x,y)\right| \leq \frac{(87178307584)\times(268435456)^3}{268435455}\left(|x|^{14} + |y|^{14}\right) \qquad \forall x, y \in \mathbb{R}. \tag{57}$$

Proof. It is easy to see that f is bounded by $\frac{268435456}{268435455}$ on $\mathbb{R}$. If $|x|^{14} + |y|^{14} = 0$ or $|x|^{14} + |y|^{14} \geq \frac{1}{268435456}$, then

$$
\begin{aligned}
\left|Df(x,y)\right| &\le \frac{(87178307584)\times(268435456)}{268435455} \\
&\le \frac{(87178307584)\times(268435456)^3}{268435455}\left(|x|^{14}+|y|^{14}\right)
\end{aligned}
$$

for all $x, y \in \mathbb{R}$. Now, suppose that $0 < |x|^{14} + |y|^{14} < \frac{1}{268435456}$. Then there exists a non-negative integer k such that

$$
\frac{1}{(268435456)^{k+2}} \le |x|^{14} + |y|^{14} < \frac{1}{(268435456)^{k+1}}. \tag{58}
$$

Hence

$$
(268435456)^K |x|^{14} < \frac{1}{268435456},\ (268435456)^K |y|^{14} < \frac{1}{268435456}
$$

and

$$
\begin{aligned}
&4^n(x+7y), 4^n(x+6y), 4^n(x+5y), 4^n(x+4y), 4^n(x+3y), 4^n(x+2y),\\
&4^n(x+y), 4^n(x-y), 4^n(x-2y), 4^n(x-3y), 4^n(x-4y), 4^n(x-5y),\\
&4^n(x-6y), 4^n(x-7y), 4^n(x), 4^n(y) \in (-1,1) \quad \text{for all } n = 0,1,\cdots,k-1.
\end{aligned}
$$

Hence $D\phi(4^n x, 4^n y) = 0$ for all $n = 0,1,\cdots,k-1$. From the definition of f and the inequality (58), we obtain that

$$
\begin{aligned}
\left|Df(x,y)\right| &\le \sum_{n=k}^{\infty} 4^{-14n}\cdot(87178307584) \\
&= \frac{(87178307584)\cdot 4^{14(1-k)}}{268435455} \\
&\le \frac{(87178307584)(268435456)^3}{268435455}\left(|x|^{14}+|y|^{14}\right).
\end{aligned}
$$

Therefore, f satisfies (57) for all $x, y \in \mathbb{R}$. Now, we claim that functional Equation (1) is not stable for $\lambda = 14$ in above Corollary (4.2) $(\alpha = \beta = p = 1)$.

Suppose on the contrary that there exists a Quattuordecic mapping $Q:\mathbb{R} \to \mathbb{R}$ and constant $d > 0$ such that $\left|f(x) - Q(x)\right| \le d|x|^{14}\ \forall x \in \mathbb{R}$. Then there exists a constant $c \in \mathbb{R}$ such that $Q(x) = cx^{14}$ for all rational numbers x (see (25)). So we obtain the following inequality

$$
\left|f(x)\right| \le (d+|c|)|x|^{14}, \quad \text{forall } x \in \mathbb{Q}. \tag{59}
$$

Let $m \in \mathbb{N}$ with $m+1 > d+|c|$. If x is a rational number in $(0, 4^{-m})$, then $4^n x \in (0,1)$ for all $n = 0,1,2,\cdots,m$ and in this case we get

$$
f(x) = \sum_{n=0}^{\infty}\frac{\varphi(4^n x)}{4^{14n}} \ge \sum_{n=0}^{\infty}\frac{(4^n x)^{14}}{4^{14n}} = (m+1)x^{14} > (d+|c|)|x|^{14}
$$

which contradicts the inequality (59).

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