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The Global Attractors and Their Hausdorff and Fractal Dimensions Estimation for the Higher-Order Nonlinear Kirchhoff-Type Equation with Strong Linear Damping

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Abstract

In this paper, we study the longtime behavior of solution to the initial boundary value problem for a class of strongly damped Higher-order Kirchhoff type equations: $u_{tt}+(-\Delta)^m u_t+\left\|D^m u\right\|^{2q}(-\Delta)^m u+g(u)=f(x)$. At first, we prove the existence and uniqueness of the solution by priori estimation and the Galerkin method. Then, we obtain to the existence of the global attractor. At last, we consider that the estimation of the upper bounds of Hausdorff and fractal dimensions for the global attractors are obtained.

Keywords

Nonlinear Higher-Order Kirchhoff Type Equation, The Existence and Uniqueness, The Global Attractors, Hausdorff Dimensions, Fractal Dimensions

1. Introduction

In this paper, we are concerned with the existence of global attractor and Hausdorff and Fractal dimensions estimation for the following nonlinear Higher-order Kirchhoff-type equations:

$$u_{tt}+(-\Delta)^m u_t+\left\|D^m u\right\|^{2q}(-\Delta)^m u+g(u)=f(x),(x,t)\in\Omega\times[0,+\infty), \tag{1.1}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, \tag{1.2}$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,\cdots,m-1,x\in\partial\Omega,t\in(0,+\infty), \tag{1.3}$$

where $m>1$ is an integer constant, and $q>0$ is a positive constant. Moreover, Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and v is the unit outward normal on $\partial\Omega$. $g(u)$ is a nonlinear function specified later.

Recently, Marina Ghisi and Massimo Gobbino [1] studied spectral gap global solutions for degenerate Kirchhoff equations. Given a continuous function $m:[0,+\infty)\to[0,+\infty)$, they consider the Cauchy problem:

$$u_{tt}(t,x)+m\left(\int_\Omega|\nabla u(t,x)|^2\,\mathrm{d}x\right)\Delta u(t,x)=0,\forall(t,x)\in\Omega\times[0,T), \tag{1.4}$$

$$u(0)=u_0,u_t(0)=u_1, \tag{1.5}$$

where $\Omega\subseteq R^n$ is an open set and ∇u and Δu denote the gradient and the Laplacian of u with respect to the space variables. They prove that for such initial data (u_0,u_1) there exist two pairs of initial data $(\overline{u}_0,\overline{u}_1),(\hat{u}_0,\hat{u}_1)$ for which the solution is global, and such that $u_0=\overline{u}_0+\hat{u}_0,u_1=\overline{u}_1+\hat{u}_1$.

Yang Zhijian, Ding Pengyan and Lei Li [2] studied Longtime dynamics of the Kirchhoff equations with fractional damping and supercritical nonlinearity:

$$u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+(-\Delta)^\alpha u_t+f(u)=g(x),x\in\Omega,t>0, \tag{1.6}$$

$$u|_{\partial\Omega}=0,u(x,0)=u_0(x),u_t(x,0)=u_1(x), \tag{1.7}$$

where $\alpha\in\left(\frac{1}{2},1\right)$, Ω is a bounded domain R_N with the smooth boundary $\partial\Omega$, and the nonlinearity $f(u)$ and external force term g will be specified. The main results are focused on the relationships among the growth exponent p of the nonlinearity $f(u)$ and well-posedness. They show that (i) even if p is up to the supercritical range, that is, $1\le p<\frac{N+4\alpha}{(N-4\alpha)^+}$, the well-posedness and the longtime behavior of the solutions of the equation are of the characters of the parabolic equation; (ii) when $\frac{N+4\alpha}{(N-4\alpha)^+}\le p<\frac{N+4}{(N-4)^+}$, the corresponding subclass G of the limit solutions exists and possesses a weak global attractor.

Yang Zhijian, Ding Pengyan and Liu Zhiming [3] studied the Global attractor for the Kirchhoff type equations with strong nonlinear damping and supercritical nonlinearity:

$$u_{tt}-\sigma\left(\|\Delta u\|^2\right)\Delta u_t-\phi\left(\|\Delta u\|^2\right)\Delta u+f(u)=h(x)\quad\text{in }\Omega\times\mathbb{R}^+, \tag{1.8}$$

$$u(x,t)|_{\partial\Omega}=0,u(x,0)=u_0(x),u_t(x,0)=u_1(x),\quad x\in\Omega. \tag{1.9}$$

where Ω is a bounded domain in R^N with the smooth boundary $\partial\Omega$, $\sigma(s)$, $\phi(s)$ and $f(s)$ are nonlinear functions, and $h(x)$ is an external force term. They prove that in strictly positive stiffness factors and supercritical nonlinearity case, there exists a global finite-dimensional attractor in the natural energy space endowed with strong topology.

Li Fucai [4] studied the global existence and blow-up of solutions for a higher-order nonlinear Kirchhoff-type hyperbolic equation:

$$u_{tt}+\left(\int_\Omega \left|D^m u\right|^2 \mathrm{d}x\right)^q (-\Delta)^m u+u_t\left|u_t\right|^r=|u|^p u, x\in\Omega, t>0, \tag{1.10}$$

$$u(x,t)=0, \frac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,m-1, x\in\partial\Omega, t>0, \tag{1.11}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), \tag{1.12}$$

where $m\ge1, p,q,r\ge0$, Ω is a bounded domain R_n, with a smooth boundary $\partial\Omega$ and a unit outer normal *v*. Setting $E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2(q+1)}\|D^m u\|_2^{2(q+1)}-\frac{1}{p+2}\|u\|_{p+2}^{p+2}$. Assume that p satisfies the condition:

$$p\le\frac{2}{N-2m}, \text{ for } N>2m; \quad p>0, \text{ for } N\le 2m. \tag{1.13}$$

Their main results are the two theorems:

Theorem 1. Suppose that $p\le r$ and condition (1.13) holds. Then for any initial data $(u_0,u_1)\in H^{2m}(\Omega)\cap H_0^m(\Omega)\times H_0^m(\Omega)$, the solution of (1.10) - (1.12) exists globally.

Theorem 2. Suppose that $p>\max\{r,2q\}$ and condition (1.12) holds. Then for any initial data $(u_0,u_1)\in H^{2m}(\Omega)\cap H_0^m(\Omega)\times H_0^m(\Omega)$, the solution of (1.10) - (1.12) blows up at finite time in L_{p+2} norm provided that $E(0)<0$.

Li Yan [5] studied The Asymptotic Behavior of Solutions for a Nonlinear Higher Order Kirchhoff Type Equation:

$$u_{tt}+\left(\int_\Omega \left|D^m u\right|^2 \mathrm{d}x\right)^q (-\Delta)^m u+\beta u_t+g(u)=0, \quad \text{in } Q=\Omega\times(0,+\infty), \tag{1.14}$$

$$u(x,t)=0, \frac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,m-1, \quad \text{on } \Sigma=\Gamma\times(0,+\infty), \tag{1.15}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad \text{in } x\in\Omega, \tag{1.16}$$

where Ω is an open bounded set of $R^n(n\ge1)$ with smooth boundary Γ and the unit normal vector. The function $g\in C^1$ satisfies the following conditions:

$$\liminf_{|s|\to\infty}\frac{G(s)}{s^2}\ge0, G(s)=\int_0^s g(r)\mathrm{d}r; \tag{1.17}$$

$$\liminf_{|s|\to\infty}\frac{|g'(s)|}{|s|^\gamma}=0, \tag{1.18}$$

where $0\le\gamma<+\infty(n=1,2), 0\le\gamma<2(n=3), \gamma=0(n\ge4)$. Furthermore, there exists $C_1>0$ such that

$$\liminf_{|s|\to\infty}\frac{sg(s)-C_1G(s)}{s^2}\ge0. \tag{1.19}$$

At last, Li Yan studied the asymptotic behavior of solutions for problem (1.14) - (1.16).

For the most of the scholars represented by Yang Zhijian have studied all kinds of low order Kirchhoff equations and only a small number of scholars have studied the blow-up and asymptotic behavior of solutions for higher-order Kirchhoff equation. So, in this context, we study the high-order Kirchhoff equation is very meaningful. In order to study the high-order nonlinear Kirchhoff equation with the damping term, we borrow some of Li Yan's [5] partial assumptions (2.1) - (2.3) for the nonlinear term g in the equation. In order to prove that the lemma 1, we have improved the results from assumptions (2.1) - (2.3) such that $0 < C_2 \le 1$. Then, under all assumptions, we prove that the equation has a unique smooth solution $(u,u_t) \in L^\infty((0,+\infty); H^{2m}(\Omega)\times H_0^m(\Omega))$ and obtain the solution semigroup $S(t): H^{2m}(\Omega)\times H_0^m(\Omega) \to H^{2m}(\Omega)\times H_0^m(\Omega)$ has global attractor $\mathcal{A}$. Finally, we prove the equation has finite Hausdorff dimensions and Fractal dimensions by reference to the literature [7].

For more related results we refer the reader to [6] [7] [8] [9] [10]. In order to make these equations more normal, in section 2 and in section 3, some assumptions, notations and the main results are stated. Under these assumptions, we prove the existence and uniqueness of solution, then we obtain the global attractors for the problems (1.1) - (1.3). According to [6] [7] [8] [9] [10], in section 4, we consider that the global attractor of the above mentioned problems (1.1) - (1.3) has finite Hausdorff dimensions and fractal dimensions.

2. Preliminaries

For convenience, we denote the norm and scalar product in $L^2(\Omega)$ by $\|.\|$ and $(.,.)$; $f = f(x)$, $L^p = L^p(\Omega)$, $H^k = H^k(\Omega)$, $H_0^k = H_0^k(\Omega)$, $\|\cdot\| = \|\cdot\|_{L^2}$, $\|\cdot\|_p = \|\cdot\|_{L^p}$.

According to [5], we present some assumptions and notations needed in the proof of our results. For this reason, we assume nonlinear term $g(u) \in C^1(\Omega)$ satisfies that

(H_1) Setting $G(s) = \int_0^s g(r)\,\mathrm{d}r$, then

$$\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \ge 0; \tag{2.1}$$

(H_2) If

$$\limsup_{|s|\to\infty} \frac{|g'(s)|}{|s|^r} = 0, \tag{2.2}$$

where $0 \le r < +\infty (n = 1,2), 0 \le r < 2 (n = 3), r = 0 (n \ge 4)$.

(H_3) There exist constant $C_0 > 0$, such that

$$\liminf_{|s|\to\infty} \frac{sg(s) - C_0 G(s)}{s^2} \ge 0. \tag{2.3}$$

(H_4) There exist constant $C_1 > 0$, such that

$$|g(s)| \le C_1 (1 + |s|^p), \tag{2.4}$$

$$|g'(s)| \le C_1 (1 + |s|^{p-1}), \tag{2.5}$$

where $1 \le p \le \frac{n}{n-2m}$;

For every $\gamma > 0$, by (H$_1$)-(H$_3$) and apply Poincaré inequality, there exist constants $C(\gamma) > 0$, such that

$$J(u) + \gamma\left\|D^m u\right\|^2 + C(\gamma) \ge 0, \quad \forall u \in H^m(\Omega), \tag{2.6}$$

$$\left(g(u), u\right) - C_2 J(u) + \gamma\left\|D^m u\right\|^2 + C(\gamma) \ge 0, \quad \forall u \in H^m(\Omega), \tag{2.7}$$

where $J(u) = \int_\Omega G(u)\mathrm{d}x, 0 < C_2 \le 1$ is independent of γ.

Lemma 1. Assume (H$_1$)-(H$_3$) hold, and $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega), f(x) \in L^2(\Omega)$. Then the solution (u, v) of the problem (1.1) - (1.3) satisfies $(u, v) \in L^\infty\left((0, +\infty); H_0^m(\Omega) \times L^2(\Omega)\right)$, and

$$\left\|D^m u\right\|^2 + \|v\|^2 \le y(0)\mathrm{e}^{-C_2\varepsilon t} + \frac{\tilde{C}}{C_2\varepsilon} + \left(2^{q^2+2q+1} - 1\right)\frac{q}{q+1}. \tag{2.8}$$

where $v = u_t + \varepsilon u$, $0 < \varepsilon < \min\left\{\frac{\lambda_1^m}{1+2\lambda_1^m}, \frac{\sqrt{1+4\lambda_1^m} - 1}{2}, \frac{\sqrt{(2+C_2)^2 + 16\lambda_1^m} - 2 - C_2}{4}\right\}$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and

$y(0) = \left\|u_1 + \varepsilon u_0\right\|^2 - \varepsilon\left\|D^m u_0\right\|^2 + \frac{1}{q+1}\left\|D^m u_0\right\|^{2q+2} + \frac{q}{q+1} + 2J(u_0) + 2C(\gamma_2)$,

$\tilde{C} = \frac{1}{\varepsilon^2}\|f\|^2 + 2\varepsilon C(\gamma_1) + q\varepsilon + \frac{qw}{q+1} + 2\varepsilon C_2 C(\gamma_2)$, $\gamma_1 = \frac{1}{2} - \frac{\varepsilon}{2\lambda^m} - \varepsilon > 0$, $\gamma_2 = \frac{1-\varepsilon}{2} > 0$,

$w = \min\left\{2\lambda_1^m - 2\varepsilon^2 - 2\varepsilon, (q+1)\varepsilon\right\}$. Thus, there exists E_0 and $t_0 = t_0(\Omega) > 0$, such that

$$\left\|(u, v)\right\|^2_{H_0^m(\Omega)\times L^2(\Omega)} = \left\|D^m u\right\|^2 + \|v\|^2 \le E_0, \quad (t > t_0). \tag{2.9}$$

Proof. We take the scalar product in L^2 of equation (1.1) with $v = u_t + \varepsilon u$. Then

$$\left(u_{tt} + (-\Delta)^m u_t + \left\|D^m u\right\|^{2q}(-\Delta)^m u + g(u), v\right) = \left(f(x), v\right). \tag{2.10}$$

After a computation in (2.10), we have

$$\left(u_{tt}, v\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 - \varepsilon\|v\|^2 + \varepsilon^2(u, v), \tag{2.11}$$

$$\left((-\Delta)^m u_t, v\right) = -\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^2 + \left\|D^m v\right\|^2 - \varepsilon^2\left\|D^m u\right\|^2, \tag{2.12}$$

$$\left(\left\|D^m u\right\|^{2q}(-\Delta)^m u, v\right) = \frac{1}{2(q+1)}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q+2} + \varepsilon\left\|D^m u\right\|^{2q+2}, \tag{2.13}$$

$$\left(g(u), v\right) = \frac{\mathrm{d}}{\mathrm{d}t}J(u) + \varepsilon\left(g(u), u\right). \tag{2.14}$$

Collecting with (2.11) - (2.14), we obtain from (2.10) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 - \varepsilon\left\|D^m u\right\|^2 + \frac{1}{q+1}\left\|D^m u\right\|^{2q+2} + 2J(u)\right) - \varepsilon\|v\|^2 + \varepsilon^2(u, v) \\ &+ \left\|D^m v\right\|^2 - \varepsilon^2\left\|D^m u\right\|^2 + \varepsilon\left\|D^m u\right\|^{2q+2} + \varepsilon\left(g(u), u\right) = \left(f(x), v\right). \end{aligned} \tag{2.15}$$

Since $v = u_t + \varepsilon u$ and

$0 < \varepsilon < \min\left\{ \frac{\lambda_1^m}{1+2\lambda_1^m}, \frac{\sqrt{1+4\lambda_1^m}-1}{2}, \frac{\sqrt{(2+C_2)^2+16\lambda_1^m}-2-C_2}{4} \right\}$, by using Hölder inequality Young's inequality and Poincaré inequality, we deal with the terms in (2.15) one by one as follow:

$$\varepsilon^2(u,v) \ge -\frac{\varepsilon^2}{2}\|u\|^2 - \frac{\varepsilon^2}{2}\|v\|^2 \ge -\frac{\varepsilon^2}{2\lambda_1^m}\left\|D^m u\right\|^2 - \frac{\varepsilon^2}{2}\|v\|^2, \tag{2.16}$$

$$\left\|D^m v\right\|^2 \ge \lambda_1^m \|v\|^2. \tag{2.17}$$

By (2.7), we can obtain

$$\varepsilon\big(g(u),u\big) \ge C_2\varepsilon J(u) - \left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{2\lambda_1^m} - \varepsilon^2\right)\left\|D^m u\right\|^2 - \varepsilon C(\gamma_1), \tag{2.18}$$

where $\gamma_1 = \frac{1}{2} - \frac{\varepsilon}{2\lambda_1^m} - \varepsilon > 0$.

Because of $f(x) \in L^2(\Omega)$, we can obtain

$$\big(f(x),v\big) \le \|f\|\|v\| \le \frac{\|f\|^2}{2\varepsilon^2} + \frac{\varepsilon^2}{2}\|v\|^2. \tag{2.19}$$

By (2.16) - (2.19), it follows from that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 - \varepsilon\left\|D^m u\right\|^2 + \frac{1}{q+1}\left\|D^m u\right\|^{2q+2} + 2J(u)\right) + \left(2\lambda_1^m - 2\varepsilon^2 - 2\varepsilon\right)\|v\|^2 \\ &-\varepsilon\left\|D^m u\right\|^2 + 2\varepsilon\left\|D^m u\right\|^{2q+2} + 2C_2\varepsilon J(u) \le \frac{1}{\varepsilon^2}\|f\|^2 + 2\varepsilon C(\gamma_1). \end{aligned} \tag{2.20}$$

By Young's inequality and $0 < \varepsilon < \frac{\lambda_1^m}{1+2\lambda_1^m} < 1$, we have

$$\frac{1}{q+1}\left\|D^m u\right\|^{2q+2} - \varepsilon\left\|D^m u\right\|^2 + \frac{q}{q+1} \ge (1-\varepsilon)\left\|D^m u\right\|^2 \ge 0, \tag{2.21}$$

$$\varepsilon\left\|D^m u\right\|^{2q+2} - \varepsilon\left\|D^m u\right\|^2 + q\varepsilon \ge 0. \tag{2.22}$$

By (2.22), we get

$$\begin{aligned} &\left(2\lambda_1^m - 2\varepsilon^2 - 2\varepsilon\right)\|v\|^2 - \varepsilon\left\|D^m u\right\|^2 + 2\varepsilon\left\|D^m u\right\|^{2q+2} + 2C_2\varepsilon J(u) + q\varepsilon \\ &= \left(2\lambda_1^m - 2\varepsilon^2 - 2\varepsilon\right)\|v\|^2 + \varepsilon(q+1)\left(\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}\right) \\ &\quad + \left(\varepsilon\left\|D^m u\right\|^{2q+2} - \varepsilon\left\|D^m u\right\|^2 + q\varepsilon\right) + 2C_2\varepsilon J(u) \\ &\ge w\left(\|v\|^2 + \frac{1}{q+1}\left\|D^m u\right\|^{2q+2}\right) + 2C_2\varepsilon J(u) \\ &\ge w\left(\|v\|^2 - \varepsilon\left\|D^m u\right\|^2 + \frac{1}{q+1}\left\|D^m u\right\|^{2q+2}\right) + 2C_2\varepsilon J(u), \end{aligned} \tag{2.23}$$

where $w = \min\left\{2\lambda_1^m - 2\varepsilon^2 - 2\varepsilon, (q+1)\varepsilon\right\}$.

By (2.21) and substituting (2.23) into (2.20), we receive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2-\varepsilon\left\|D^m u\right\|^2+\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}+\frac{q}{q+1}+2J(u)\right)\\&+w\left(\|v\|^2-\varepsilon\left\|D^m u\right\|^2+\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}+\frac{q}{q+1}\right)+2C_2\varepsilon J(u)\\&\le\frac{1}{\varepsilon^2}\|f\|^2+2\varepsilon C(\gamma_1)+q\varepsilon+\frac{qw}{q+1}.\end{aligned}\tag{2.24}$$

Since $0<\varepsilon<\frac{\sqrt{(2+C_2)^2+16\lambda_1^m}-2-C_2}{4}$ and $0<C_2<1$, we get

$$w=\min\left\{2\lambda_1^m-2\varepsilon^2-2\varepsilon,(q+1)\varepsilon\right\}\ge C_2\varepsilon.\tag{2.25}$$

By (2.6) and (2.21), we have

$$\begin{aligned}&-\varepsilon\left\|D^m u\right\|^2+\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}+\frac{q}{q+1}+2J(u)+2C(\gamma_2)\\&\ge(1-\varepsilon)\left\|D^m u\right\|^2+2J(u)+2C(\gamma_2)\ge 0,\end{aligned}\tag{2.26}$$

where $\gamma_2=\frac{1-\varepsilon}{2}>0$.

Combining with (2.25) and (2.26), formula (2.24) into

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2-\varepsilon\left\|D^m u\right\|^2+\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}+\frac{q}{q+1}+2J(u)+2C(\gamma_2)\right)\\&+C_2\varepsilon\left(\|v\|^2-\varepsilon\left\|D^m u\right\|^2+\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}+\frac{q}{q+1}+2J(u)+2C(\gamma_2)\right)\\&\le\frac{1}{\varepsilon^2}\|f\|^2+2\varepsilon C(\gamma_1)+q\varepsilon+\frac{qw}{q+1}+2\varepsilon C_2C(\gamma_2).\end{aligned}\tag{2.27}$$

We set $y(t)=\|v\|^2-\varepsilon\left\|D^m u\right\|^2+\frac{1}{q+1}\left\|D^m u\right\|^{2q+2}+\frac{q}{q+1}+2J(u)+2C(\gamma_2)$. Then, (2.27) is simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+C_2\varepsilon y(t)\le\tilde{C},\tag{2.28}$$

where $\tilde{C}=\frac{1}{\varepsilon^2}\|f\|^2+2\varepsilon C(\gamma_1)+q\varepsilon+\frac{qw}{q+1}+2\varepsilon C_2C(\gamma_2)$.

From conclusion (2.26), we know $y(t)\ge 0$. So, by Gronwall's inequality, we obtain

$$y(t)\le y(0)\mathrm{e}^{-C_2\varepsilon t}+\frac{\tilde{C}}{C_2\varepsilon},\tag{2.29}$$

where $y(0)=\left\|u_1+\varepsilon u_0\right\|^2-\varepsilon\left\|D^m u_0\right\|^2+\frac{1}{q+1}\left\|D^m u_0\right\|^{2q+2}+\frac{q}{q+1}+2J(u_0)+2C(\gamma_2)$.

By generalized Young's inequality, we have $\left\|D^m u\right\|^2\le\frac{1}{2^{q+1}(q+1)}\left\|D^m u\right\|^{2q+2}+2^{\frac{q+1}{q}}\frac{q}{q+1}$.

Then, we get

$$\frac{1}{(q+1)}\left\|D^m u\right\|^{2q+2}\ge 2^{q+1}\left\|D^m u\right\|^2-2^{\frac{q^2+2q+1}{q}}\frac{q}{q+1}.\tag{2.30}$$

By (2.26) and (2.30), we have

$$\begin{aligned} y(t) &= \|v\|^2 - \varepsilon\|D^m u\|^2 + \frac{1}{q+1}\|D^m u\|^{2q+2} + 2J(u) + \frac{q}{q+1} + 2C(\gamma_2) \\ &\geq \|v\|^2 + (1-\varepsilon)\|D^m u\|^2 + \left(2^{q+1}-1\right)\|D^m u\|^2 \\ &\quad - \frac{2^{q^2+2q+1}q}{q+1} + \frac{q}{q+1} + 2J(u) + 2C(\gamma_2) \\ &\geq \|v\|^2 + \left(2^{q+1}-1\right)\|D^m u\|^2 + \left(1-2^{q^2+2q+1}\right)\frac{q}{q+1} \\ &\geq \min\left\{1, 2^{q+1}-1\right\}\left(\|v\|^2 + \|D^m u\|^2\right) + \left(1-2^{q^2+2q+1}\right)\frac{q}{q+1} \\ &= \left(\|v\|^2 + \|D^m u\|^2\right) + \left(1-2^{q^2+2q+1}\right)\frac{q}{q+1}. \end{aligned} \tag{2.31}$$

Combining with (2.29) and (2.31),we obtain

$$\|D^m u\|^2 + \|v\|^2 \leq y(0)\mathrm{e}^{-C_2\varepsilon t} + \frac{\tilde{C}}{C_2\varepsilon} + \left(2^{q^2+2q+1}-1\right)\frac{q}{q+1}, \tag{2.32}$$

Then,

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H_0^m(\Omega)\times L^2(\Omega)} = \|D^m u\|^2 + \|v\|^2 \leq \frac{\tilde{C}}{C_2\varepsilon} + \left(2^{q^2+2q+1}-1\right)\frac{q}{q+1}. \tag{2.33}$$

So, there exist E_0 and $t_0 = t_0(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H_0^m(\Omega)\times L^2(\Omega)} = \|D^m u\|^2 + \|v\|^2 \leq E_0, \quad (t > t_0). \tag{2.34}$$

Lemma 2. In addition to the assumptions of Lemma 1, (H_1) - (H_4) hold. If (H_5): $f(x) \in H_0^m(\Omega)$, and $(u_0, u_1) \in H^{2m}(\Omega) \times H_0^m(\Omega)$. Then the solution (u,v) of the problems (1.1) - (1.3) satisfies $(u,v) \in L^\infty\left((0,+\infty); H^{2m}(\Omega)\times H_0^m(\Omega)\right)$, and

$$\|D^{2m}u\|^2 + \|D^m v\|^2 \leq \frac{z(0)}{T}\mathrm{e}^{-\alpha_1 t} + \frac{\frac{1}{\varepsilon^2}\|D^m f\|^2 + C_3}{\alpha_1 T}. \tag{2.35}$$

where $(-\Delta)^m v = (-\Delta)^m u_t + \varepsilon(-\Delta)^m u$, λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $z(0) = \|D^m u_1 + \varepsilon D^m u_0\|^2 + \left(\|D^m u_0\|^{2q} - \varepsilon\right)\|D^{2m}u_0\|^2$, $\alpha_1 = \min\left\{\lambda_1^m - 2\varepsilon - 2\varepsilon^2, M\right\}$, $T = \min\left\{1, \inf_{t\geq 0}\|D^m u\|^{2q} - \varepsilon\right\}$. Thus, there exists E_1 and $t_1 = t_1(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H^{2m}(\Omega)\times H_0^m(\Omega)} = \|D^{2m}u\|^2 + \|D^m v\|^2 \leq E_1, \quad (t > t_1). \tag{2.36}$$

Proof. Taking L^2-inner product by $(-\Delta)^m v = (-\Delta)^m u_t + \varepsilon(-\Delta)^m u$ in (1.1), we have

$$\left(u_{tt} + (-\Delta)^m u_t + \|D^m u\|^{2q}(-\Delta)^m u + g(u), (-\Delta)^m v\right) = \left(f(x), (-\Delta)^m v\right). \tag{2.37}$$

After a computation in (2.37) one by one, as follow

$$\begin{aligned} \left(u_{tt}, (-\Delta)^m v\right) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^m v\|^2 - \varepsilon\|D^m v\|^2 + \varepsilon^2\left(D^m u, D^m v\right) \\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^m v\|^2 - \varepsilon\|D^m v\|^2 - \frac{\varepsilon^2}{2\lambda_1^m}\|D^{2m}u\|^2 - \frac{\varepsilon^2}{2}\|D^m v\|^2, \end{aligned} \tag{2.38}$$

$$\left((-\Delta)^m u_t,(-\Delta)^m v\right)=\left\|D^{2m}v\right\|^2-\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2-\varepsilon^2\left\|D^{2m}u\right\|^2, \tag{2.39}$$

$$\begin{aligned}&\left(\left\|D^m u\right\|^{2q}(-\Delta)^m u,(-\Delta)^m v\right)\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^m u\right\|^{2q}\left\|D^{2m}u\right\|^2\right)-\frac{\left\|D^{2m}u\right\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q}+\varepsilon\left\|D^m u\right\|^{2q}\left\|D^{2m}u\right\|^2.\end{aligned} \tag{2.40}$$

By Young's inequality, we get

$$\left(g(u),(-\Delta)^m v\right)\ge-\left\|g(u)\right\|\left\|D^{2m}v\right\|\ge-\frac{\left\|g(u)\right\|^2}{2}-\frac{\left\|D^{2m}v\right\|^2}{2}. \tag{2.41}$$

Next to estimate $\left\|g(u)\right\|^2$ in (2.41). By (H$_4$): $\left|g(s)\right|\le C_1\left(1+|s|^p\right)$ and Young's inequality, we have

$$\begin{aligned}\left\|g(u)\right\|^2&\le\int_\Omega C_1^2\left(1+|u|^p\right)^2\mathrm{d}x\\&\le\int_\Omega\left(C_1^2+2C_1^2|u|^p+C_1^2|u|^{2p}\right)\mathrm{d}x\\&\le\int_\Omega\left(2C_1^2+2C_1^2|u|^{2p}\right)\mathrm{d}x\\&\le 2C_1^2|\Omega|+2C_1^2\left\|u\right\|_{L^{2p}(\Omega)}^{2p}.\end{aligned} \tag{2.42}$$

By $1\le p\le\dfrac{n}{n-2m}$ and Embeding Theorem, then $H_0^m(\Omega)\to L^{2p}(\Omega)$. So there exists $K>0$, such that $\left\|u\right\|_{L^{2p}(\Omega)}\le K\left\|D^m u\right\|$. $\left\|D^m u\right\|$ bounded by lemma 1. Then, (2.42) turns into

$$\left\|g(u)\right\|^2\le C_3\left(p,C_1,K,|\Omega|\right). \tag{2.43}$$

Collecting with (2.43), from (2.41) we have

$$\left(g(u),(-\Delta)^m v\right)\ge-\frac{C_3}{2}-\frac{\left\|D^{2m}v\right\|^2}{2}. \tag{2.44}$$

By $f(x)\in H_0^m(\Omega)$ and Young's inequality, we obtain

$$\left(f(x),(-\Delta)^m v\right)=\left(D^m f(x),D^m v\right)\le\frac{1}{2\varepsilon^2}\left\|D^m f\right\|^2+\frac{\varepsilon^2}{2}\left\|D^m v\right\|^2. \tag{2.45}$$

Integrating (2.38) - (2.40), (2.44) - (2.45), from (2.37) entails

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|D^m v\right\|^2+\left(\left\|D^m u\right\|^{2q}-\varepsilon\right)\left\|D^{2m}u\right\|^2\right]+\left\|D^{2m}v\right\|^2-\left(2\varepsilon+2\varepsilon^2\right)\left\|D^m v\right\|^2\\&+\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q}+2\varepsilon\left\|D^m u\right\|^{2q}-2\varepsilon^2-\frac{\varepsilon^2}{\lambda_1^m}\right)\left\|D^{2m}u\right\|^2\le\frac{1}{\varepsilon^2}\left\|D^m f\right\|^2+C_3\end{aligned} \tag{2.46}$$

By Poincaré inequality, such that $\lambda_1^m\left\|D^m v\right\|^2\le\left\|D^{2m}v\right\|^2$. So, (2.46) turns into

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|D^m v\right\|^2+\left(\left\|D^m u\right\|^{2q}-\varepsilon\right)\left\|D^{2m}u\right\|^2\right]+\left(\lambda_1^m-2\varepsilon-2\varepsilon^2\right)\left\|D^m v\right\|^2\\&+\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q}+2\varepsilon\left\|D^m u\right\|^{2q}-2\varepsilon^2-\frac{\varepsilon^2}{\lambda_1^m}\right)\left\|D^{2m}u\right\|^2\le\frac{1}{\varepsilon^2}\left\|D^m f\right\|^2+C_3\end{aligned} \tag{2.47}$$

First, we take proper ε, such that $\lambda_1^m - 2\varepsilon - 2\varepsilon^2 > 0$ and $\left\|D^m u\right\|^{2q} - \varepsilon > 0$ by Lamma 1. Then, we assume that there exists $M > 0$, such that $M - 2\varepsilon > 0$ and $0 < M\left(\left\|D^m u\right\|^{2q} - \varepsilon\right) \le -\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q} + 2\varepsilon\left\|D^m u\right\|^{2q} - 2\varepsilon^2 - \frac{\varepsilon^2}{\lambda_1^m}$. Then, formula is simplified to

$$\left(M - 2\varepsilon\right)\left\|D^m u\right\|^{2q} + \frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q} \le M\varepsilon - 2\varepsilon^2 - \frac{\varepsilon^2}{\lambda_1^m}. \tag{2.48}$$

By Gronwall's inequality, we get

$$\varepsilon < \left\|D^m u\right\|^{2q} \le \left\|D^m u_0\right\|^{2q} \mathrm{e}^{(2\varepsilon - M)t} + \frac{M\varepsilon - 2\varepsilon^2 - \frac{\varepsilon^2}{\lambda_1^m}}{M - 2\varepsilon}. \tag{2.49}$$

On account of Lemma 1, we know $\left\|D^m u\right\|^{2q}$ is bounded. So the hypothesis is true. Namely, we prove that there are $M > 0$, makes

$$0 < M\left(\left\|D^m u\right\|^{2q} - \varepsilon\right) \le -\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^{2q} + 2\varepsilon\left\|D^m u\right\|^{2q} - 2\varepsilon^2 - \frac{\varepsilon^2}{\lambda_1^m}. \tag{2.50}$$

Substituting (2.50) into (2.47), we receive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|D^m v\right\|^2 + \left(\left\|D^m u\right\|^{2q} - \varepsilon\right)\left\|D^{2m} u\right\|^2\right] + \left(\lambda_1^m - 2\varepsilon - 2\varepsilon^2\right)\left\|D^m v\right\|^2 \\ &+ M\left(\left\|D^m u\right\|^{2q} - \varepsilon\right)\left\|D^{2m} u\right\|^2 \le \frac{1}{\varepsilon^2}\left\|D^m f\right\|^2 + C_3.\end{aligned} \tag{2.51}$$

Taking $\alpha_1 = \min\left\{\lambda_1^m - 2\varepsilon - 2\varepsilon^2, M\right\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) + \alpha_1 z(t) \le \frac{1}{\varepsilon^2}\left\|D^m f\right\|^2 + C_3, \tag{2.52}$$

where $z(t) = \left\|D^m v\right\|^2 + \left(\left\|D^m u\right\|^{2q} - \varepsilon\right)\left\|D^{2m} u\right\|^2$. By Gronwall's inequality, we have

$$z(t) \le z(0)\mathrm{e}^{-\alpha_1 t} + \frac{\frac{1}{\varepsilon^2}\left\|D^m f\right\|^2 + C_3}{\alpha_1}, \tag{2.53}$$

where $z(0) = \left\|D^m u_1 + \varepsilon D^m u_0\right\|^2 + \left(\left\|D^m u_0\right\|^{2q} - \varepsilon\right)\left\|D^{2m} u_0\right\|^2$.

Let $T = \min\left\{1, \inf_{t \ge 0}\left\|D^m u\right\|^{2q} - \varepsilon\right\}$, so we get

$$\left\|D^m v\right\|^2 + \left\|D^{2m} u\right\|^2 \le \frac{z(0)}{T}\mathrm{e}^{-\alpha_1 t} + \frac{\frac{1}{\varepsilon^2}\left\|D^m f\right\|^2 + C_3}{\alpha_1 T}, \tag{2.54}$$

Then

$$\overline{\lim_{t \to \infty}}\left\|(u, v)\right\|^2_{H^{2m}(\Omega) \times H_0^m(\Omega)} = \left\|D^{2m} u\right\|^2 + \left\|D^m v\right\|^2 \le \frac{\frac{1}{\varepsilon^2}\left\|D^m f\right\|^2 + C_3}{\alpha_1 T}. \tag{2.55}$$

So, there exists E_1 and $t_1 = t_1(\Omega) > 0$, such that

$$\left\|(u, v)\right\|^2_{H^{2m}(\Omega) \times H_0^m(\Omega)} = \left\|D^{2m} u\right\|^2 + \left\|D^m v\right\|^2 \le E_1, \quad \left(t > t_1\right). \tag{2.56}$$

3. Global Attractor

3.1. The Existence and Uniqueness of Solution

Theorem 3.1. Assume (H_1) - (H_4) hold, and $(u_0,u_1)\in H^{2m}(\Omega)\times H_0^m(\Omega)$, $f(x)\in H_0^m(\Omega)$, $v=u_t+\varepsilon u$. So Equation (1.1) exists a unique smooth solution

$$(u(x,t),v(x,t))\in L^\infty((0,+\infty);H^{2m}(\Omega)\times H_0^m(\Omega)). \tag{3.1}$$

Proof. By the Galerkin method, **Lemma 1** and **Lemma 2**, we can easily obtain the existence of Solutions. Next, we prove the uniqueness of Solutions in detail.

Assume u,v are two solutions of the problems (1.1) - (1.3), let $w=u-v$, then $w(x,0)=w_0(x)=0, w_t(x,0)=w_1(x)=0$ and the two equations subtract and obtain

$$w_{tt}+(-\Delta)^m w_t+\|D^m u\|^{2q}(-\Delta)^m u-\|D^m v\|^{2q}(-\Delta)^m v+g(u)-g(v)=0. \tag{3.2}$$

By multiplying (3.2) by w_t, we get

$$\left(w_{tt}+(-\Delta)^m w_t+\|D^m u\|^{2q}(-\Delta)^m u-\|D^m v\|^{2q}(-\Delta)^m v+g(u)-g(v),w_t\right)=0, \tag{3.3}$$

$$(w_{tt},w_t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2, \tag{3.4}$$

$$\left((-\Delta)^m w_t,w_t\right)=\|D^m w_t\|^2, \tag{3.5}$$

$$\begin{aligned}
&\left(\|D^m u\|^{2q}(-\Delta)^m u-\|D^m v\|^{2q}(-\Delta)^m v,w_t\right)\\
&=\left(\|D^m u\|^{2q}(-\Delta)^m w,w_t\right)+\left(\|D^m u\|^{2q}-\|D^m v\|^{2q}\right)\left((-\Delta)^m v,w_t\right)\\
&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^m u\|^{2q}\|D^m w\|^2-q\|D^m u\|^{2q-1}\|D^m u_t\|\|D^m w\|^2\\
&\quad+\left(\|D^m u\|^{2q}-\|D^m v\|^{2q}\right)\left((-\Delta)^m v,w_t\right).
\end{aligned} \tag{3.6}$$

Exploiting (3.4) - (3.6), we receive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\|D^m u\|^{2q}\|D^m w\|^2\right)+2\|D^m w_t\|^2\\
&=2q\|D^m u\|^{2q-1}\|D^m u_t\|\|D^m w\|^2-2\left(\|D^m u\|^{2q}-\|D^m v\|^{2q}\right)\left((-\Delta)^m v,w_t\right)\\
&\quad-2\left(g(u)-g(v),w_t\right).
\end{aligned} \tag{3.7}$$

In (3.7), according to Lemma 1 and Lemma 2, such that

$$\begin{aligned}
&2q\|D^m u\|^{2q-1}\|D^m w\|^2-2\left(\|D^m u\|^{2q}-\|D^m v\|^{2q}\right)\left((-\Delta)^m v,w_t\right)\\
&\le 2q\|D^m u\|^{2q-1}\|D^m w\|^2+4q\left\|\|D^m u\|+\theta\left(\|D^m v\|-\|D^m u\|\right)\right\|^{2q-1}\|D^m w\|\|\Delta^m v\|\|w_t\|\\
&\le C_4(q)\|D^m w\|^2+C_5(q,\theta)\|D^m w\|\|w_t\|\\
&\le\left(C_4(q)+\frac{C_5(q,\theta)}{2}\right)\|D^m w\|^2+\frac{C_5(q,\theta)}{2}\|w_t\|^2,
\end{aligned} \tag{3.8}$$

where $0<\theta<1, C_4(q)>0$ and $C_5(q,\theta)>0$ are constants.

By (H_4), we obtain

$$\begin{aligned}
&-2\left(g(u)-g(v),w_t\right)\\
&=-2\left(g'\left(\theta u+(1-\theta)v\right)w,w_t\right)\\
&\le 2\left\|g'\left(\theta u+(1-\theta)v\right)w\right\|\left\|w_t\right\|\\
&\le 2\left(\int_\Omega C_1^2(1+\left|\theta u+(1-\theta)v\right|^{p-1}dx\right)^{\frac{1}{2}}\|w\|\|w_t\|\\
&\le 2C_1\left(1+\left\|\left(\left|\theta u+(1-\theta)v\right|^{p-1}\right)\right\|\right)\|w\|\|w_t\|\\
&\le 2C_1C_6\left(\theta,p,\lambda_1,m\right)\left\|D^m w\right\|\left\|w_t\right\|\\
&\le C_1C_6\left(\theta,p,\lambda_1,m\right)\left\|w_t\right\|^2+C_1C_6\left(\theta,p,\lambda_1,m\right)\left\|D^m w\right\|^2,
\end{aligned}\tag{3.9}$$

where $C_6=C_6\left(\theta,p,\lambda_1,m\right)>0$ is constant.

From the above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2\right)\le\left(C_4(q)+\frac{C_5(\theta,q)}{2}+C_1C_6\right)\left(\|w_t\|^2+\left\|D^m w\right\|^2\right).\tag{3.10}$$

For (3.10), because $\left\|D^m u\right\|^{2q}$ is bounded. Then, there exists $\varepsilon>0$, such that $\left\|D^m u\right\|^{2q}\ge\varepsilon$. So, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2\right)\\
&\le\left(C_4(q)+\frac{C_5(\theta,q)}{2}+C_1C_6)\right)\|w_t\|^2\\
&\quad+\left(\frac{C_4(q)}{\varepsilon}+\frac{C_5(\theta,q)}{2\varepsilon}+\frac{C_1C_6}{\varepsilon}\right)\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2\\
&\le C_7\left(\|w_t\|^2+\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2\right),
\end{aligned}\tag{3.11}$$

where $C_7=\min\left\{C_4(q)+\frac{C_5(\theta,q)}{2}+C_1C_6,\frac{C_4(q)}{\varepsilon}+\frac{C_5(\theta,q)}{2\varepsilon}+\frac{C_1C_6}{\varepsilon}\right\}$. By using Gronwall's inequality for (3.11), we obtain

$$0\le\|w_t\|^2+\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2\le\left(\|w_t(0)\|^2+\left\|D^m u(0)\right\|^{2q}\left\|D^m w(0)\right\|^2\right)\mathrm{e}^{C_7t}=0.\tag{3.12}$$

Hence , we can get $\|w_t\|^2+\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2=0.$ That shows that

$$\|w_t\|^2=0,\quad\left\|D^m u\right\|^{2q}\left\|D^m w\right\|^2=0.\tag{3.13}$$

That is

$$w(x,t)=0.\tag{3.14}$$

Therefore

$$u=v.\tag{3.15}$$

So we get the uniqueness of the solution.

3.2. Global Attractor

Theorem 3.2. [10] Let E be a Banach space, and $\{S(t)\}(t\geq 0)$ are the semigroup operator on E. $S(t):E\to E, S(t+\tau)=S(t)S(\tau)(\forall t,\tau\geq 0), S(0)=I$, where I is a unit operator.Set $S(t)$ satisfy the follow conditions:

1) $S(t)$ is uniformly bounded, namely $\forall R>0,\|u\|_E\leq R$, it exists a constant $C(R)$, so that

$$\|S(t)u\|_E\leq C(R)\quad (t\in[0,+\infty)); \tag{3.16}$$

2) It exists a bounded absorbing set $B_0\subset E$, namely, $\forall B\subset E$, it exists a constant t_0, so that

$$S(t)B\subset B_0\quad (t\geq t_0); \tag{3.17}$$

where B_0 and B are bounded sets.

3) When $t>0$, $S(t)$ is a completely continuous operator. Therefore, the semigroup operator S(t) exists a compact global attractor $\mathcal{A}$.

Theorem 3.3. Under the assume of **Lemma 1**, **Lemma 2** and **Theorem 3.1**, equations have global attractor

$$\mathcal{A}=\omega(B_0)=\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}S(t)B_0}, \tag{3.18}$$

where $B_0=\left\{(u,v)\in H^{2m}(\Omega)\times H_0^m(\Omega):\|(u,v)\|^2_{H^{2m}\times H_0^m}=\|u\|^2_{H^{2m}}+\|v\|^2_{H_0^m}\leq R_0+R_1\right\}$, B_0 is the bounded absorbing set of $H^{2m}\times H_0^m$ and satisfies

1) $S(t)\mathcal{A}=\mathcal{A}, t>0$;

2) $\lim\limits_{t\to\infty} dist(S(t)B,\mathcal{A})=0$, here $B\subset H^{2m}\times H_0^m$ and it is a bounded set,

$$dist(S(t)B,\mathcal{A})=\sup_{x\in B}\left(\inf_{y\in\mathcal{A}}\|S(t)x-y\|_{H^{2m}\times H_0^m}\right)\to 0, t\to\infty. \tag{3.19}$$

Proof. Under the conditions of Theorem 3.1, it exists the solution semigroup S(t), $S(t):H^{2m}\times H_0^m\to H^{2m}\times H_0^m$, here $E=H^{2m}(\Omega)\times H_0^m(\Omega)$.

(1) From Lemma 1 to Lemma 2, we can get that $\forall B\subset H^{2m}(\Omega)\times H_0^m(\Omega)$ is a bounded set that includes in the ball $\left\{\|(u,v)\|_{H^{2m}\times H_0^m}\leq R\right\}$,

$$\begin{aligned}\|S(t)(u_0,v_0)\|^2_{H^{2m}\times H_0^m}&=\|u\|^2_{H^{2m}}+\|v\|^2_{H_0^m}\leq\|u_0\|^2_{H^{2m}}+\|v_0\|^2_{H_0^m}+C\\&\leq R^2+C,(t\geq 0,(u_0,v_0)\in B).\end{aligned} \tag{3.20}$$

This shows that $S(t)(t\geq 0)$ is uniformly bounded in $H^{2m}(\Omega)\times H_0^m(\Omega)$.

(2) Furthermore, for any $(u_0,v_0)\in H^{2m}(\Omega)\times H_0^m(\Omega)$, when $t\geq\max\{t_0,t_1\}$, we have

$$\|S(t)(u_0,v_0)\|^2_{H^{2m}\times H_0^m}=\|u\|^2_{H^{2m}}+\|v\|^2_{H_0^m}\leq R_0+R_1. \tag{3.21}$$

So we get B_0 is the bounded absorbing set.

(3) Since $E_1:=H^{2m}(\Omega)\times H_0^m(\Omega)\to E_0:=H^{2m}(\Omega)\times L^2(\Omega)$ is compact embedded, which means that the bounded set in E_1 is the compact set in E_0, so the semigroup operator S(t) exists a compact global attractor $\mathcal{A}$.

4. The Estimates of the Upper Bounds of Hausdorff and Fractal Dimensions for the Global Attractor

We rewrite the problems (1.1) - (1.3):

$$u_{tt}+A^m u_t+\left\|A^{\frac{m}{2}}u\right\|^{2q}A^m u+g(u)=f(x)\quad \text{in } \Omega\times R^+, \tag{4.1}$$

$$u(x,0)=u_0(x);u_t(x,0)=u_1(x),\quad x\in\Omega, \tag{4.2}$$

$$u(x,t)\big|_{\partial\Omega}=0,\frac{\partial^i u}{\partial v^i}=0(i=1,\cdots,m-1),\quad \text{in } \partial\Omega\times R^+. \tag{4.3}$$

Let $Au=-\Delta u$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, q is positive constant, and m is positive integer. The linearized equations of the above equations as follows:

$$U_t+AU=FU, \tag{4.4}$$

$$U_0=\xi,U_t(0)=\zeta. \tag{4.5}$$

Let $U_0\in H_0^m(\Omega)$, $U(t)$ is the solution of problems (4.4) - (4.5). We can prove that the problems (4.4) - (4.5) have a unique solution $U\in L^\infty\left(0,T,H_0^m(\Omega)\right),U_t\in L^\infty\left(0,T,L^2(\Omega)\right)$. The equation (4.4) is the linearized equation by the Equation (4.17). Define the mapping $Ls(t)_{u_0}:Ls(t)_{u_0}\zeta=U(t)$, here $u(t)=s(t)u_0$, let $\varphi_0=(u_0,u_1)$, $\overline{\varphi_0}=\varphi_0+\{\xi,\zeta\}=\{u_0+\xi,u_1+\zeta\}$, let $\|\varphi_0\|_{E_0}\le R_1$, $\left\|\overline{\varphi_0}\right\|_{E_0}\le R_2$, $E_0=H_0^m(\Omega)\times Ł^2(\Omega)$, $S(t)\varphi_0=\varphi(t)=\{u(t),u_t(t)\}$, $S(t)\overline{\varphi_0}=\left\{\varphi(t),\overline{\varphi_t}(t)\right\}$.

Lemma 4.1 [6] Assume H is a Hilbert space, E_0 is a compact set of H. $S(t):E_0\to H$ is a continuous mapping, satisfy the follow conditions.

1) $S(t)E_0=E_0,t>0$;

2) If $S(t)$ is Fréchet differentiable, it exists is a bounded linear differential operator $L(t,\varphi_0)\in C\left(R^+;L(E_0,E_0)\right),\forall t>0$, that is

$$\frac{\left\|S(t)\overline{\varphi_0}-S(t)\varphi_0-L(t,\varphi_0)(u,v)\right\|_{E_0}^2}{\left\|\{\xi,\zeta\}\right\|_{E_0}^2}\to 0,\{\xi,\zeta\}\to 0.$$

The proof of lemma 4.1 see ref. [6] is omitted here. According to Lemma 4.1, we can get the following theorem :

Theorem 4.1. [6] [7] Let $\mathcal{A}$ is the global attractor that we obtain in section 3.In that case, $\mathcal{A}$ has finite Hausdorff dimensions and Fractal dimensions in $H^{2m}(\Omega)\times H_0^m(\Omega)$,that is $d_H(\mathcal{A})\le\frac{n}{5},d_F(\mathcal{A})\le\frac{6n}{5}$.

Proof. Firstly, we rewrite the equations (4.1), (4.2) into the first order abstract evolution equations in E_0.

Let $\Psi=R_\varepsilon\varphi=\{u,u_t+\varepsilon u\}$, let $R_\varepsilon:\{u,u_t\}\to\{u,u_t+\varepsilon u\}$, is an isomorphic mapping. So let $\mathcal{A}$ is the global attractor of $\{S(t)\}$, then $R_\varepsilon\mathcal{A}$ is also the global attractor of $\{S_\varepsilon(t)\}$, and they have the same dimensions. Then Ψ satisfies as follows:

$$\Psi_t + \Lambda_\varepsilon \Psi + \overline{g}(\Psi) = \overline{f}, \tag{4.6}$$

$$\Psi(0) = \{u_0, u_1 + \varepsilon u_0\}^{\mathrm{T}}, \tag{4.7}$$

where $\Psi = \{u, u_t + \varepsilon u\}^{\mathrm{T}}, \overline{g}(\Psi) = \{0, g(u)\}^{\mathrm{T}}, \overline{f} = \{0, f(x)\}^{\mathrm{T}}$,

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \left\|A^{\frac{m}{2}}u\right\|^{2q} A^m - \varepsilon A^m + \varepsilon^2 I & A^m - \varepsilon I \end{pmatrix}, \tag{4.8}$$

$$\Psi_t := F(\Psi) = \overline{f} - \Lambda_\varepsilon \Psi - \overline{g}(\Psi), \tag{4.9}$$

$$P_t = F_t(\Psi), \tag{4.10}$$

$$P_t + \Lambda_\varepsilon P + \overline{g}_t(\Psi) P = 0, \tag{4.11}$$

where $P = \{U, U_t + \varepsilon U\}^{\mathrm{T}}, \overline{g}_t(\Psi)P = \{0, g_t(u)U\}^{\mathrm{T}}$. The initial condition (4.5) can be written in the following form:

$$P(0) = \omega, \omega = \{\xi, \zeta\} \in E_0. \tag{4.12}$$

We take $n \in N$, then consider the corresponding n solutions: $(P = P_1, P_2, \cdots, P_n; P_j \in E_0)$ of the initial values: $(\omega = \omega_1, \omega_2, \cdots, \omega_n; \omega_j \in E_0)$ in the Equations (4.10) - (4.11). So there is $\left|P_1(t) \wedge P_2(t) \wedge \cdots \wedge P_n(t)\right|_{\wedge^n_{E_0}} = \left|\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n\right|_{\wedge^n_{E_0}} \cdot \mathrm{e}^{\int_0^t TrF_t(S_\varepsilon(\tau)\Psi_0)\cdot Q_n(\tau)\mathrm{d}\tau}$. from $\psi(\tau) = S_\varepsilon(\tau)\Psi_0$, we get $S_\varepsilon(\tau): \{u_0, v_1 = u_1 + \varepsilon u_0\} \to \{u(\tau), v(\tau) = u_t(\tau) + \varepsilon u(\tau)\}$, $\psi(\tau) = \{u(\tau), v_t(\tau) = u_t(\tau) + \varepsilon u(\tau)\}$, here u is the solution of problems (4.1)-(4.3); $\wedge$ represents the outer product, Tr reprsents the trace, $Q_n(\tau) = Q_n(\tau, \Psi_0; \omega_1, \omega_2, \cdots, \omega_n)$ is an orthogonal projection from the space $E_0 = \mathcal{V} \times \mathcal{H}$ to the subspace spanned by $\{P_1(\tau), P_2(\tau), \cdots, P_n(\tau)\}$.

For a given time τ, let $\phi_j(\tau) = \{\xi_j(\tau), \zeta_j(\tau)\}, j = 1, 2, \cdots, n$. $\{\phi_j(\tau)\}_{j=1,2,\cdots,n}$ is the standard orthogonal basis of the space $Q_n(\tau)_{E_0} = span\left[P_1(\tau), P_2(\tau), \cdots, P_n(\tau)\right]$.

From the above, we have

$$\begin{aligned} TrF_t(\Psi(\tau)) \cdot Q_n(\tau) &= \sum_{j=1}^{\infty} \left(F_t(\Psi(\tau)) \cdot Q_n(\tau)\phi_j(\tau), \phi_j(\tau)\right)_{E_0} \\ &= \sum_{j=1}^{n} \left(F_t(\Psi(\tau))\phi_j(\tau), \phi_j(\tau)\right)_{E_0}, \end{aligned} \tag{4.13}$$

where $(\cdot,\cdot)_{E_0}$ is the inner product in E_0.Then $\left(\{\xi, \zeta\}, \{\overline{\xi}, \overline{\zeta}\}\right)_{E_0} = (\xi, \overline{\xi}) + (\zeta, \overline{\zeta})$; $\left(F_t(\Psi)\phi_j, \phi_j\right)_{E_0} = -\left(\Lambda_\varepsilon \phi_j, \phi_j\right)_{E_0} - \left(g_t(u)\xi_j, \xi_j\right)$.

$$\begin{aligned} \left(\Lambda_\varepsilon \phi_j, \phi_j\right) &= \varepsilon\left\|\xi_j\right\|^2 + \left(\varepsilon^2 - 1\right)\left(\xi_j, \zeta_j\right) + \left(\left\|A^{\frac{m}{2}}u\right\|^{2q} - \varepsilon\right)\left(A^m \xi_j, \zeta_j\right) + \left(A^m \zeta_j, \zeta_j\right) - \varepsilon\left\|\zeta_j\right\|^2 \\ &= \varepsilon\left\|\xi_j\right\|^2 + \left(\varepsilon^2 - 1\right)\left(\xi_j, \zeta_j\right) + \lambda_j\left(\left\|A^{\frac{m}{2}}u\right\|^{2q} - \varepsilon\right)\left(\xi_j, \zeta_j\right) + \lambda_j\left\|\zeta_j\right\|^2 - \varepsilon\left\|\zeta_j\right\|^2 \\ &\geq a\left(\left\|\xi_j\right\|^2 + \left\|\zeta_j\right\|^2\right), \end{aligned} \tag{4.14}$$

where

$$a := \min\left\{\frac{2\varepsilon+\left[1-\varepsilon^2+\left(\varepsilon-\left\|A^{\frac{m}{2}}u\right\|^{2q}\right)\lambda_j\right]}{2},\frac{2\left(\lambda_j-\varepsilon\right)+\left[1-\varepsilon^2+\left(\varepsilon-\left\|A^{\frac{m}{2}}u\right\|^{2q}\right)\lambda_j\right]}{2}\right\}$$

Now, suppose that $\{u_0,u_1\}\in\mathcal{A}$, according to theorem 3.3, $\mathcal{A}$ is a bounded absorbing set in E_1. $\Psi(t)=\{u(t),u_t(t)+\varepsilon u(t)\}\in E_1, u(t)\in D(A); D(A)=\{u\in\mathcal{V}, Au\in\mathcal{H}\}$.

Then there is a $s\in[0,1]$ to make the mapping $g_t: D(A)\to\rho(\mathcal{V}_s,\mathcal{H})$. At the same time, there are the following results:

$$\begin{aligned}&R_A=\sup_{\{\xi,\zeta\}\in\mathcal{A}}|A\xi|<\infty;\\&\sup_{\substack{u\in D(A)\\|Au|<R_A}}\left|g_t(u)\right|_{\rho(\mathcal{V}_s,\mathcal{H})}\le r<\infty\end{aligned}\tag{4.15}$$

where $\left\|g_t(u)\xi_j,\zeta_j\right\|$ meets: $\left\|g_t(u)\xi_j,\zeta_j\right\|\le r\left\|\xi_j\right\|_s\left\|\zeta_j\right\|$. Comprehensive above can be obtained:

$$\begin{aligned}\left(F_t(\Psi)\phi_j,\phi_j\right)_{E_0}&\le -a\left(\left\|\xi_j\right\|^2+\left\|\zeta_j\right\|^2\right)+r\left\|\xi_j\right\|_s\left\|\zeta_j\right\|.\\&\le-\frac{a}{2}\left(\left\|\xi_j\right\|^2+\left\|\zeta_j\right\|^2\right)+\frac{r^2}{2a}\left\|\xi_j\right\|_s^2.\end{aligned}\tag{4.16}$$

$\left\|\xi_j\right\|^2+\left\|\zeta_j\right\|^2=\left\|\phi_j\right\|_{E_0}^2=1$, due to $\left\{\phi_j(\tau)\right\}_{j=1,2,\cdots,n}$ is a standard orthogonal basis in $Q_n(\tau)_{E_0}$. So

$$\sum_{j=1}^{n}\left(F_t(\Psi(\tau))\phi_j(\tau),\phi_j(\tau)\right)_{E_0}\le-\frac{na}{2}+\frac{r^2}{2a}\left\|\xi_j\right\|_s^2.\tag{4.17}$$

Almost to all t, making

$$\sum_{j=1}^{n}\left\|\xi_j\right\|_s^2\le\sum_{j=1}^{n-1}\lambda_j^{s-1}.\tag{4.18}$$

So

$$TrF_t(\Psi(\tau))\cdot Q_n(\tau)\le-\frac{na}{2}+\frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1}.\tag{4.19}$$

Let us assume that $\{u_0,u_1\}\in\mathcal{A}$, is equivalent to $\Psi_0=\{u_0,u_1+\varepsilon u_0\}\in R_\varepsilon\mathcal{A}$. Then

$$\begin{aligned}&q_n(t)=\sup_{\Psi_0\in R_\varepsilon\mathcal{A}}\sup_{\substack{\omega\in E_0\\\|\omega\|_{E_0}\le1}}\left(\frac{\int_0^t TrF_t\left(S_\varepsilon(\tau)\Psi_0\right)\cdot Q_n(\tau)\mathrm{d}\tau}{t_0}\right), j=1,2,\cdots,n,\\&q_n=\limsup_{t\to\infty}q_n(t).\end{aligned}\tag{4.20}$$

According to (4.19), (4.20), so

$$\begin{aligned}&q_n(t)\le-\frac{na}{2}+\frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1},\\&q_n\le-\frac{na}{2}+\frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1}.\end{aligned}\tag{4.21}$$

Therefore, the Lyapunov exponent of $\mathcal{A}$ (or $R_\varepsilon \mathcal{A}$) is uniformly bounded.

$$\mu_1 + \mu_2 + \cdots + \mu_n \le -\frac{na}{2} + \frac{r^2}{2a}\sum_{j=1}^{n}\lambda_j^{s-1}. \tag{4.22}$$

From what has been discussed above, it exists $n \ge 1$, a and r are constants, then

$$\frac{1}{n}\sum_{j=1}^{n}\lambda_j^{s-1} \le \frac{a^2}{6r^2}, \tag{4.23}$$

$$q_n \le -\frac{na}{2}\left(1 - \frac{r^2}{a^2}\sum_{j=1}^{n}\lambda_j^{s-1}\right) \le -\frac{5na}{12}, \tag{4.24}$$

$$\left(q_j\right)_+ \le \frac{r^2}{2a}\sum_{i=1}^{j}\lambda_i^{s-1} \le \frac{na}{12}, \; j = 1, 2, \cdots, n, \tag{4.25}$$

$$\max_{1 \le j \le n-1} \frac{\left(q_j\right)_+}{\left|q_n\right|} \le \frac{1}{5}. \tag{4.26}$$

According to the reference [6] [7], we immediately to the Hausdorff dimension and fractal dimension are respectively $d_H(\mathcal{A}) \le \frac{n}{5}, d_F(\mathcal{A}) \le \frac{6n}{5}$.

5. Conclusion

In this paper, we prove that the higher-order nonlinear Kirchhoff equation with linear damping in $L^\infty\left((0,+\infty); H^{2m}(\Omega) \times H_0^m(\Omega)\right)$ has a unique smooth solution (u, u_t). Further, we obtain the solution semigroup $S(t): H^{2m}(\Omega) \times H_0^m(\Omega) \to H^{2m}(\Omega) \times H_0^m(\Omega)$ has global attractor $\mathcal{A}$. Finally, we prove the equation has finite Hausdorff dimensions and Fractal dimensions in $L^\infty\left((0,+\infty); H^{2m}(\Omega) \times H_0^m(\Omega)\right)$.

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