Log-Concavity of Centered Polygonal Figurate Number Sequences

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Received 28 May 2016; accepted 23 June 2016; published 27 June 2016

Abstract

This paper investigates the log-concavity of the centered *m*-gonal figurate number sequences. The author proves that for $m \geq 3$, the sequence $\{\mathcal{C}_n(m)\}_{n\geq 1}$ of centered *m*-gonal figurate numbers is a log-concave.

Keywords

Log-Concavity, Figurate Numbers, Centered Polygonal, Number Sequences

Subject Areas: **Discrete Mathematics, Combinatorial Sequences, Recurrences**

1. Introduction

For $n \geq 1$ and $m \geq 3$, let $\mathcal{C}_n(m)$ denote the n^{th} term of the centered *m*-gonal figurate number sequence. E. Deza and M. Deza [1] stated that $\mathcal{C}_n(m)$ could be defined by the following recurrence relation:

$$\mathcal{C}_{n+1}(m) = \mathcal{C}_n(m) + mn \tag{1}$$

where $\mathcal{C}_1(m) = 1$. E. Deza and M. Deza [1] also gave different properties of $\mathcal{C}_n(m)$ and obtained

$$\mathcal{C}_n(m) = 1 + \frac{m(n-1)n}{2} = \frac{mn^2 - mn + 2}{2} \tag{2}$$

where $n \geq 1$ and $m \geq 3$. For $m \geq 3$, some terms of the sequence $\{\mathcal{C}_n(m)\}_{n\geq 1}$ are as follows:

$$1, 1+m, 1+3m, 1+6m, 1+10m, 1+15m, 1+21m, 1+28m, \cdots.$$

Some scholars have been studying the log-concavity (or log-convexity) of different numbers sequences such as Fibonacci & Hyperfibonacci numbers, Lucas & Hyperlucas numbers, Bell numbers, Hyperpell numbers, Motzkin numbers, Fine numbers, Franel numbers of order 3 & 4, Apéry numbers, Large Schröder numbers,

How to cite this paper: Gedefa, F.T. (2016) Log-Concavity of Centered Polygonal Figurate Number Sequences. *Open Access Library Journal*, **3**: e2774. http://dx.doi.org/10.4236/oalib.1102774

Central Delannoy numbers, Catalan-Larcombe-French numbers sequences, and so on (see for instance [2]-[9]).

To the best of the author's knowledge, among all the aforementioned works on the log-concavity and log-convexity of number sequences, no one has studied the log-concavity (or log-convexity) of centered *m*-gonal figurate number sequences. In [1] [10] [11], some properties of centered figurate numbers are given. The main aim of this paper is to discuss properties related to the sequence $\{\mathcal{C}_n(m)\}_{n\geq 1}$. Now we recall some definitions involved in this paper.

Definition 1. *Let* $\{s_n\}_{n\geq 0}$ *be a sequence of positive numbers. If for all* $i\geq 1$, $s_i^2 \geq s_{i-1}s_{i+1}$, *the sequence* $\{s_n\}_{n\geq 0}$ *is called log-concave.*

Definition 2. *Let* $\{s_n\}_{n\geq 0}$ *be a sequence of positive numbers. If for all* $i\geq 1$, $s_i^2 \leq s_{i-1}s_{i+1}$, *the sequence* $\{s_n\}_{n\geq 0}$ *is called log-convex. In case of equality,* $s_i^2 = s_{i-1}s_{i+1}, i\geq 1$, *we call the sequence* $\{s_n\}_{n\geq 0}$ *geometric or log-straight.*

Definition 3. *Let* $\{s_n\}_{n\geq 0}$ *be a sequence of positive numbers. The sequence* $\{s_n\}_{n\geq 0}$ *is log-concave* (*log-convex*) *if and only if its quotient sequence* $\left\{\frac{s_{n+1}}{s_n}\right\}_{n\geq 0}$ *is non-increasing* (*non-decreasing*).

Log-concavity and log-convexity are important properties of combinatorial sequences and they play a crucial role in many fields, for instance economics, probability, mathematical biology, quantum physics and white noise theory [2] [12]-[18].

2. Log-Concavity of Centered *m*-gonal Figurate Number Sequences

In this section, we state and prove the main results of this paper.

Theorem 4. *For* $m\geq 3$ *and* $n\geq 3$, *the following recurrence formulas for centered m-gonal number sequences hold*:

$$\mathcal{C}_n(m) = R(n)\mathcal{C}_{n-1}(m) + S(n)\mathcal{C}_{n-2}(m) \tag{3}$$

with the initial conditions $\mathcal{C}_1(m)=1, \mathcal{C}_2(m)=1+m$ and the recurrence of its quotient sequence is given by

$$x_{n-1} = R(n) + \frac{S(n)}{x_{n-2}} \tag{4}$$

with the initial condition $x_1 = 1+m$.

Proof. By (1), we have

$$\mathcal{C}_{n+1}(m) = \mathcal{C}_n(m) + mn \tag{5}$$

It follows that

$$\mathcal{C}_{n+2}(m) = \mathcal{C}_{n+1}(m) + m(n+1) \tag{6}$$

Rewriting (5) and (6) for $n\geq 3$, we have

$$\mathcal{C}_{n-1}(m) = \mathcal{C}_{n-2}(m) + m(n-2) \tag{7}$$

$$\mathcal{C}_n(m) = \mathcal{C}_{n-1}(m) + m(n-1) \tag{8}$$

Multiplying (7) by $m(n-1)$ and (8) by $m(n-2)$, and subtracting as to cancel the non homogeneous part, one can obtain the homogeneous second-order linear recurrence for $\mathcal{C}_n(m)$:

$$\mathcal{C}_n(m) = \left[\frac{2n-3}{n-2}\right]\mathcal{C}_{n-1}(m) - \left[\frac{n-1}{n-2}\right]\mathcal{C}_{n-2}(m), \forall n,m\geq 3. \tag{9}$$

By denoting

$$\frac{2n-3}{n-2} = R(n)$$

and

$$-\frac{n-1}{n-2}=S(n),$$

one can obtain

$$\mathcal{C}_n(m)=R(n)\mathcal{C}_{n-1}(m)+S(n)\mathcal{C}_{n-2}(m),\forall n,m\geq 3 \tag{10}$$

with given initial conditions $\mathcal{C}_1(m)=1$ and $\mathcal{C}_2(m)=1+m$.

By dividing (10) through by $\mathcal{C}_{n-1}(m)$, one can also get the recurrence of its quotient sequence x_{n-1} as

$$x_{n-1}=R(n)+\frac{S(n)}{x_{n-2}},n\geq 3 \tag{11}$$

with initial condition $x_1=1+m.$ □

Lemma 5. *For the centered m-gonal figurate number sequence* $\{\mathcal{C}_n(m)\}_{n\geq 1}$, *let* $x_n=\frac{\mathcal{C}_{n+1}(m)}{\mathcal{C}_n(m)}$ for $n\geq 1$ and $m\geq 3$. Then we have $1<x_n\leq 1+m$ for $n\geq 1$.

Proof. Assume $x_n\neq 1$ for $n\geq 1$ and $m\geq 3$. Otherwise,

$$1=x_n=\frac{\mathcal{C}_{n+1}(m)}{\mathcal{C}_n(m)}=\frac{2+mn(n+1)}{2+mn(n-1)}. \tag{12}$$

It follows that $-1=1$ which not true. Now it is clear that $x_n\neq 1$ and

$$x_1=1+m, x_2=3-\frac{2}{1+m}, x_3=2-\frac{1}{1+3m}>1, \text{ for } m\geq 3. \tag{13}$$

Assume that $x_n>1$ for all $n\geq 3$. It follows from (11) that

$$x_n=\frac{2n-1}{n-1}-\frac{n}{(n-1)x_{n-1}},n\geq 2 \tag{14}$$

For $n\geq 3$, by (14), we have

$$x_{n+1}-1=\frac{n+1}{n}-\frac{n+1}{nx_n} \tag{15}$$

$$=\frac{(n+1)x_n-(n+1)}{nx_n} \tag{16}$$

$$=\frac{(n+1)(x_n-1)}{nx_n} \tag{17}$$

$$>0 \text{ for } m\geq 3.$$

Hence $x_n>1$ for $n\geq 1$ and $m\geq 3.$
Similarly, it is known that

$$x_1=1+m, x_2=3-\frac{2}{1+m}, x_3=2-\frac{1}{1+3m}<1+m, \text{ for } m\geq 3. \tag{18}$$

Assume that $x_n\leq 1+m$ for all $n\geq 3$. It follows from (11) that

$$x_n=\frac{2n-1}{n-1}-\frac{n}{(n-1)x_{n-1}},n\geq 2 \tag{19}$$

For $n\geq 3$, by (19), we have

$$x_{n+1}-(1+m)=\frac{n+1-mn}{n}-\frac{n+1}{nx_n} \tag{20}$$

$$= \frac{(n+1-mn)x_n - (n+1)}{nx_n} \tag{21}$$

$$< -\frac{m}{x_n} < 0 \text{ for } m \geq 3.$$

Hence $x_n \leq 1+m$ for $n \geq 1$ and $m \geq 3$. □

Thus, in general, from the above two cases it follows that $1 < x_n \leq 1+m$ for $n \geq 1$ and $m \geq 3$.

Lemma 6. *For the centered m-gonal figurate number sequence* $\{\mathcal{C}_n(m)\}_{n\geq 1}$, *the quotient sequence* $\{x_n\}_{n\geq 1}$, *given in* (4), *is a decreasing sequence for* $m \geq 3$.

Proof. Let $\{x_n\}_{n\geq 1}$ be a quotient sequence given in (4). We prove by induction that the sequence $\{x_n\}_{n\geq 1}$ is decreasing. Indeed, since $x_1 = 1+m, x_2 = 3 - \frac{2}{1+m}, x_3 = 2 - \frac{1}{1+3m}$, we have $x_1 > x_2 > x_3$. Next we assume that $x_n < x_{n-1}$.

By using (11), one can obtain

$$x_n = \frac{2n-1}{n-1} - \frac{n}{(n-1)x_{n-1}}, n \geq 2 \tag{22}$$

with initial condition $x_1 = 1+m$.

For $n \geq 3$, by (22), we get

$$x_{n+1} - x_n = \frac{2n+1}{n} - \frac{n+1}{nx_n} - \frac{2n-1}{n-1} + \frac{n}{(n-1)x_{n-1}} \tag{23}$$

$$= \frac{2n+1}{n} - \frac{2n-1}{n-1} - \frac{n+1}{nx_n} + \frac{n}{(n-1)x_{n-1}} \tag{24}$$

$$= \frac{2n+1}{n} - \frac{2n-1}{n-1} + \frac{1}{x_n}\left[\frac{n}{n-1} - \frac{n+1}{n}\right] + \frac{n}{n-1}\left[\frac{1}{x_{n-1}} - \frac{1}{x_n}\right] \tag{25}$$

$$= -\frac{1}{n(n-1)} + \frac{1}{n(n-1)x_n} + \frac{n}{n-1}\left[\frac{1}{x_{n-1}} - \frac{1}{x_n}\right] \tag{26}$$

$$= -[\frac{x_n - 1}{n(n-1)x_n}] + \frac{n}{n-1}\left[\frac{1}{x_{n-1}} - \frac{1}{x_n}\right] < 0. \tag{27}$$

By Lemma 5 and induction assumption, one can get $x_{n+1} - x_n < 0$ for $n \geq 3$.

Thus, the sequence $\{x_n\}_{n\geq 1}$ is decreasing for $m \geq 3$. □

Theorem 7 *For* $m \geq 3$, *the sequence* $\{\mathcal{C}_n(m)\}_{n\geq 1}$ *of centered m-gonal figurate numbers is a log-concave*.

Proof. Let $\{\mathcal{C}_n(m)\}_{n\geq 1}$ be a sequence of centered *m*-gonal figurate numbers and $\{x_n\}_{n\geq 1}$ its quotient sequence, given by (4). To prove the log-concavity of $\{\mathcal{C}_n(m)\}_{n\geq 1}$ for all $m \geq 3$, it suffices to show that the quotient sequence $\{x_n\}_{n\geq 1}$ is decreasing.

By Lemma 6, the quotient sequence $\{x_n\}_{n\geq 1}$ is decreasing. Thus, by definition 3, the sequence $\{\mathcal{C}_n(m)\}_{n\geq 1}$ of centered *m*-gonal figurate numbers is a log-concave for $m \geq 3$. This completes the proof of the theorem. □

3. Conclusion

In this paper, we have discussed the log-behavior of centered *m*-gonal figurate number sequences. We have also proved that for $m \geq 3$, the sequence $\{\mathcal{C}_n(m)\}_{n\geq 1}$ of centered *m*-gonal figurate numbers is a log-concave.

Acknowledgements

The author is grateful to the anonymous referees for their valuable comments and suggestions.

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