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Sectorial Approach of the Gradient Observability of the Hyperbolic Semilinear Systems Intern and Boundary Cases

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Abstract

The aim of this paper is to study the notion of the gradient observability on a subregion ω of the evolution domain Ω and also we consider the case where the subregion of interest is a boundary part of the system evolution domain for the class of semilinear hyperbolic systems. We show, under some hypotheses, that the flux reconstruction is guaranteed by means of the sectorial approach combined with fixed point techniques. This leads to several interesting results which are performed through numerical examples and simulations.

Keywords

Distributed Systems, Semilinear Hyperbolic Systems, Boundary Reconstruction, Regional Boundary Gradient Observability, Regional Gradient Observability, Gradient Observability, Fixed Point, Sectorial Operator

1. Introduction

The regional observability is one of the most important notions of systems theory. It consists to reconstruct the trajectory only in a subregion in the whole domain. This concept has been widely developed see [1] [2]. Afterwards, the concept of regional gradient observability for parabolic systems has been developed see [3]-[7] and for hyperbolic systems see [8] [9], it concerns the reconstruction of the gradient conditions initials only in a critical subregion interior to the system domain without the knowledge of the conditions initials.

The aim of this papers is to study the regional gradient observability of an important class of semilinear hyperbolic systems. For the sake of brevity and simplicity, we shall focus our attention on the case where the

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dynamic of the system is a sectorial operator linear and generating an analytical semigroup $\left(\bar{S}(t)\right)_{t\geq 0}$ on the Hilbert space.

The plan of the paper is as follows: Section 2 is devoted to the presentation of problem of regional gradient of semilinear hyperbolic systems, and then we give definitions and propositions of this new concept. Section 3 concerns the sectorial approach. Section 4 concerns the numerical approach which gives algorithm can simulated by a numerical example.

2. Position of the Problem

Let Ω be an open bounded subset of $\mathbb{R}^n \left(n=1,2,3\right)$. For $T>0$, we denote $Q=\Omega\times\left]0,T\right[$, $\Sigma=\partial\Omega\times\left]0,T\right[$ and we consider the following hyperbolic semi-linear system

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x,t)=\mathcal{A}y(x,t)+\mathcal{N}y(x,t) & \text{in } Q \\ y(x,0)=y^0(x), \frac{\partial y}{\partial t}(x,0)=y^1(x) & \text{in } \Omega \\ y(\xi,t)=0 & \text{on } \Sigma \end{cases} \tag{1}$$

and the linear part of the system (1) is

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x,t)=\mathcal{A}y(x,t) & \text{in } Q \\ y(x,0)=y^0(x), \frac{\partial y}{\partial t}(x,0)=y^1(x) & \text{in } \Omega \\ y(\xi,t)=0 & \text{on } \Sigma \end{cases} \tag{2}$$

where $\mathcal{A}$ is an elliptic and second order operator and $\mathcal{N}$ is a nonlinear operator assumed to be locally Lipschitzian, system (1) is augmented with the output function given by

$$z(t)=Cy(t) \tag{3}$$

where $C:H_0^1(\Omega)\to\mathbb{R}^q$ (resp. $C:H^2(\Omega)\cap H_0^1(\Omega)\to\mathbb{R}^q$ if the subregion of interest is a boundary part Γ of the system evolution domain Ω) is a linear operator, and depends on the number q and the nature of the considered sensors. The observation space is $\mathcal{O}=L^2\left(0,T;\mathbb{R}^q\right)$ and assumes that

$$\left(y^0,y^1\right)\in X=H_0^1(\Omega)\times H_0^1(\Omega) \quad \left(\text{resp. } \left(y^0,y^1\right)\in X=\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\right).$$

Let $\bar{y}(t)=\begin{bmatrix} y(t) \\ \frac{\partial y(t)}{\partial t}\end{bmatrix}, \bar{y}^0=\begin{bmatrix} y^0 \\ y^1\end{bmatrix}, \bar{\mathcal{A}}=\begin{bmatrix} 0 & I \\ \mathcal{A} & 0\end{bmatrix}, \bar{\mathcal{N}}\begin{bmatrix} y_1 \\ y_2\end{bmatrix}=\begin{bmatrix} 0 \\ \mathcal{N}y_1\end{bmatrix}$

For $\left(y_1,y_2\right)\in\mathcal{F}=H_0^1(\Omega)\times L^2(\Omega)$ the system (2) is equivalent to

$$\begin{cases} \frac{\partial\bar{y}}{\partial t}(t)=\bar{\mathcal{A}}\bar{y}(t) & 0<t<T \\ \bar{y}(0)=\bar{y}^0 \end{cases} \tag{4}$$

and the system (1) is equivalent to

$$\begin{cases} \frac{\partial\bar{y}}{\partial t}(t)=\bar{\mathcal{A}}\bar{y}(t)+\bar{\mathcal{N}}\,\bar{y}(t) & 0<t<T \\ \bar{y}(0)=\bar{y}^0 \end{cases} \tag{5}$$

augmented with the output function

$$\bar{z}(t) = \bar{C}\bar{y}(t) \tag{6}$$

with $\bar{C} = (C, 0)$ the system (4) has a unique solution see [10]-[12] that can be expressed as $\bar{y}(t) = \bar{S}(t)\bar{y}^0$, $\left(\bar{S}(t)\right)_{t\geq 0}$ is the semigroup generated by the operator $\bar{\mathcal{A}}$.

Let's consider a basis of eigenfunctions of the operator $\mathcal{A}$, with the condition of Dirichlet which noted Φ_{mj} and eigenvalues associated are λ_m with multiplicity r_m.

We can write for any $(y_1, y_2) \in \mathcal{F}$

$$\bar{S}(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sum_m^{\infty}\sum_{j=1}^{r_m}\left[\langle y_1, \Phi_{mj}\rangle \cos\sqrt{-\lambda_m}t + \frac{1}{\sqrt{-\lambda_m}}\langle y_2, \Phi_{mj}\rangle \sin\sqrt{-\lambda_m}t\right]\Phi_{mj} \\ \sum_m^{\infty}\sum_{j=1}^{r_m}\left[-\sqrt{-\lambda_m}\langle y_1, \Phi_{mj}\rangle \sin\sqrt{-\lambda_m}t + \langle y_2, \Phi_{mj}\rangle \cos\sqrt{-\lambda_m}t\right]\Phi_{mj} \end{pmatrix}.$$

The system (5) has a unique solution that can be expressed as follows see [13]

$$\bar{y}(t) = \bar{S}(t)\bar{y}^0 + \int_0^t \bar{S}(t-s)\bar{\mathcal{N}}\bar{y}(s)\,\mathrm{d}s, \tag{7}$$

then the output Equation (6) can be expressed by

$$\bar{z}(t) = \overline{CS}(t)\bar{y}^0 = \bar{K}(t)\bar{y}^0,\ t \in \left]0, T\right[.$$

Let $\bar{K}$ be the observation operator defined by

$$\begin{aligned} \bar{K} : X &\to \mathcal{O} \\ \bar{z} &\mapsto \overline{CS}(.)\bar{z}, \end{aligned}$$

which is linear and bounded with the adjoint $\bar{K}^*$ given by

$$\begin{aligned} \bar{K}^* : \mathcal{O} &\to X \\ \bar{z}^* &\mapsto \int_0^T \bar{S}^*(t)\bar{C}^*\bar{z}^*(t)\,\mathrm{d}t. \end{aligned}$$

Consider the operator $\bar{\nabla}$ given by the formula

$$\begin{aligned} \bar{\nabla} : X &\to \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \\ (y_1, y_2) &\mapsto \bar{\nabla}(y_1, y_2) = (\nabla y_1, \nabla y_2), \end{aligned}$$

where

$$\begin{aligned} \nabla : \tilde{X} &\to \left(L^2(\Omega)\right)^n \\ y &\mapsto \nabla y = \left(\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n}\right). \end{aligned}$$

$\tilde{X} = H_0^1(\Omega)$ (resp. $\tilde{X} = H^2(\Omega) \cap H_0^1(\Omega)$ if the subregion of interest is a boundary part Γ of the system evolution domain Ω.)

$\bar{\nabla}^*$ is the adjoint of $\bar{\nabla}$.

The initial condition $\bar{y}^0$ (initial state y^0 and initial speed y^1) and $\bar{\nabla}\bar{y}^0$ its gradient are assumed unknown. For $\omega \subset \Omega$ an open subregion of Ω, consider the restriction operators

$$\begin{aligned} \bar{\chi}_\omega : \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n &\to \left(L^2(\omega)\right)^n \times \left(L^2(\omega)\right)^n \\ (\xi, \xi') &\mapsto \bar{\chi}_\omega(\xi, \xi') = (\xi, \xi')\big|_\omega, \end{aligned}$$

$$\begin{aligned} \chi_\omega : \left(L^2(\Omega)\right)^n &\to \left(L^2(\Omega)\right)^n \\ \xi &\mapsto \chi_\omega \xi = \xi\big|_\omega, \end{aligned}$$

with $\bar{\chi}_\omega^*$ is the adjoint of $\bar{\chi}_\omega$ (resp. χ_ω^* is the adjoint of χ_ω).
(resp. For $\Gamma \subset \partial\Omega$, consider

$$\bar{\chi}_\Gamma : \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \times \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \to \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$$
$$(\xi,\xi') \mapsto \bar{\chi}_\Gamma(\xi,\xi') = (\xi,\xi')|_\Gamma,$$

$$\chi_\Gamma : \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \to \left(H^{\frac{1}{2}}(\Gamma)\right)^n \quad \text{and} \quad \tilde{\chi}_\Gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\Gamma)$$
$$\xi \mapsto \chi_\Gamma \xi = \xi|_\Gamma \qquad\qquad \xi \mapsto \tilde{\chi}_\Gamma \xi = \xi|_\Gamma,$$

with $\bar{\chi}_\Gamma^*$ (resp. χ_Γ^* and $\tilde{\chi}_\Gamma^*$) is the adjoint of $\bar{\chi}_\Gamma$ (resp. χ_Γ and $\tilde{\chi}_\Gamma$) which is the restriction operator.

The trace operator is defined by

$$\bar{\gamma} : \left(H^1(\Omega)\right)^n \times \left(H^1(\Omega)\right)^n \to \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \times \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$$
$$\left(y^1, y^2\right) \mapsto \bar{\gamma}\left(y^1, y^2\right) = \left(\gamma y^1, \gamma y^2\right),$$

with

$$\gamma : \left(H^1(\Omega)\right)^n \to \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$$
$$z \mapsto \gamma z = \left(\gamma_0 z_1, \cdots, \gamma_0 z_n\right),$$

and $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator of order zero which is linear, continuous, and surjective. γ^* (resp. γ_0^*) denote the adjoint of operator γ (resp. γ_0).

Finally, we reconstruct the operator as follows

$$\bar{H}_\omega = \bar{\chi}_\omega \bar{\nabla} \bar{K}^* \text{ from } \mathcal{O} \text{ into } \left(L^2(\omega)\right)^n.$$

$$\left(\text{resp. } \bar{H}_\Gamma = \bar{\chi}_\Gamma \bar{\gamma} \bar{\nabla} \bar{K}^* \text{ from } \mathcal{O} \text{ into } \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n\right).$$

Definition 1

- The system (2) together with the output (3) is said to be exactly (resp. weakly) *G*-observable in ω if $Im\left(\bar{H}\right) = \left(L^2(\omega)\right)^n$ (resp. $\overline{Im\left(\bar{H}\right)} = \left(L^2(\omega)\right)^n$.
- The system (2) together with the output (3) is said to be exactly (resp. weakly) *G*-observable in Γ if $Im\left(\bar{H}\right) = \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$ (resp. $\overline{Im\left(\bar{H}\right)} = \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$).

Remark 1.

- If the system (2) together with the output (3) is exactly *G*-observable on Γ (resp. in ω) then it is weakly *G*-observable on Γ.
- For $\Gamma_2 \subset \Gamma_1 \subset \partial\Omega$ the system (2) together with the output (3) is exactly (resp. weakly) *G*-observable on Γ_1 then it is exactly (resp. weakly) *G*-observable on Γ_2. see [9].

Definition 2 *The semilinear system* (1) *augmented by the output function* (3) *is said to be gradient observable or G-observable on* Γ (*resp. in* ω) *if we can reconstruct the gradient of its state and speed on a subregion* Γ *of* $\partial\Omega$ (*resp. in* ω *of* Ω).

Let the gradient $\bar{\nabla}\bar{y}^0 = \left(\nabla y^0, \nabla y^1\right)$ of the initial condition $\bar{y}^0 = \left(y^0, y^1\right)$ be decomposed as follows:

$$\overline{\nabla}\overline{y}^0 = \begin{cases} \overline{y}_1^0 & \text{in } \Gamma \\ \overline{y}_2^0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases} \tag{8}$$

where $\overline{y}_1^0 = \left(y_1^0, y_1^1\right)$, $\overline{y}_2^0 = \left(y_2^0, y_2^1\right)$ and

$$\nabla y^0 = \begin{cases} y_1^0 & \text{in } \Gamma \\ y_2^0 & \text{on } \partial\Omega \setminus \Gamma \end{cases}, \qquad \nabla y^1 = \begin{cases} y_1^1 & \text{in } \Gamma \\ y_2^1 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

Problem (*)

Given system (1) augmented by the output (3) on $]0,T[$, is it possible to reconstruct $\overline{y}_1^0 = \left(y_1^0, y_1^1\right)$ which is the gradient of initial condition of (1) in ω? (resp. on Γ.)

3. Sectorial Case

In this section, we study Problem (*) under some supplementary hypothesis on $\mathcal{A}$ and the nonlinear operator $\mathcal{N}$.

With the same notations as in the previous case, we reconsider the semilinear system described by the Equation (5) augmented by the output (6) where one suppose that the operator $\overline{\mathcal{A}}$ generates an analytic semigroup $\left(\overline{S}(t)\right)_{t\geq 0}$ in the state space *E*.

Let's consider $\overline{\mathcal{A}}_1 = \overline{\mathcal{A}} + aI$ such that $Re\left(\sigma\left(\overline{\mathcal{A}}_1\right)\right) > \delta > 0$ with *a* is a positive real number and $Re\left(\sigma\left(\overline{\mathcal{A}}_1\right)\right)$ denotes the real part of spectrum of $\overline{\mathcal{A}}_1$. Then for $0 \leq \alpha < 1$, we define the fractional power $\left(\overline{\mathcal{A}}_1\right)^\alpha$ as a closed operator with domain $E^\alpha = D\left(\overline{\mathcal{A}}_1^\alpha\right)$ which is a dense Banach space on *E* endowed with the graph norm

$$\|\cdot\|_{E^\alpha} = \left\|\overline{\mathcal{A}}_1^\alpha(\cdot)\right\|_E.$$

Let us consider $V = Im\left(\chi_\Gamma \overline{\gamma}\overline{\nabla}\overline{K}^*\right)$ then the objective is to study the Problem (*) in *V* endowed with the norm

$$\|\cdot\|_V = \left\|\overline{H}^*(\cdot)\right\|_{\mathcal{O}}, \tag{9}$$

we have

$$\left\|\overline{S}(t)\right\|_{\mathcal{L}\left(E,E^\alpha\right)} \leq ct^{-\alpha}\exp(a-\alpha)t = g_1(t),$$

where *c* is a constant. For more details, see ([2] [11] [14]).

For $r, s > 1$, assume that

$$\begin{cases} \dfrac{1}{r} + \dfrac{1}{s} = 1 \\ g_1 \in L^r(0,T), \end{cases} \tag{10}$$

and the operator $\overline{\mathcal{N}} : L^r\left(0,T;E^\alpha\right) \to L^s(0,T;E)$ is well defined and satisfies the following conditions:

$$\begin{cases} \left\|\overline{\mathcal{N}}x - \overline{\mathcal{N}}y\right\|_{L^s(0,T;E)} \leq k\left(\|x\|,\|y\|\right)\|x-y\|_{L^r\left(0,T;E^\alpha\right)} \ \forall x,y \in L^r\left(0,T;E^\alpha\right) \\ \overline{\mathcal{N}}(0) = 0 \quad \text{with } k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \\ \text{such that } \lim\limits_{(\theta_1,\theta_2)\to(0,0)} k\left(\theta_1,\theta_2\right) = 0. \end{cases} \tag{11}$$

This hypothesis are verified by many important class of semi linear hyperbolic systems. Various examples are given and discussed in ([14]-[16]).

We show that there exists a set of admissible initial gradient states and admissible initial gradient speed, admissible in the sense that allows to obtain system (2) weakly *G*-observable.

Let's consider

$$\varphi\left(\overline{y}_1^0,.\right):L^r\left(0,T;E^\alpha\right)\to L^r\left(0,T;E^\alpha\right),$$

given by

$$\varphi\left(\overline{y}_1^0,\overline{y}\right)(t)=\overline{S}(t)\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_\Gamma^*\overline{y}_1^0+\overline{S}(t)\tilde{y}_2^0+L(t)\overline{N}\overline{y}\quad t\in[0,T],$$

where $\overline{y}_1^0$ is the restriction in Γ and $\tilde{y}_2^0$ is the residual part of the initial gradient condition $\overline{\nabla}\overline{y}^0$ given by (8). we assume that

$$\left\|\overline{S}(t)\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_\Gamma^*\right\|_{\mathcal{L}\left(V,E^\alpha\right)}\le g_2(t)\quad\text{with } g_2\in L^2(0,T),\tag{12}$$

then we have the following result

Proposition 1 *Suppose that the system* (2) *is weakly G-observable on* Γ, *and* (10), (11) *and* (12) *satisfied, then the following assertion hold*:

- There exists $a_1>0$ and $m>0$ such that for all $\overline{y}_1^0\in B(0,m)\subset V$ the function $\varphi\left(\overline{y}_1^0,.\right)$ has a unique fixed point $\overline{y}$ in the ball $B(0,a_1)\subset L^r\left(0,T;E^\alpha\right)$ solution of the system (5).
- There exist $m=m(a_1)$ and $m_1=m_1(a_1)$ such that $\left\|\tilde{y}_2^0\right\|_E\le m_1$, the mapping f is Lipschitzian where

$$\begin{aligned}f:B(0,m)&\to B(0,a_1)\\ \overline{y}_1^0&\mapsto\overline{y}.\end{aligned}$$

Proof.

- Since $\lim_{(\theta_1,\theta_2)\to(0,0)}k(\theta_1,\theta_2)=0$, then there exists $a_1\in\,]0,1[$ such that

$$\alpha_1=T^{\frac{1}{r}}\left\|g_1\right\|_{L^r(0,T)}\sup_{\theta_1,\theta_2\le a_1}k(\theta_1,\theta_2)<1,$$

and we have $\alpha_1\in\,]0,1[$.

Let us consider $\overline{y}$ and $\overline{x}$ in $B(0,a_1)\subset L^r\left(0,T;E^\alpha\right)$ and $\overline{y}_1^0\in V$ we have

$$\begin{aligned}&\left\|\varphi\left(\overline{y}_1^0,\overline{y}\right)-\varphi\left(\overline{y}_1^0,\overline{x}\right)\right\|_{L^r\left(0,T;E^\alpha\right)}\\&=\left\|L(.)\left(\overline{N}\overline{y}-\overline{N}\overline{x}\right)\right\|_{L^r\left(0,T;E^\alpha\right)}\\&=\left(\int_0^T\left\|L(t)\left(\overline{N}\overline{y}-\overline{N}\overline{x}\right)\right\|_{E^\alpha}^r\mathrm{d}t\right)^{\frac{1}{r}},\end{aligned}$$

where

$$\begin{aligned}&\left\|L(t)\left(\overline{N}\overline{y}-\overline{N}\overline{x}\right)\right\|_{E^\alpha}\\&=\left\|\int_0^t\overline{S}(t-\tau)\left(\overline{N}\overline{y}(\tau)-\overline{N}\overline{x}(\tau)\right)\mathrm{d}\tau\right\|_{E^\alpha}\\&\le\int_0^T\left\|\overline{S}(t-\tau)\left(\overline{N}\overline{y}(\tau)-\overline{N}\overline{x}(\tau)\right)\right\|_{E^\alpha}\mathrm{d}\tau\\&\le\int_0^T\left\|\overline{S}(t-\tau)\right\|_{\mathcal{L}\left(E,E^\alpha\right)}\left\|\left(\overline{N}\overline{y}(\tau)-\overline{N}\overline{x}(\tau)\right)\right\|_E\mathrm{d}\tau\\&\le\int_0^T g_1(t)\left\|\left(\overline{N}\overline{y}(\tau)-\overline{N}\overline{x}(\tau)\right)\right\|_E\mathrm{d}\tau.\end{aligned}$$

Using Holder's inequality we take

$$\left\|L(t)\left(\overline{N}\overline{y}-\overline{N}\overline{x}\right)\right\|_{E^\alpha}^r\le\left\|g_1(t)\right\|_{L^r(0,T)}^r\left\|\overline{N}\overline{y}-\overline{N}\overline{x}\right\|_{L^s(0,T;E)}^r,$$

and using (11), we have

$$\begin{aligned}\left\|\varphi\left(\bar{y}_1^0,\bar{y}\right)-\varphi\left(\bar{y}_1^0,\bar{x}\right)\right\|_{L^r\left(0,T;E^\alpha\right)} &\le \left(\int_0^T \left\|g_1(t)\right\|_{L^r(0,T)}^r \left\|\bar{N}\bar{y}-\bar{N}\bar{x}\right\|_{L^s(0,T;E)}^r \mathrm{d}t\right)^{\frac{1}{r}}\\ &\le T^{\frac{1}{r}}\left\|g_1(t)\right\|_{L^r(0,T)}\left\|\bar{N}\bar{y}-\bar{N}\bar{x}\right\|_{L^s(0,T;E)}\\ &\le T^{\frac{1}{r}}\left\|g_1\right\|_{L^r(0,T)} k\left(\|\bar{y}\|,\|\bar{x}\|\right)\|\bar{y}-\bar{x}\|_{L^r\left(0,T;E^\alpha\right)}\\ &\le T^{\frac{1}{r}}\left\|g_1\right\|_{L^r(0,T)} \sup_{\|\bar{y}\|,\|\bar{x}\|\le a_1} k\left(\|\bar{y}\|,\|\bar{x}\|\right)\|\bar{y}-\bar{x}\|_{L^r\left(0,T;E^\alpha\right)}\\ &\le \alpha_1\|\bar{y}-\bar{x}\|_{L^r\left(0,T;E^\alpha\right)}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left\|\varphi\left(\bar{y}_1^0,\bar{y}\right)\right\|_{L^r\left(0,T;E^\alpha\right)} &= \left\|\bar{S}(.)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\bar{y}_1^0+\bar{S}(.)\tilde{y}_2^0+L(.)\bar{N}\bar{y}\right\|_{L^r\left(0,T;E^\alpha\right)}\\ &\le \left\|\bar{S}(.)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\bar{y}_1^0\right\|_{L^r\left(0,T;E^\alpha\right)}+\left\|\bar{S}(.)\tilde{y}_2^0\right\|_{L^r\left(0,T;E^\alpha\right)}+\left\|L(.)\bar{N}\bar{y}\right\|_{L^r\left(0,T;E^\alpha\right)}\\ &\le \left(\int_0^T\left\|\bar{S}(t)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\bar{y}_1^0\right\|_{E^\alpha}^r \mathrm{d}t\right)^{\frac{1}{r}}+\left(\int_0^T\left\|\bar{S}(t)\tilde{y}_2^0\right\|_{E^\alpha}^r \mathrm{d}t\right)^{\frac{1}{r}}+\left(\int_0^T\left\|L(t)\bar{N}\bar{y}\right\|_{E^\alpha}^r \mathrm{d}t\right)^{\frac{1}{r}},\end{aligned}$$

but we have

$$\left\|\bar{S}(t)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\bar{y}_1^0\right\|_{E^\alpha}\le\left\|\bar{S}(t)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\right\|_{\mathcal{L}\left(v,E^\alpha\right)}\left\|\bar{y}_1^0\right\|_v\le g_2(t)\left\|\bar{y}_1^0\right\|_v,$$

and

$$\left\|\bar{S}(t)\tilde{y}_2^0\right\|_{E^\alpha}\le\left\|\bar{S}(t)\right\|_{E^\alpha}\left\|\tilde{y}_2^0\right\|_E\le g_1(t)\left\|\tilde{y}_2^0\right\|_E.$$

Using Holder's inequality, we obtain

$$\begin{aligned}\left\|L(t)\bar{N}\bar{y}\right\|_{E^\alpha} &= \left\|\int_0^T\bar{S}(t-\tau)\bar{N}\bar{y}(\tau)\mathrm{d}\tau\right\|_{E^\alpha}\\ &\le \int_0^T\left\|\bar{S}(t-\tau)\right\|_{\mathcal{L}\left(E,E^\alpha\right)}\left\|\bar{N}\bar{y}(\tau)\right\|_E \mathrm{d}\tau\\ &\le \left\|g_1(t)\right\|_{L^r(0,T)}\left\|\bar{N}\bar{y}\right\|_{L^s\left(0,T;E^\alpha\right)}\\ &\le \left\|g_1(t)\right\|_{L^r(0,T)} k\left(\|\bar{y}\|,0\right)\|\bar{y}\|_{L^r\left(0,T;E^\alpha\right)}\\ &\le \left\|g_1(t)\right\|_{L^r(0,T)} \sup_{\theta\le a_1} k(\theta,0)\|\bar{y}\|_{L^r\left(0,T;E^\alpha\right)},\end{aligned}$$

then we have

$$\begin{aligned}\left\|\bar{S}(.)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\bar{y}_1^0\right\|_{\mathcal{L}\left(E,E^\alpha\right)} &= \left(\int_0^T\left\|\bar{S}(t)\bar{\nabla}^*\bar{\gamma}^*\bar{\chi}_\Gamma^*\bar{y}_1^0\right\|_{E^\alpha}^r \mathrm{d}t\right)^{\frac{1}{r}}\\ &\le \int_0^T g_2^r(t)\left\|\bar{y}_1^0\right\|_V^r \mathrm{d}t\\ &\le \left\|g_2(t)\right\|_{L^r(0,T)}\left\|\bar{y}_1^0\right\|_V,\end{aligned}$$

$$\begin{aligned}\left\|\bar{S}(.)\tilde{y}_2^0\right\|_{\mathcal{L}\left(E,E^\alpha\right)} &= \left(\int_0^T\left\|\bar{S}(t)\bar{y}_2^0\right\|_{E^\alpha}^r \mathrm{d}t\right)^{\frac{1}{r}}\\ &\le \left(\int_0^T g_1^r(t)\left\|\tilde{y}_2^0\right\|_E^r \mathrm{d}t\right)^{\frac{1}{r}}\\ &\le \left\|g_1(t)\right\|_{L^r(0,T)}\left\|\tilde{y}_2^0\right\|_E,\end{aligned}$$

and

$$\left\|L(.)\bar{N}\bar{y}\right\|_{\mathcal{L}\left(E,E^{\alpha}\right)}=\left(\int_0^T\left\|L(t)\bar{N}\bar{y}\right\|_{E^{\alpha}}^r\mathrm{d}t\right)^{\frac{1}{r}}$$
$$\leq T^{\frac{1}{r}}\left\|g_1(t)\right\|_{L^r(0,T)}\sup_{\theta\leq a_1}k(\theta,0)\left\|\bar{y}\right\|_{\mathcal{L}\left(0,T;E^{\alpha}\right)}$$
$$\leq\alpha_2\left\|\bar{y}\right\|_{\mathcal{L}\left(0,T;E^{\alpha}\right)},$$

where $\alpha_2=T^{\frac{1}{r}}\left\|g_1(t)\right\|_{L^r(0,T)}\sup_{\theta\leq a_1}k(\theta,0)$.

Finally

$$\left\|\varphi\left(\bar{y}_1^0,\bar{y}\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}\leq\left\|g_2(t)\right\|_{L^r(0,T)}\left\|\bar{y}_1^0\right\|_V+\left\|g_1(t)\right\|_{L^r(0,T)}\left\|\tilde{y}_2^0\right\|_E+\alpha_2\left\|\bar{y}\right\|_{\mathcal{L}\left(0,T;E^{\alpha}\right)}.$$

Let's consider $m=m(a_1)=\dfrac{a_1(1-\alpha_2)-\left\|\tilde{y}_2^0\right\|_E\left\|g_1(t)\right\|_{L^r(0,T)}}{\left\|g_2(t)\right\|_{L^r(0,T)}}$,

and $m_1=m_1(a_1)=\dfrac{a_1(1-\alpha_2)}{\left\|g_1(t)\right\|_{L^r(0,T)}}$, $1-\alpha_2>0$, then $m>0$.

It is sufficient to take $\left\|\tilde{y}_2^0\right\|_E<m_1$ and $\left\|\bar{y}_1^0\right\|_V<m$, then for all $\bar{y}\in B(0,a_1)$ we have $\varphi\left(\bar{y}_1^0,\bar{y}\right)\in B(0,a_1)$.

- Let $\bar{y}_1$ and $\bar{x}_1$ be the solution of the system (5) corresponding respectively to the initial gradient condition, we suppose that we have the same residual part ($\tilde{y}_2^0=\tilde{x}_2^0$), then for $\bar{y}_1^0,\bar{x}_1^0\in B(0,m)$ we have

$$\left\|f\left(\bar{y}_1^0\right)-f\left(\bar{x}_1^0\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}=\left\|\varphi\left(\bar{y}_1^0,\bar{y}\right)-\varphi\left(\bar{x}_1^0,\bar{x}\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}=\left\|\bar{y}-\bar{x}\right\|_{L^r\left(0,T;E^{\alpha}\right)}$$
$$=\left\|\varphi\left(\bar{y}_1^0,\bar{y}\right)-\varphi\left(\bar{x}_1^0,\bar{x}\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}$$
$$=\left\|\bar{S}(.)\bar{\nabla}^*\bar{\gamma}^*\chi_{\Gamma}^*\left(\bar{y}_1^0-\bar{x}_1^0\right)+L(.)\left(\bar{N}\bar{y}-\bar{N}\bar{x}\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}$$
$$\leq\left\|\bar{y}_1^0-\bar{x}_1^0\right\|_V+\left\|L(.)\left(\bar{N}\bar{y}-\bar{N}\bar{x}\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)},$$

but we have

$$\left\|L(.)\left(\bar{N}\bar{y}-\bar{N}\bar{x}\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}\leq T^{\frac{1}{r}}\left\|g_1\right\|_{L^r(0,T)}\left\|\bar{N}\bar{y}-\bar{N}\bar{x}\right\|_{L^s(0,T;E)}$$
$$\leq T^{\frac{1}{r}}\left\|g_1\right\|_{L^r(0,T)}\sup_{\theta_1,\theta_2\leq a_1}K(\theta_1,\theta_2)\left\|\bar{y}-\bar{x}\right\|_{L^r\left(0,T;E^{\alpha}\right)}$$
$$\leq\alpha_1\left\|\bar{y}-\bar{x}\right\|_{L^r\left(0,T;E^{\alpha}\right)},$$

and we deduce that

$$\left\|f\left(\bar{y}_1^0\right)-f\left(\bar{x}_1^0\right)\right\|_{L^r\left(0,T;E^{\alpha}\right)}\leq\frac{\left\|g_2\right\|_{L^r(0,T)}}{1-\alpha_1}\left\|\bar{y}_1^0-\bar{x}_1^0\right\|_V. \tag{13}$$

Finally, f is Lipschitzian in $B(0,m)$.

Remark 2 *The given results show that there exists a set of admissible gradient initial state. If the gradient initial state is taken in* $B(0,m)$, *with a bounded residual part then the system* (5) *has only one solution in* $B(0,a_1)$.

Here, we show that if the measurements are in $B(0,\rho)$, with ρ is suitably chosen then the gradient initial state can be obtained as a solution of a fixed point problem.

Let us consider the mapping

$$\varphi_1\left(\bar z,\bar y_1^0\right)=\left(\bar K\bar\nabla^*\bar\gamma^*\bar\chi_\Gamma^*\right)^\dagger\left(\bar z-\bar{CS}(.)\tilde y_2^0-\bar{CL}(.)\bar Nf\left(\bar y_1^0\right)\right), \tag{14}$$

and assume that $\forall\bar y_1^0\in V,\bar{CL}(.)\bar Nf\left(\bar y_1^0\right)\in\operatorname{Im}\left(\bar K\bar\nabla^*\bar\gamma^*\bar\chi_\Gamma^*\right)$.

Then we have the following result.

Proposition 2 *Assume that*

$$\exists c_1>0 \quad \text{such that} \quad \left\|\bar{CL}(.)\bar y\right\|_Y\le c_1\left\|\bar y\right\|_{L^s(0,T;E)} \quad \forall\bar y\in L^s\left(0,T;E\right), \tag{15}$$

$$\text{and} \quad \exists c_2>0 \quad \text{such that} \quad \left\|\bar{CS}(.)\bar y\right\|_Y\le c_2\left\|\bar y\right\|_E \quad \forall\bar y\in E \tag{16}$$

and if the linear system (2) is weakly G-observable on Γ and (11) holds, then there exists $a_2>0$ and $\rho=\rho(a_2)>0$, such that for all $\bar z\in B(0,\rho)\subset Y$, the function (14) admit a unique fixed point in $B(0,m)$ which correspond to the gradient initial condition $\bar y_0^1$ observed on Γ. Furthermore, the function $h:\bar z\in B(0,\rho)\to\bar y_1^0\in B(0,m)$ is Lipschitzian.

Proof. Let us consider $\bar y_1^0$ and $\bar x_1^0$ in $B(0,m)\subset V$, using ((9),(11), (13), (15) and (16)) we have

$$\begin{aligned}\left\|\varphi_1\left(\bar z,\bar y_1^0\right)-\varphi_1\left(\bar z,\bar x_1^0\right)\right\|_V&=\left\|\left(\bar K\bar\nabla^*\bar\gamma^*\bar\chi_\Gamma^*\right)^\dagger\left(\bar{CL}(.)\left(\bar Nf\left(\bar y_1^0\right)-\bar Nf\left(\bar x_1^0\right)\right)\right)\right\|_V\\&=\left\|\left(\bar K\bar\nabla^*\bar\gamma^*\bar\chi_\Gamma^*\right)\left(\bar K\bar\nabla^*\bar\gamma^*\bar\chi_\Gamma^*\right)^\dagger\left(\bar{CL}(.)\left(\bar Nf\left(\bar y_1^0\right)-\bar Nf\left(\bar x_1^0\right)\right)\right)\right\|_Y\\&\le c_1\left\|\bar Nf\left(\bar y_1^0\right)-\bar Nf\left(\bar x_1^0\right)\right\|_{L^s(0,T;E)}\\&\le c_1k\left(\left\|f\left(\bar y_1^0\right)\right\|,\left\|f\left(\bar x_1^0\right)\right\|\right)\left\|f\left(\bar y_1^0\right)-f\left(\bar x_1^0\right)\right\|_{L^r\left(0,T;E^\alpha\right)}.\end{aligned}$$

Or $\lim_{(\theta_1,\theta_2)\to(0,0)}k\left(\theta_1,\theta_2\right)=0$, then there exists $a_2>0$ such that

$$\alpha_3=c_1\sup_{\theta_1,\theta_2\le a_2}K\left(\theta_1,\theta_2\right)\frac{\left\|g_1\right\|_{L^r(0,T)}}{1-\alpha_1}<1,$$

and we have $\alpha_3\in\left]0,1\right[$.

Then we obtain

$$\left\|\varphi_1\left(\bar z,\bar y_1^0\right)-\varphi_1\left(\bar z,\bar x_1^0\right)\right\|_V\le\alpha_3\left\|\bar y_1^0-\bar x_1^0\right\|_V.$$

On the other hand, using the inequalities (11), (15) and (16), we have

$$\begin{aligned}\left\|\varphi_1\left(\bar z,\bar y_1^0\right)\right\|_V&=\left\|\bar z-\bar{CS}(.)\tilde y_2^0-\bar{CL}(.)\bar Nf\left(\bar y_1^0\right)\right\|_Y\\&\le\left\|\bar z\right\|_Y+\left\|\bar{CS}(.)\tilde y_2^0\right\|_Y+\left\|\bar{CL}(.)\bar Nf\left(\bar y_1^0\right)\right\|_Y\\&\le\left\|\bar z\right\|_Y+c_2\left\|\tilde y_2^0\right\|_E+c_1\left\|\bar Nf\left(\bar y_1^0\right)\right\|_{L^s(0,T;E)}\\&\le\left\|\bar z\right\|_Y+c_2\left\|\tilde y_2^0\right\|_E+c_1k\left(\left\|f\left(\bar y_1^0\right)\right\|,0\right)\left\|f\left(\bar y_1^0\right)\right\|_{L^r\left(0,T;E^\alpha\right)}\\&\le\left\|\bar z\right\|_Y+c_2\left\|\tilde y_2^0\right\|_E+c_1a_2\sup_{\theta\le a_2}k\left(\theta,0\right).\end{aligned}$$

Let's consider $\rho=\rho\left(a_2\right)=m-c_2\left\|\tilde y_2^0\right\|_E-c_1a_2\sup_{\theta\le a_2}k\left(\theta,0\right)$.

In order to have $\varphi_1\left(\bar z,\bar y_1^0\right)\in B\left(0,m\right)$, it suffices to consider $\left\|\bar z\right\|_Y\le\rho$.

For $\bar z_1,\bar z_2\in B\left(0,\rho\right)$, we have

$$h(\overline{z}_1)-h(\overline{z}_2)=\overline{y}_{1,1}^0-\overline{y}_{1,2}^0=\varphi_1(\overline{z}_1,\overline{y}_{1,1}^0)-\varphi_1(\overline{z}_2,\overline{y}_{1,2}^0)$$
$$=\varphi_1(\overline{z}_1,h(\overline{z}_1))-\varphi_1(\overline{z}_2,h(\overline{z}_2)),$$

which gives

$$\|h(\overline{z}_1)-h(\overline{z}_2)\|_V\le\|\varphi_1(\overline{z}_1,h(\overline{z}_1))-\varphi_1(\overline{z}_1,h(\overline{z}_2))\|_V+\|\varphi_1(\overline{z}_1,h(\overline{z}_2))-\varphi_1(\overline{z}_2,h(\overline{z}_2))\|_V$$
$$\le\alpha_3\|h(\overline{z}_1)-h(\overline{z}_2)\|_V+\|\overline{z}_1-\overline{z}_2\|_Y.$$

Then

$$\|h(\overline{z}_1)-h(\overline{z}_2)\|_V\le\frac{1}{1-\alpha_3}\|\overline{z}_1-\overline{z}_2\|_Y,$$

which shows that h is Lipschitzian.

Remark 3 *We can consider the regional intern problem in a subregion* ω *of* Ω (*see* [17]).

4. Numerical Approach

We show the existence of a sequence of the initial gradient state and initial gradient speed which converges respectively to the regional initial gradient state and initial gradient speed to be observed on Γ.

Proposition 3 *We suppose that the hypothesis of the proposition* (3.2) *are verified, then for* $\overline{z}\in B(0,\rho)$, *the sequence of the initial gradient condition defined in* $B(0,m)\subset V$ *by*

$$\begin{cases}\tilde{y}_{1,0}^0=0\\ \tilde{y}_{1,n+1}^0=\left(\overline{H}^*\right)^\dagger\left(\overline{z}-\overline{CS}(.)\tilde{y}_2^0-\overline{C}L(.)\overline{N}f\left(\tilde{y}_{1,n}^0\right)\right),\end{cases}\tag{17}$$

converges to $\overline{y}_1^0$ the regional initial gradient condition (the regional initial gradient state y_1^0 and the regional initial gradient speed y_1^1) to be observed on Γ. where $\tilde{y}_2^0$ is the residual part of the initial gradient condition.

Proof. We have,

$$\left\|\tilde{y}_{1,n+1}^0-\tilde{y}_{1,n}^0\right\|_V=\left\|\varphi_1\left(\overline{z},\tilde{y}_{1,n+1}^0\right)-\varphi_1\left(\overline{z},\tilde{y}_{1,n-1}^0\right)\right\|_V$$
$$\le\alpha_3\left\|\tilde{y}_{1,n}^0-\tilde{y}_{1,n-1}^0\right\|_V\le\alpha_3^2\left\|\tilde{y}_{1,n-1}^0-\tilde{y}_{1,n-2}^0\right\|_V$$
$$\le\cdots\le\alpha_3^n\left\|\tilde{y}_{1,1}^0-\tilde{y}_{1,0}^0\right\|_V=\alpha_3^n\left\|\tilde{y}_{1,1}^0\right\|_V,$$

or $\lim_{p\to+\infty}\alpha_3^p=0$, then $\exists n_0\in\mathbb{N}\quad\forall n\in\mathbb{N},\ \forall p\ge n_0$.

$$\left\|\tilde{y}_{1,n+p}^0-\tilde{y}_{1,p}^0\right\|_V\le\left\|\tilde{y}_{1,n+p}^0-\tilde{y}_{1,n+p-1}^0\right\|_V+\left\|\tilde{y}_{1,n+p-1}^0-\tilde{y}_{1,n+p-2}^0\right\|_V+\left\|\tilde{y}_{1,n+p-2}^0-\tilde{y}_{1,n+p-3}^0\right\|_V+\cdots+\left\|\tilde{y}_{1,p+1}^0-\tilde{y}_{1,p}^0\right\|_V$$
$$\le\alpha_3^{n+p-1}\left\|\tilde{y}_{1,1}^0\right\|_V+\alpha_3^{n+p-2}\left\|\tilde{y}_{1,1}^0\right\|_V+\alpha_3^{n+p-3}\left\|\tilde{y}_{1,1}^0\right\|_V+\cdots+\alpha_3^p\left\|\tilde{y}_{1,1}^0\right\|_V$$
$$=\alpha_3^p\left\|\tilde{y}_{1,1}^0\right\|_V\sum_{j=0}^{n-1}\alpha_3^j=\alpha_3^p\left\|\tilde{y}_{1,1}^0\right\|_V\left(\frac{1-\alpha_3^n}{1-\alpha_3}\right)\le\alpha_3^p\left\|\tilde{y}_{1,1}^0\right\|_V\left(\frac{1}{1-\alpha_3}\right).$$

Then $\left(\tilde{y}_{1,n}^0\right)_{n\ge0}$ is a Cauchy sequence on V and is convergent.

We consider $\overline{y}_n=f\left(\tilde{y}_{1,n}^0\right)$ and $\overline{z}_n=\overline{C}\overline{y}_n$ with

$$\begin{cases}\overline{y}_n=\overline{S}(.)\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_\Gamma^*\tilde{y}_{1,n}^0+\overline{S}(.)\tilde{y}_2^0+L(.)\overline{N}f\left(\tilde{y}_{1,n}^0\right)\\ \overline{z}_n=\overline{H}^*\tilde{y}_{1,n}^0+\overline{CS}(.)\tilde{y}_2^0+\overline{C}L(.)\overline{N}f\left(\tilde{y}_{1,n}^0\right),\end{cases}$$

we have $\left(\overline{H}^*\right)^\dagger\left(\overline{z}-\overline{z}_n\right)=\tilde{y}_{1,n+1}^0-\tilde{y}_{1,n}^0$.

So

$$\overline{z}-\overline{z}_n=\overline{z}-\overline{CS}(.)\tilde{y}_2^0-\overline{C}L(.)\overline{N}f\left(\tilde{y}_{1,n}^0\right)-\overline{H}^*\tilde{y}_{1,n}^0=\overline{H}^*\left(\tilde{y}_{1,n+1}^0-\tilde{y}_{1,n}^0\right),$$

then

$$\left\|\overline{z}-\overline{z}_n\right\|_Y=\left\|\tilde{y}^0_{1,n+1}-\tilde{y}^0_{1,n}\right\|_V\le\alpha_3^n\left\|\tilde{y}^0_{1,1}\right\|_V,$$

which show that the sequence $\overline{z}_n$ converges to $\overline{z}$ in Y on the other hand, we have

$$\left\|\overline{y}^0_1-\tilde{y}^0_{1,n}\right\|_V=\left\|h(\overline{z})-h(\overline{z}_n)\right\|_V\le\frac{1}{1-\alpha_3}\left\|\overline{z}-\overline{z}_n\right\|_Y,$$

hence $\tilde{y}^0_{1,n}$ converges to the regional initial gradient $\overline{y}^0_1$ to be observed on Γ.

Algorithm

Let's consider $r_{n+1}=\overline{z}-\overline{CS}(.)\tilde{y}^0_2-\overline{C}L(.)\overline{N}f\left(\tilde{y}^0_{1,n}\right)$, then we have

$$\tilde{y}^0_{1,n+1}=\left(\overline{H}^*\right)^\dagger r_{n+1}\quad\text{and}\quad\overline{z}_n=\overline{H}^*\tilde{y}^0_{1,n}+\overline{z}-r_{n+1}=r_n+\overline{z}-r_{n+1}.$$

Thus, we obtain the following algorithm:

Step 1 : The initial condition $\bar{y}^0_1=(y^0,y^1)$, the subregion Γ, The domain D and the function of distribution f and the accuracy threshold ε,
$r_1=\bar{z}-\bar{C}\bar{S}(.)\tilde{y}^0_2-\bar{C}L(.)\bar{N}f(\tilde{y}^0_{1,0})$.

Step 2 : Repeat

$\ominus$ $\tilde{y}^0_{1,n}=(\bar{H}^*)^\dagger r_n$

$\ominus$

$$\begin{cases}\dfrac{\partial\bar{y}_n(t)}{\partial t} & =\bar{\mathcal{A}}\bar{y}_n(t)+\bar{\mathcal{N}}\bar{y}_n(t)\\ \bar{y}_n(0) & =\overline{\nabla}^*\bar{\gamma}^*\bar{\chi}^*_\Gamma\tilde{y}^0_{1,n}+\tilde{y}^0_2\end{cases}$$

$\ominus$ $\bar{z}_n=\bar{C}\bar{y}_n$

Until $\|\bar{z}-\bar{z}_n\|_Y\le\varepsilon$.

Step 3 : $\tilde{y}^0_{1,n}$ which corresponds to the initial gradient condition to be observed $\bar{y}^0_1$ on Γ.

else $r_{n+1}=r_n+\bar{z}+\bar{z}_n$ and go to step 2.

5. Simulations

In this part, we give a numerical illustrating example and the simulations are related to the choice of the subregion, the sensor location.

5.1. Internal Subregion Target

Consider the one dimensional semilinear hyperbolic system

$$\begin{cases}\dfrac{\partial^2y}{\partial t^2}(x,t)=\dfrac{\partial^2y}{\partial x^2}(x,t)+\sum_{k=1}^{\infty}\left|\langle y(t),\varphi_k\rangle\right|\langle y(t),\varphi_k\rangle\varphi_k(x) & \text{in }]0,1[\times]0,T[\\ y(x,0)=y^0(x),\dfrac{\partial y}{\partial t}(x,0)=y^1(x) & \text{in }]0,1[\\ y(\xi,t)=0\quad\text{on }]0,T[,\end{cases}\tag{18}$$

augmented with the output function described by a pointwise sensor located in $b=0.20$ and $T=2$

$$z(t)=y(b,t).\tag{19}$$

where $(\varphi_k)_{k\ge1}$ is a complete set of $L^2(\Omega)$. Let's consider

$y^0_1=0.4\log(0.8x+1)(1-x)+0.5$ the initial gradient state initial,

$y^1_1=(x(1-x))^2+0.4$ the initial gradient speed to be observed in a subregion $\omega=]0.4,0.5[$.

Using the previous algorithm, we obtain the following figures.

- **Figure 1** shows that the estimate gradient state is very close to the real initial gradient state in Ω.
- **Figure 2** shows that the estimate gradient speed is very close to the real initial gradient speed in Ω.

5.2. Boundary Subregion Target

Consider the two dimensional system described in $\Omega =]0,1[\times]0,1[$ by

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x_1,x_2,t) = 0.01\left[\dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t)\right] + \sum_{i,j=1}^{\infty}\left|\langle y,\varphi_{ij}\rangle\right|\langle y,\varphi_{ij}\rangle\varphi_{ij} & \text{in }]0,1[\times]0,T[\\ y(x_1,x_2,0) = y^0(x_1,x_2), \dfrac{\partial y}{\partial t}(x_1,x_2,0) = y^1(x_1,x_2) & \text{in }]0,1[\\ y(\xi,\eta,t) = 0 & \text{on }]0,T[, \end{cases} \tag{20}$$

where $(\varphi_{ij})_{ij}$ is a complete set of $H^1(\Omega)$.

The system (20) augmented by output function described by a pointwise sensor located in *b*.

$$z(t) = y(b,t), \quad t \in]0,T[, \tag{21}$$

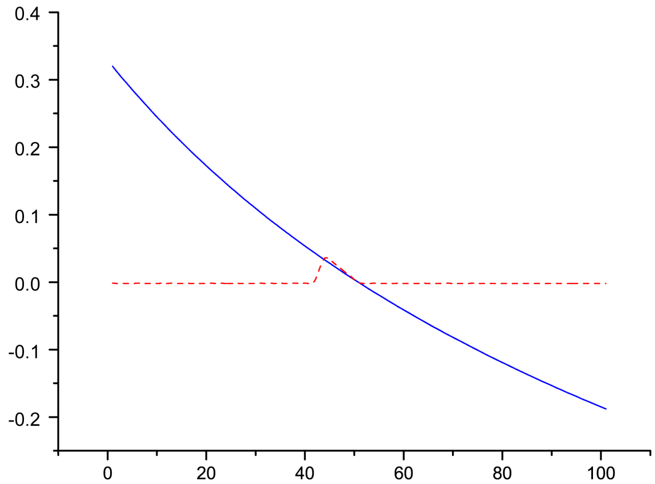

Figure 1. The estimated initial gradient state in ω.

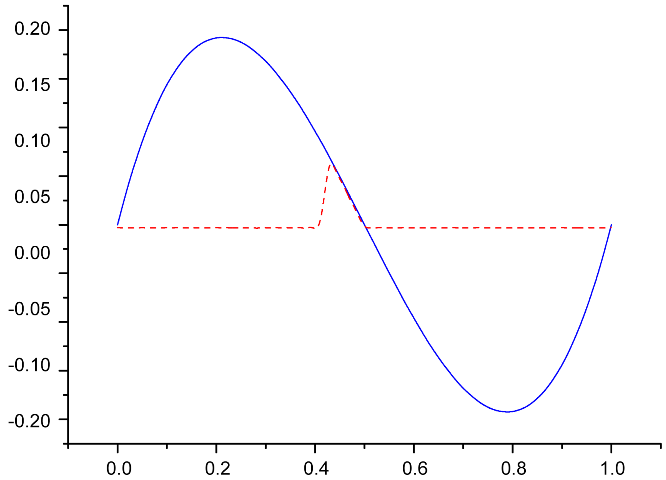

Figure 2. The estimated initial gradient speed in ω.

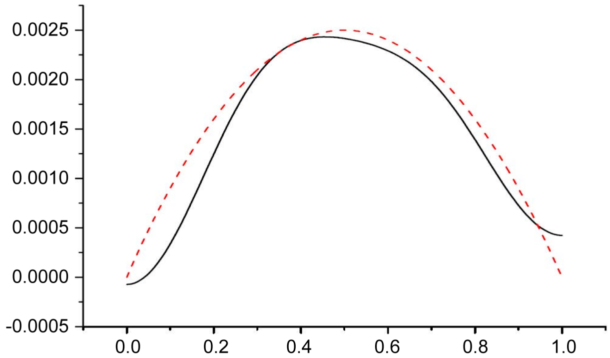

Figure 3. The estimated initial gradient state on Γ.

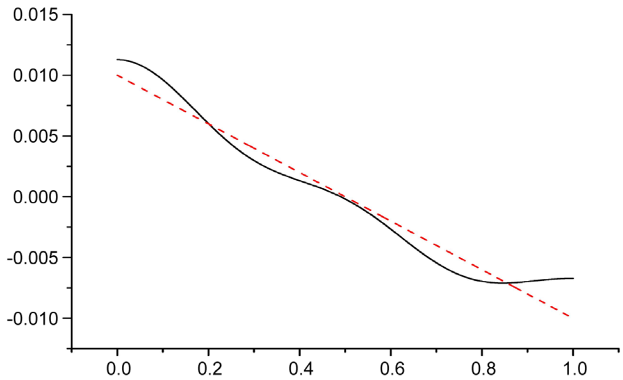

Figure 4. The estimated initial gradient speed on Γ.

with

- $T = 2$, the sensor located at $b = (0.3, 0.6)$.
- $\omega =]0, 0.2[\times]0,1[$ is the intern region.
- $\Gamma = \{0\} \times]0,1[$ is the boundary region.
- The initials gradient conditions

$$\begin{cases} \nabla y^0(x_1, x_2) = \big((2x_1 - 1)x_2(x_2 - 1); (2x_2 - 1)x_1(x_1 - 1)\big) \\ \nabla y^1(x_1, x_2) = \big((1 - x_1)(x_1 - 1)(2x_2 - 1); (1 - x_2)(x_2 - 1)(2x_1 - 1)\big), \end{cases}$$

to be observed on Γ.

Using the previous algorithm, we obtain the following results:

- **Figure 3** shows that the estimate boundary gradient state is very close to the real initial boundary gradient state on Γ.
- **Figure 4** shows that the estimate boundary gradient speed is very close to the real initial boundary gradient speed on Γ.

6. Conclusion

The question of the regional internal and boundary gradient observability for semilinear hyperbolic systems was discussed and solved using the sectorial approach, which used sectorial properties of dynamical operators, the fixed point techniques and the properties of the linear part of the considered system. Many questions remain open, such as the case of the regional gradient observability of semilinear systems using Hilbert Uniqueness Method (HUM) and the constrained observability of semilinear hyperbolic system.

References

[1] Zerrik, E., Bourray, H. and Boutoulout, A. (2002) Regional Boundary Observability, Numerical Approach. *Interna-*

tional Journal of Applied Mathematics and Computer Science, **12**, 143-151.

[2] Zerrik, E., Bourray, H. and El Jai, A. (2007) Regional Observability for Semilinear Distributed Parabolic Systems. *Journal of Dynamical and Control Systems*, **3**, 413-430.

[3] Boutoulout, A., Bourray, H. and El Alaoui, F.Z. (2013) Boundary Gradient Observability for Semilinear Parabolic Systems: Sectorial Approach. *Mathematical Sciences Letters*, **2**, 45-54. http://dx.doi.org/10.12785/msl/020106

[4] Boutoulout, A., Bourray, H. and El Alaoui, F.Z. (2012) Regional Gradient Observability for Distributed Semilinear Parabolic Systems. *Journal of Dynamical and Control Systems*, **18**, 159-179. http://dx.doi.org/10.1007/s10883-012-9138-3

[5] Zerrik, E. and Bourray, H. (2003) Gradient Observability for Diffustion System. *International Journal of Applied Mathematics and Computer Science*, **13**, 139-150.

[6] Zerrik, E., Bourray, H. and El Jai, A. (2003) Regional Flux Reconstruction for Parabolic Systems. *International Journal of Systems Science*, **34**, 641-650. http://dx.doi.org/10.1080/00207720310001601028

[7] Boutoulout, A., Bourray, H., Benhadid, S. and El Alaoui, F.Z. (2014) Regional Gradient Observability for Distributed Semilinear Parabolic Systems. *International Journal of Control*, **87**, 1-13.

[8] Boutoulout, A. and Khazari, A. (2013) Gradient Observability for Hyperbolic System. *International Review of Automatic Control* (*IREACO*), **6**, 263-274.

[9] Boutoulout, A., Bourray, H. and Khazari, A. (2014) Flux Observability for Hyperbolic Systems. *Applied Mathematics & Information Sciences*, **2**, 13-24.

[10] Pazy, A. (1990) Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York.

[11] Zeidler, E. (1990) Nonlinear Functional Analysis and Its Applications II/A Linear Applied Functional Analysis. Springer, Berlin.

[12] Curtain, R.F. and Zwart, H. (1995) An Introduction to Infinite Dimensional Linear Systems Theory. Texts in Applied Mathematics, Springer-Verlag, New York, 138. http://dx.doi.org/10.1007/978-1-4612-4224-6

[13] Lions, J.L. (1988) Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués. Tome 1, Masson, Paris.

[14] Henry, D. (1981) Geometric Theory of Semilinear Parabolic Systems. *Lecture Notes in Mathematics* 840, Springer-Verlag Berlin Heidelberg New York. http://dx.doi.org/10.1007/BFb0089647

[15] Kassara, K. and El Jai, A. (1983) Algorithme pour la commande d'une classe de systmes paramtre rpartis non linaires. *Revue Marocaine d'Automatique, d'Informatique et de Traitement de Signal*, **1**, 3-24.

[16] Lions, J.L. and Magenes, E. (1968) Problèmes aux limites non homogènes et applications. Vols. 1 et 2, Dunod, Paris.

[17] Khazari, A. and Boutoulout, A. (2014) Gradient Observability for Semilinear Hyperbolic Systems: Sectorial Approach. *Intelligent Control and Automation*, **6**, 170-181. http://dx.doi.org/10.4236/ica.2014.53019