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Dynamics of Income Distribution—A Diffusion Analysis

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Abstract

The study is motivated by the observation that the distribution of income across countries varies as a function of time. It would not be unreasonable to assume that there exists a statistical equilibrium distribution of income with a certain mean and variance, towards which the ensemble of countries considered tend to converge, and there is a speed of adjustment towards this said equilibrium. In order to quantify this process, the evolution through time of income around its trend is modeled using a classic stochastic differential equation. The model describes the diffusion of shocks across space, via an income adjustment process with noise. The dynamics rely on two opposing flows: (i) a factor equalization process, and (ii) a counteracting diffusion process. It is hypothesized that these flows follow simple evolutionary laws that can be described with five parameters—parameters that can be estimated from historical data with some accuracy. The dynamic behavior of the model is analytically derived. Both the extent and speed of adjustment of income are analyzed. An empirical application of the proposed model to the evolution of the distribution of income for 25 countries in the European Union tests the validity of the proposed method and suggests that diffusion may be a preferable technique for the analysis of income dynamics.

Keywords: Cross-sectional Distribution of Income, Diffusion

1. Introduction

While much attention has been devoted in the economics literature to the explanation of the shape of income distribution at a given point in time by reference to steady state arguments [1-3], the dynamics in question have been relatively ignored. The main objective of this paper is to help fill this gap by introducing a diffusion model that allows to incorporate dynamics which are typically neglected when looking at convergence of incomes. We explore a representation for the dynamics in the evolving distributions where the growth distribution of incomes can be generated by a single stochastic process in which income follows a Brownian motion. An empirical application to personal income for 25 countries in the European Union helps fill a second gap in the literature, as only few diffusion studies have employed real statistical data when analyzing income dynamics.

2. Theoretical Framework

Consider a region consisting of a constant number of countries with different levels of personal income (wages). The set of personal incomes forms a distribution which evolves over time. Given wage differentials, labor (capital) migrates from low-wage (high-wage) to high wage (low-wage) countries. As capital/labor ratio declines in high-wage countries, wage growth decelerates and the opposite in low-wage areas. This process is reinforced as technology and knowledge capital flow from rich to poor countries [4]. A counteracting force exists, in the form of economic, political and institutional blockages and faster population growth which cause bottlenecks and generate divergence in the system [5,6]. In view of the above, it is assumed that there exists an equilibrium distribution of personal income with a certain unknown mean and variance, determined by the tension between counteracting forces of convergence (migration of labor and flow of ideas) and divergence (bottlenecks to flow of labor, capital and ideas). Given an exogenous shock, the ensemble of country incomes considered tend to converge to this long-run equilibrium[1].

[1]Equilibrium in this paper refers to a statistical equilibrium, which is characterized by a stationary probability distribution of personal incomes. This equilibrium can also be associated with level of personal income which is in line with potential incomes and output which in themselves are due to natural, technological and institutional constraints.

3. The Model

A classic attempt to describe the distribution of income is due to Gibrat [7], where it is proposed that income is governed by multiplicative random processes, with luck having a permanent effect[2]. It would not be unreasonable however, to assume that due to competition amongst suppliers and demanders of labor, labor which is paid an unreasonably low wage would search and find a better-paid job, and labor who is paid an unreasonably high wage, would be pressured to accept a lower wage, or risk getting replaced by someone who is willing to accept the lower wage. Thus, one might include a drift term in the model, representing mean reversion. In general, one can study a Markov process generated by a matrix of transitions from one income to another, where the Markov process can be treated as income diffusion. Then one can apply the general Fokker-Planck equation to describe evolution in time of personal income. Hence, assuming that personal income behaves like a stochastic process and that it is continuous and Markovian, we consider the most natural candidate; a classical linear stochastic differential equation driven by Gaussian white noise:

$$dS_t + \lambda\left(u - S_t\right) d_t = \sqrt{2\varepsilon} dB_t \quad (1)$$

where S_t is personal income. λ denotes velocity of adjustment to stationary equilibrium interpreted as income adjustment rate[3], u denotes the mean of the stationary equilibrium distribution, $\varepsilon > 0$ is a constant diffusion parameter, and B_t is the Brownian motion.

More precisely, for the drift spread, it is assumed that there exists some equilibrium distribution of wages with a certain mean and variance, towards which the ensemble of agents gravitate. Drift is driven by diminishing returns to capital, which in turn leads to factor price equalization. For the diffusion spread, noise is generated by diffusion of knowledge and learning [10], [11], and limited by the presence of obstacles in the form of trade barriers and such. Random effects cause a spread of personal income from high density towards lower density. This happens as a result of learning and the phenomenon of catch up, themselves due to economic integration [12]. The income adjustment process is thus interpreted as depending on learning speed which in turn is proportional to the mobility of factors of production and the speed with which diminishing returns set in[4]. This learning process generates randomness in the system [13].

[2]See [8] and [9] for a review.
[3]For simplicity we assume this rate to be constant.
[4]Convergence in the context of our model would mean collapsing of the cross-sectional distribution.

3.1. Analysis of the Model

The second order partial differential equation associated with equation (1) can be expressed by:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial s}\left(\lambda\left(u - s\right) f\right) = \varepsilon \frac{\partial^2 f}{\partial s^2} \quad (2)$$

where f denotes probability density and s denotes personal income[5].

The time-development of the distribution can be expressed by:

$$f\left(s,t\right) = Ne^{\lambda t}\sqrt{\frac{a}{a+\beta}}e^{-\frac{(s-ut)^2}{2\sigma_t^2}} \quad (3)$$

where

$$a = \frac{\sigma_0^2}{2}$$

$$\beta = \frac{\varepsilon}{2\lambda}\left(e^{2\lambda t} - 1\right)$$

$$ut = E\left[f\right]_t = u\left(1 - e^{-\lambda t}\right) + u_0 e^{-\lambda t}$$

$$\sigma_t^2 = \sigma_0^2 e^{-2\lambda t} + \frac{\varepsilon}{\lambda}\left(1 - e^{-2\lambda t}\right)$$

u_0 represents the initial mean of personal income distribution and σ_0^2 represents the initial variance of the distribution and N is the normalization constant[6].

4. Empirical Application

4.1. Data and Descriptive Statistics

Our data consists of labor compensation per hour of work ($US PPP adjusted)[7], for 25 countries in the European Union, recorded from 1976 up to 2007. Data was compiled from Eurostat. The countries are: Austria, Belgium, Bulgaria, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Latvia, Lithuania, Luxembourg, Netherlands, Poland, Portugal, Slovakia, Slovenia, Spain, Sweden, United Kingdom. **Table 1** shows the descriptive of log labor compensation per hour of work.

[5]The process derived from the diffusion model evolves according to an Ornstein-Uhlenbeck, but with a transition, such that the mean tends to u, instead of 0. This type of model has been widely used in Biomathematics [14], [15], [16], [17], [18] and [19].
[6]By considerations of analytic tractability, the initial distribution is approximated to be normal.
[7]Controlling for inflation could produce different results.

Table 1. Descriptive analysis.

	N	Years	Minimum	Maximum	Mean	Std. Deviation
All data	516	1976-2007	8.484	10.986	9.962	0.546
Used for fit	498	1978-2007	8.484	10.986	9.991	0.533

Table 2. Parameter estimates.

Parameter	Value	Std Error	t-value
λ	0.182	0.042	4.321
u	10.113	0.040	248.264
u_0	8.882	0.204	43.410
σ_o^2	−0.084	0.058	−1.450
ε	0.008	0.002	2.844

4.2. Estimation

A second order partial differential equation model has been proposed to express the dynamics of personal income. The model has five parameters: u_0, u, ε, σ_0^2, and λ. u_0 denotes the initial mean of the distribution, and u denotes where the initial mean is heading. σ_0 is the standard deviation at time zero, ε represents the diffusion parameter, and λ determines the income adjustment rate. To examine the behavior of the model as $t\to\infty$, one observes that $f(s,t)\to f_\infty(s)$ where

$$f(s,t)\to t_{\to\infty} N\sqrt{\frac{2a\lambda}{\varepsilon}}e^{-\frac{(s-u)^2}{2\frac{\varepsilon}{\lambda}}} = f_\infty(s) \qquad (4)$$

and that $\lim_{t\to\infty} E[f]_t = u$

The model has been applied to the wage distribution of the population as a function of time. **Tables 2** reports estimates for the five model parameters u_0, u, ε, σ_0^2, and λ, along with the standard errors and t-values, using the first and second moments of the distribution.

Figure 1 graphically illustrates the evolution of the density over time, superimposed on histograms which describe the time evolution of the distribution in the data (for selected years). The solid and dotted curves in this Figure illustrate the distributions as predicted by the model and data respectively. The vertical axes denote frequency, and the horizontal axes measure wages in logarithms.

The following observations can be made concerning our results:

1). The mean and variance of the distribution are clearly evolving, corresponding to our theoretical predictions.

2). The value for the income adjustment rate λ is positive as expected.

3). The value for the diffusion parameter ε is small and positive as expected.

4). The diffusive limit, *i.e.*, the limit as $t\to\infty$ of the variance is: $\lim_{t\to\infty}\sigma_t^2 = \varepsilon/\lambda$ The results predict that if we start with a normal distribution and let the model drive the distribution, the distribution variance will tend toward a constant ε/λ and concentrated around a mean u.

5. Final Remarks

A methodology has been proposed which is a more transparent way to quantify the dynamics of income, as it avoids the complications associated with dynamic inference and statistical regression fallacy inherent in standard cross-section tests [20-22]. The present study is in the spirit of probabilistic models of Krugman [23] who studies city sizes, Axtell [24] and Hashemi [25,26] who study firm sizes, and Hashemi [27,28] who studies income and rate of unemployment. Our suggestion is that these models could provide interesting insights as well, if applied to spatial dynamics of personal income.

One fruitful extension of the present study would be to relax the homogeneity assumptions of the model parameters. Another interesting extension would be to examine if the driving forces of convergence and divergence are the same for other components of income such as profit, interest and rent. We hope the present study illustrates that the endeavor is promising.

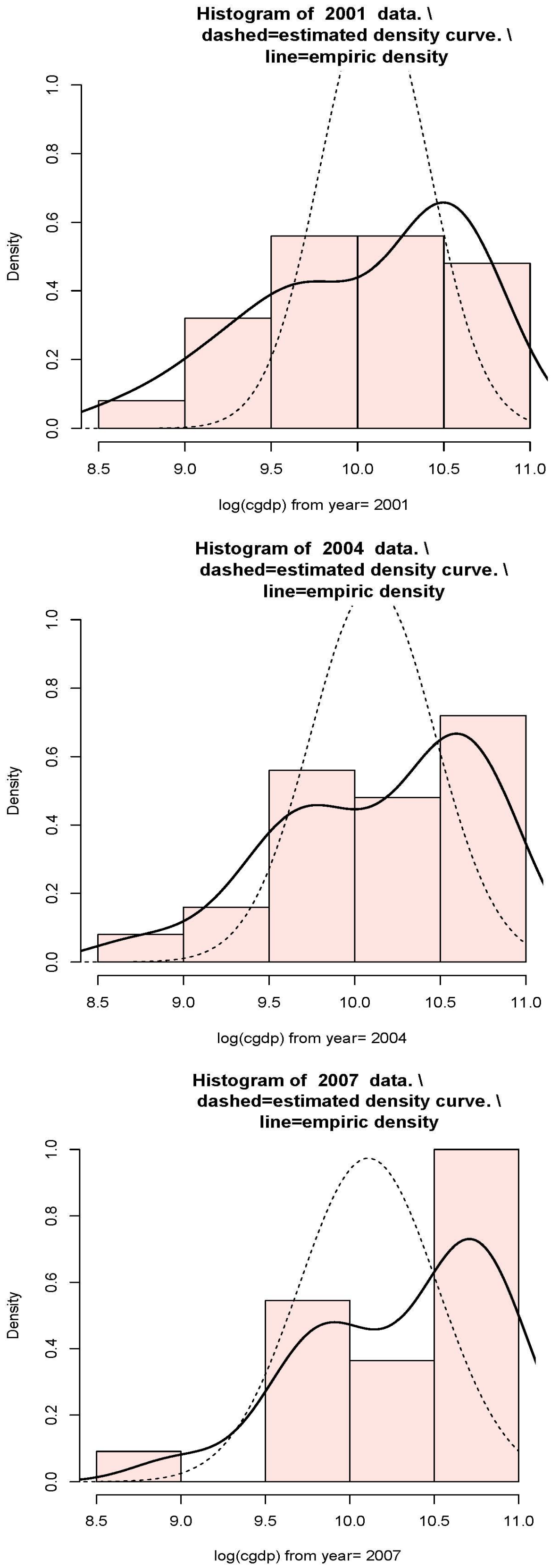

Figure 1. Actual vs. predicted distribution.

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