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About the Riemann Hypothesis

Jinhua Fei

Changling Company of Electronic Technology, Baoji, China
Email: feijinhuayoujian@msn.com

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Abstract

The Riemann hypothesis is part of Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems. It is also one of the Clay Mathematics Institute's Millennium Prize Problems. Some mathematicians consider it the most important unresolved problem in pure mathematics. Many mathematicians made a lot of efforts; they don't have to prove the Riemann hypothesis. In this paper, I use the analytic methods to deny the Riemann Hypothesis; if there's something wrong, please criticize and correct me.

Keywords

Riemann Hypothesis, Disavowal

1. Introduction

Riemann Hypothesis was posed by Riemann in early 50's of the 19th century in his thesis titled "The Number of Primes Less than a Given Number". It is one of the unsolved "super" problems of mathematics. The Riemann Hypothesis is closely related to the well-known Prime Number Theorem. The Riemann Hypothesis states that all the nontrivial zeros of the zeta-function lie on the "critical line" $\left\{s:\operatorname{Re}s=\frac{1}{2}\right\}$. In this paper, we use the analytical methods, and refute the Riemann Hypothesis. For convenience, we will abbreviate the Riemann Hypothesis as RH.

2. Some Theorems in the Classic Theory

In this paper, $\Gamma(s)$ is the Euler gamma function, $\zeta(s)$ is the Riemann zeta function.

Lemma 2.1. If $\operatorname{Re}w>0$, then

$$\frac{1}{2\pi i}\int_{(2)}\Gamma(s)w^{-s}\mathrm{d}s=\exp(-w)$$

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where Re w is the real part of complex number w.

Let $\eta > 0$ be given, when $|s| \ge \eta$ and $|\arg s| \le \pi - \eta$, then

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right).$$

If $-4 \le \sigma \le 4, |t| \ge 1$, then

$$\Gamma(\sigma + it) = \sqrt{2\pi}|t|^{\sigma - \frac{1}{2}} \exp\left(-\frac{\pi}{2}|t| + it(\log|t| - 1) + i\lambda\frac{\pi}{2}\left(\sigma - \frac{1}{2}\right)\right) + O\left(|t|^{\sigma - \frac{3}{2}} \exp\left(-\frac{\pi}{2}|t|\right)\right)$$

where $\lambda = 1$ if $t \ge 1$, $\lambda = -1$ if $t \le -1$.

See [1] page 523, page 525.

Lemma 2.2. If $\operatorname{Re} s > 1$, then

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n)$ is the Mangoldt function.

Let s is any complex number, we have

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + c_1 + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}s + 1\right)$$

where ρ be the nontrivial zeros of $\zeta(s)$, c_1 be the positive constant.

We write $s = \sigma + it$. If $-1 \le \sigma \le 2, -\pi < \operatorname{Im}\{\log(s-1)\} \le \pi$, $-\pi < \operatorname{Im}\{\log(s-\rho)\} \le \pi$, then

$$\log\zeta(s) = -\log(s-1) + \sum_{|\gamma - t| \le 1} \log(s-\rho) + O(\log(|t| + 2))$$

where Im s is the imaginary part of complex number s.

See [2] page 4, page 31, page 218.

Lemma 2.3. Let $N(T)$ is the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1, 0 < t < T$. then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

where $S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right)$.

See [3] page 98.

Lemma 2.4. Assume that RH, If $x \ge 2$, then

$$\psi(x) = \sum_{2 \le n \le x} \Lambda(n) = x + R(x)$$

where $R(x) \ll x^{\frac{1}{2}} \log^2 x$.

See [3] page 113.

3. Some Preparation Work

Lemma 3.1. Assume that RH, and $0 < \delta \le \frac{1}{50}$, then

$$\int_{\frac{1}{2}+\delta}^{2} |\log\zeta(\sigma + i\gamma_0)| \mathrm{d}\sigma \ll 1 \quad \text{and} \quad \int_{-1}^{\frac{1}{2}-\delta} |\log\zeta(\sigma + i\gamma_0)| \mathrm{d}\sigma \ll 1$$

where γ_0 is the ordinate of nontrivial first zero of $\zeta(s)$, $\gamma_0 \approx 14.134725\cdots$.

Proof. By Lemma 2.2 and RH, we have

$$\log\zeta(\sigma+i\gamma_0)\ll\sum_{|\gamma-\gamma_0|\le 1}\left|\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)\right|+O(\log\gamma_0)$$

because

$$\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)=\frac{1}{2}\log\left(\left(\sigma-\frac{1}{2}\right)^2+(\gamma_0-\gamma)^2\right)+i\,Arg\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)$$

$$\left|\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)\right|\ll\left|\log\left(\left(\sigma-\frac{1}{2}\right)^2+(\gamma_0-\gamma)^2\right)\right|+1$$

and

$$\log\left(\sigma-\frac{1}{2}\right)^2\le\log\left(\left(\sigma-\frac{1}{2}\right)^2+(\gamma_0-\gamma)^2\right)\le\log\left(\frac{9}{4}+1\right)$$

$$\left|\log\left(\left(\sigma-\frac{1}{2}\right)^2+(\gamma_0-\gamma)^2\right)\right|\ll\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|+1$$

therefore

$$\left|\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)\right|\ll\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|+1.$$

And because

$$\begin{aligned}\int_{\frac{1}{2}+\delta}^{2}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|\mathrm{d}\sigma&=\int_{\frac{1}{2}+\delta}^{\frac{3}{2}}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|\mathrm{d}\sigma+\int_{\frac{3}{2}}^{2}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|\mathrm{d}\sigma\\&=-2\int_{\frac{1}{2}+\delta}^{\frac{3}{2}}\log\left(\sigma-\frac{1}{2}\right)\mathrm{d}\sigma+2\int_{\frac{3}{2}}^{2}\log\left(\sigma-\frac{1}{2}\right)\mathrm{d}\sigma\\&=-2\int_{\delta}^{1}\log\sigma\,\mathrm{d}\sigma+2\int_{1}^{\frac{3}{2}}\log\sigma\,\mathrm{d}\sigma\\&=2\delta\log\delta+2\int_{\delta}^{1}\mathrm{d}\sigma+O(1)=O(1)\end{aligned}$$

therefore

$$\begin{aligned}\int_{\frac{1}{2}+\delta}^{2}\left|\log\zeta(\sigma+i\gamma_0)\right|\mathrm{d}\sigma&\ll\sum_{|\gamma-\gamma_0|\le 1}\int_{\frac{1}{2}+\delta}^{2}\left|\log\left(\sigma-\frac{1}{2}+i\gamma_0-i\gamma\right)\right|\mathrm{d}\sigma+1\\&\ll\int_{\frac{1}{2}+\delta}^{2}\left|\log\left(\sigma-\frac{1}{2}\right)^2\right|\mathrm{d}\sigma+1\ll 1.\end{aligned}$$

Similarly, we have

$$\int_{-1}^{\frac{1}{2}-\delta}\left|\log\zeta(\sigma+i\gamma_0)\right|\mathrm{d}\sigma\ll 1.$$

This completes the proof of Lemma 3.1.

Throughout the paper, we write

$$z=a+\mathrm{i}b,\quad a=\frac{1}{T},\quad T\ge 50,\quad b=2\pi.$$

It is easy to see that

$$\operatorname{arctg}\frac{b}{a}=\frac{\pi}{2}-h,\qquad h=\sum_{k=0}^{\infty}(-1)^k\frac{a^{2k+1}}{(2k+1)b^{2k+1}},\qquad \frac{1}{4\pi T}\le h\le\frac{1}{\pi T}.$$

Lemma 3.2. We calculate the three complex numbers.
Because

$$a+ib=\left(a^2+b^2\right)^{\frac{1}{2}}\exp\left(i\operatorname{arctg}\frac{b}{a}\right)=\left(a^2+b^2\right)^{\frac{1}{2}}\exp\left(i\frac{\pi}{2}-ih\right)$$

therefore when t is the real number, we have

$$z^{\frac{3}{4}-it}=\left(a^2+b^2\right)^{\frac{3}{8}-i\frac{t}{2}}\exp\left(i\frac{3\pi}{8}-i\frac{3}{4}h+\frac{\pi}{2}t-th\right)\ll\exp\left(\frac{\pi}{2}t-th\right)$$

$$z^{-\frac{1}{2}-it}=\left(a^2+b^2\right)^{-\frac{1}{4}-i\frac{t}{2}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}+\frac{\pi}{2}t-th\right)\ll\exp\left(\frac{\pi}{2}t-th\right)$$

$$z^{-\frac{1}{2}+it}=(a+ib)^{-\frac{1}{2}+it}=\left(a^2+b^2\right)^{-\frac{1}{4}+i\frac{t}{2}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}-\frac{\pi}{2}t+th\right)\ll\exp\left(-\frac{\pi}{2}t+th\right)$$

the three complex numbers required below.
Lemma 3.3.

$$\int_{\left(-\frac{3}{4}\right)}\Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}\,\mathrm{d}s\ll 1$$

Proof. By Lemma 2.1 and Lemma 3.2, we have

$$\begin{aligned}\int_{\left(-\frac{3}{4}\right)}\Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}\,\mathrm{d}s&=i\int_{-\infty}^{+\infty}\Gamma\left(-\frac{3}{4}+it\right)\frac{\zeta'}{\zeta}\left(-\frac{3}{4}+it\right)(a+ib)^{\frac{3}{4}-it}\,\mathrm{d}t\\&\ll\int_{-\infty}^{+\infty}\left|\Gamma\left(-\frac{3}{4}+it\right)\frac{\zeta'}{\zeta}\left(-\frac{3}{4}+it\right)\right|\left|(a+ib)^{\frac{3}{4}-it}\right|\mathrm{d}t\\&\ll\int_{-\infty}^{+\infty}(|t|+2)^{-\frac{5}{4}}\log(|t|+2)\exp(-th)\,\mathrm{d}t\\&\ll\int_{-\infty}^{+\infty}(|t|+2)^{-\frac{5}{4}}\log(|t|+2)\,\mathrm{d}t\ll 1.\end{aligned}$$

This completes the proof of Lemma 3.3.
Lemma 3.4.

$$\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)(a+ib)^{-\frac{1}{2}-it}\log\frac{t}{2\pi}\,\mathrm{d}t\ll\log^2 T$$

Proof. By Lemma 2.1 and Lemma 3.2, we have

$$\begin{aligned}&\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)(a+ib)^{-\frac{1}{2}-it}\log\frac{t}{2\pi}\,\mathrm{d}t\\&=\sqrt{2\pi}\left(a^2+b^2\right)^{-\frac{1}{4}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}\right)\int_{\gamma_0}^{+\infty}\exp\left(-th+it(\log t-1)\right)\left(a^2+b^2\right)^{-i\frac{t}{2}}\log\left(\frac{t}{2\pi}\right)\mathrm{d}t\\&\quad+O\left(\left(a^2+b^2\right)^{-\frac{1}{4}}\int_{\gamma_0}^{\infty}t^{-1}\exp(-th)\log t\,\mathrm{d}t\right)\\&=I_1\left(\sqrt{2\pi}\left(a^2+b^2\right)^{-\frac{1}{4}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}\right)\right)+I_2\end{aligned}$$

we write

$$r=\left(a^2+b^2\right)^{\frac{1}{2}},\qquad 2\pi\le r\le 2\pi+1$$

$$\begin{aligned}
I_1&=\int_{\gamma_0}^{+\infty}\exp\left(-th+it\left(\log t-\log r-1\right)\right)\log\left(\frac{t}{2\pi}\right)\mathrm{d}t\\
&=\int_{\gamma_0}^{+\infty}\frac{\exp(-th)}{i\log t-i\log r}\log\left(\frac{t}{2\pi}\right)\mathrm{d}\exp\left(it\left(\log t-\log r-1\right)\right)\\
&=-i\int_{\gamma_0}^{+\infty}\frac{\exp(-th)}{\log t-\log r}\left(\log t-\log 2\pi\right)\mathrm{d}\exp\left(it\log\frac{t}{re}\right)\\
&=-i\int_{\gamma_0}^{+\infty}\left(\exp(-th)+\frac{\exp(-th)}{\log t-\log r}\left(\log t-\log 2\pi\right)\right)\mathrm{d}\exp\left(it\log\frac{t}{re}\right)\\
&=O(1)+i\int_{\gamma_0}^{+\infty}\left(-h\exp(-th)+\left(-h\frac{\exp(-th)}{\log t-\log r}-\frac{\exp(-th)}{t\left(\log t-\log r\right)^2}\right)\log\frac{r}{2\pi}\right)\exp\left(it\log\frac{t}{re}\right)\mathrm{d}t\\
&=O(1)+i\int_{\gamma_0}^{+\infty}\left(-h\exp(-th)+\left(-h\frac{\exp(-th)}{\log t-\log r}-\frac{\exp(-th)}{t\left(\log t-\log r\right)^2}\right)\log\frac{r}{2\pi}\right)\exp\left(it\log\frac{t}{re}\right)\mathrm{d}t\\
&\ll\int_{\gamma_0}^{+\infty}\left(h\exp(-th)+\frac{1}{t\left(\log t-\log r\right)^2}\right)\mathrm{d}t\ll 1
\end{aligned}$$

$$\begin{aligned}
I_2&\ll\int_{\gamma_0}^{h^{-2}}t^{-1}\exp(-th)\log t\mathrm{d}t+\int_{h^{-2}}^{\infty}t^{-1}\exp(-th)\log t\mathrm{d}t\\
&\ll\int_{\gamma_0}^{h^{-2}}t^{-1}\log t\mathrm{d}t+h^2\log h^{-1}\int_{h^{-2}}^{\infty}\exp(-th)\mathrm{d}t\ll(\log h)^2\ll\log^2 T
\end{aligned}$$

This completes the proof of Lemma 3.4.

Lemma 3.5.

$$\int_{\gamma_0}^{+\infty}\left|\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}-\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\log z\right|t^{-1}\mathrm{d}t\ll\log^2 T$$

Proof. When $t\ge\gamma_0$, by Lemma 2.1, we have

$$\begin{aligned}
\Gamma'\left(\frac{1}{2}+it\right)&\ll\left|\Gamma\left(\frac{1}{2}+it\right)\log\left(\frac{1}{2}+it\right)\right|+\left|\Gamma\left(\frac{1}{2}+it\right)\Big/\left(\frac{1}{2}+it\right)\right|\\
&\ll\exp\left(-\frac{\pi}{2}t\right)\log t+t^{-1}\exp\left(-\frac{\pi}{2}t\right)\ll\exp\left(-\frac{\pi}{2}t\right)\log t.
\end{aligned}$$

By Lemma 2.1 and Lemma 3.2, we have

$$\begin{aligned}
&\int_{\gamma_0}^{+\infty}\left|\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}-\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\log z\right|t^{-1}\mathrm{d}t\\
&\ll\int_{\gamma_0}^{+\infty}t^{-1}\exp(-th)\log t\mathrm{d}t\ll\int_{\gamma_0}^{h^{-2}}t^{-1}\exp(-th)\log t\mathrm{d}t+\int_{h^{-2}}^{+\infty}t^{-1}\exp(-th)\log t\mathrm{d}t\\
&\ll\int_{\gamma_0}^{h^{-2}}t^{-1}\log t\mathrm{d}t+h^2\log h^{-2}\int_{h^{-2}}^{+\infty}\exp(-th)\mathrm{d}t\ll\log^2 T.
\end{aligned}$$

This completes the proof of Lemma 3.5.

Lemma 3.6. Assume that RH, then

$$\int_{\gamma_0}^{+\infty}\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}S(t)\,\mathrm{d}t\ll 1 \quad \text{and} \quad \int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}S(t)\,\mathrm{d}t\ll 1$$

where $S(t)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right)$.

Proof. By Lemma 3.2, it is easy to see that

$$\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}=\Gamma'\left(\frac{1}{2}+it\right)\left(a^2+b^2\right)^{-\frac{1}{4}-i\frac{t}{2}}\exp\left(-i\frac{\pi}{4}+i\frac{h}{2}+\frac{\pi}{2}t-th\right)$$

$$=\Gamma'\left(\frac{1}{2}+it\right)\left(a^2+b^2\right)^{-\frac{1}{4}-i\frac{t}{2}}\exp\left(\frac{\pi}{2}t-th\right)\left(\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)+i\sin\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right).$$

We write

$$H(s)=\Gamma'(s)\left(a^2+b^2\right)^{-\frac{s}{2}}$$

$$G_1(s)=H(s)\left(\exp\left(\left(-i\frac{\pi}{2}+ih\right)s\right)+\exp\left(\left(-i\frac{\pi}{2}+ih\right)(s-1)\right)\right)$$

$$G_2(s)=H(s)\left(\exp\left(\left(-i\frac{\pi}{2}+ih\right)s\right)-\exp\left(\left(-i\frac{\pi}{2}+ih\right)(s-1)\right)\right)$$

$$G_3(s)=H(1-s)\left(\exp\left(\left(-i\frac{\pi}{2}+ih\right)s\right)+\exp\left(\left(-i\frac{\pi}{2}+ih\right)(s-1)\right)\right)$$

$$G_4(s)=H(1-s)\left(\exp\left(\left(-i\frac{\pi}{2}+ih\right)s\right)-\exp\left(\left(-i\frac{\pi}{2}+ih\right)(s-1)\right)\right).$$

It is easy to see that

$$G_1\left(\frac{1}{2}+it\right)=2\Gamma'\left(\frac{1}{2}+it\right)\left(a^2+b^2\right)^{-\frac{1}{4}-i\frac{t}{2}}\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)$$

$$G_2\left(\frac{1}{2}+it\right)=2i\Gamma'\left(\frac{1}{2}+it\right)\left(a^2+b^2\right)^{-\frac{1}{4}-i\frac{t}{2}}\exp\left(\frac{\pi}{2}t-ht\right)\sin\left(-\frac{\pi}{4}+\frac{h}{2}\right)$$

$$G_3\left(\frac{1}{2}+it\right)=2\Gamma'\left(\frac{1}{2}-it\right)\left(a^2+b^2\right)^{-\frac{1}{4}+i\frac{t}{2}}\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)$$

$$G_4\left(\frac{1}{2}+it\right)=2i\Gamma'\left(\frac{1}{2}-it\right)\left(a^2+b^2\right)^{-\frac{1}{4}+i\frac{t}{2}}\exp\left(\frac{\pi}{2}t-ht\right)\sin\left(-\frac{\pi}{4}+\frac{h}{2}\right).$$

Assume that RH and $0<\delta\le\frac{1}{50}$, by the contour integration method, we have

$$\int_{\frac{1}{2}-\delta+i\gamma_0}^{\frac{1}{2}-\delta+i\infty}G_1(s)\log\zeta(s)\,\mathrm{d}s+\int_{-1+i\infty}^{-1+i\gamma_0}G_1(s)\log\zeta(s)\,\mathrm{d}s+\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0}G_1(s)\log\zeta(s)\,\mathrm{d}s=0$$

$$\int_{\frac{1}{2}-\delta+i\gamma_0}^{\frac{1}{2}-\delta+i\infty}G_1(s)\log\zeta(s)\,\mathrm{d}s=-\int_{-1+i\infty}^{-1+i\gamma_0}G_1(s)\log\zeta(s)\,\mathrm{d}s-\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0}G_1(s)\log\zeta(s)\,\mathrm{d}s=J_1+J_2.$$

By Lemma 2.1 and Lemma 3.2,

$$J_1 = -\int_{-1+i\infty}^{-1+i\gamma_0} G_1(s)\log\zeta(s)\,\mathrm{d}s \ll \int_{\gamma_0}^{\infty} \left|G_1(-1+it)\right|\left|\log\zeta(-1+it)\right|\mathrm{d}t$$
$$\ll \int_{\gamma_0}^{\infty} \left|\Gamma'(-1+it)\right|\exp\left(\frac{\pi}{2}t - th\right)(\log t)\,\mathrm{d}t \ll \int_{\gamma_0}^{\infty} t^{-\frac{3}{2}}(\log t)^2 \exp(-th)\,\mathrm{d}t$$
$$\ll \int_{\gamma_0}^{\infty} t^{-\frac{3}{2}}(\log t)^2\,\mathrm{d}t \ll 1.$$

By Lemma 2.1, Lemma 3.1 and Lemma 3.2, we have

$$J_2 = -\int_{-1+i\gamma_0}^{\frac{1}{2}-\delta+i\gamma_0} G_1(s)\log\zeta(s)\,\mathrm{d}s \ll \int_{-1}^{\frac{1}{2}-\delta_0} \left|G_1(\sigma+i\gamma_0)\right|\left|\log\zeta(\sigma+i\gamma_0)\right|\mathrm{d}\sigma$$
$$\ll \int_{-1}^{\frac{1}{2}-\delta_0} \left|\log\zeta(\sigma+i\gamma_0)\right|\mathrm{d}\sigma \ll 1.$$

When $\delta \to 0$, we have

$$\int_{\gamma_0}^{\infty} G_1\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t \ll 1.$$

Similarly,

$$\int_{\gamma_0}^{\infty} G_2\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t \ll 1.$$

Assume that RH and $0 < \delta \le \frac{1}{50}$, by the contour integration method, we have

$$\int_{2+i\gamma_0}^{2+i\infty} G_3(s)\log\zeta(s)\,\mathrm{d}s + \int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G_3(s)\log\zeta(s)\,\mathrm{d}s + \int_{\frac{1}{2}+\delta+i\gamma_0}^{2+i\gamma_0} G_3(s)\log\zeta(s)\,\mathrm{d}s = 0$$

$$\int_{\frac{1}{2}+\delta+i\infty}^{\frac{1}{2}+\delta+i\gamma_0} G_3(s)\log\zeta(s)\,\mathrm{d}s$$
$$= -\int_{2+i\gamma_0}^{2+i\infty} G_3(s)\log\zeta(s)\,\mathrm{d}s - \int_{\frac{1}{2}+\delta+i\gamma_0}^{2+i\gamma_0} G_3(s)\log\zeta(s)\,\mathrm{d}s$$

same as above

$$\int_{\frac{1}{2}+\delta+i\gamma_0}^{\frac{1}{2}+\delta+i\infty} G_3(s)\log\zeta(s)\,\mathrm{d}s \ll 1.$$

When $\delta \to 0$, we have

$$\int_{\gamma_0}^{\infty} G_3\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t \ll 1.$$

Similarly,

$$\int_{\gamma_0}^{\infty} G_4\left(\frac{1}{2}+it\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t \ll 1.$$

Synthesize the above conclusion, we have

$$\int_{\gamma_0}^{\infty}\left(G_1\left(\frac{1}{2}+it\right)+G_3\left(\frac{1}{2}+it\right)\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t$$

$$=2\int_{\gamma_0}^{\infty}\left(H\left(\frac{1}{2}+it\right)+H\left(\frac{1}{2}-it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t$$

$$=4\int_{\gamma_0}^{\infty}\left(\operatorname{Re}H\left(\frac{1}{2}+it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right|+i\arg\zeta\left(\frac{1}{2}+it\right)\right)\mathrm{d}t$$

therefore

$$\int_{\gamma_0}^{\infty}\left(\operatorname{Re}H\left(\frac{1}{2}+it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)S(t)\mathrm{d}t\ll 1.$$

Similarly,

$$\int_{\gamma_0}^{\infty}\left(G_1\left(\frac{1}{2}+it\right)-G_3\left(\frac{1}{2}+it\right)\right)\log\zeta\left(\frac{1}{2}+it\right)\mathrm{d}t$$

$$=4i\int_{\gamma_0}^{\infty}\left(\operatorname{Im}H\left(\frac{1}{2}+it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right|+i\arg\zeta\left(\frac{1}{2}+it\right)\right)\mathrm{d}t$$

therefore

$$\int_{\gamma_0}^{\infty}\left(\operatorname{Im}H\left(\frac{1}{2}+it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\cos\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)S(t)\mathrm{d}t\ll 1.$$

Similarly,

$$\int_{\gamma_0}^{\infty}\left(\operatorname{Re}H\left(\frac{1}{2}+it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\sin\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)S(t)\mathrm{d}t\ll 1$$

$$\int_{\gamma_0}^{\infty}\left(\operatorname{Im}H\left(\frac{1}{2}+it\right)\right)\left(\exp\left(\frac{\pi}{2}t-ht\right)\sin\left(-\frac{\pi}{4}+\frac{h}{2}\right)\right)S(t)\mathrm{d}t\ll 1.$$

Therefore

$$\int_{\gamma_0}^{+\infty}\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}S(t)\mathrm{d}t\ll 1.$$

We use the same process, we can get

$$\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}S(t)\mathrm{d}t\ll 1.$$

This completes the proof of Lemma 3.6.

Lemma 3.7. Assume that RH, we have

$$\sum_{-\infty<\gamma<+\infty}\Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma}\ll\log^2 T$$

where γ be the ordinates of the nontrivial zeros of $\zeta(s)$.

Proof.

$$\sum_{-\infty<\gamma<+\infty}\Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma}$$

$$=\sum_{\gamma_0<\gamma<+\infty}\Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma}+\sum_{\gamma_0<\gamma<+\infty}\Gamma\left(\frac{1}{2}-i\gamma\right)(a+ib)^{-\frac{1}{2}+i\gamma}=A_1+A_2$$

$$A_1=\sum_{\gamma_0\le\gamma<+\infty}\Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma}=\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\mathrm{d}N(t)$$

by Lemma 2.3, the above formula

$$=\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\mathrm{d}\left(\frac{t}{2\pi}\log\frac{t}{2\pi}-\frac{t}{2\pi}+\frac{7}{8}+S(t)+O\left(\frac{1}{t}\right)\right)$$

$$=\frac{1}{2\pi}\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\left(\log\frac{t}{2\pi}\right)\mathrm{d}t+\int_{\gamma_0}^{+\infty}\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\mathrm{d}\left(S(t)+O\left(\frac{1}{t}\right)\right).$$

By Lemma 3.4, the above formula

$$=-\int_{\gamma_0}^{+\infty}\left(i\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}-i\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\log z\right)S(t)\mathrm{d}t$$

$$+O\left(\int_{\gamma_0}^{+\infty}\left|\Gamma'\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}+\Gamma\left(\frac{1}{2}+it\right)z^{-\frac{1}{2}-it}\log z\right|\left(\frac{1}{t}\right)\mathrm{d}t\right)+O\left(\log^2 T\right)$$

by Lemma 3.5 and Lemma 3.6, above formulas $\ll\log^2 T$.

By Lemma 2.1 and Lemma 3.2, we have

$$A_2=\sum_{\gamma_0\le\gamma<+\infty}\Gamma\left(\frac{1}{2}-i\gamma\right)(a+ib)^{-\frac{1}{2}+i\gamma}\ll\sum_{\gamma_0\le\gamma<+\infty}\exp(-\pi\gamma+\gamma h)\ll 1.$$

This completes the proof of Lemma 3.7.

Lemma 3.8. Assume that RH, if $T\ge 2$, then

$$\sum_{n=2}^{\infty}\Lambda(n)\exp\left(-\frac{n}{T}\right)=T+O\left(T^{\frac{1}{2}}\log^2 T\right).$$

Proof. By Lemma 2.4, we have

$$\sum_{n=2}^{\infty}\Lambda(n)\exp\left(-\frac{n}{T}\right)=\int_2^{\infty}\exp\left(-\frac{x}{T}\right)\mathrm{d}\psi(x)=\int_2^{\infty}\exp\left(-\frac{x}{T}\right)\mathrm{d}\left(x+R(x)\right)$$

$$=\int_2^{\infty}\exp\left(-\frac{x}{T}\right)\mathrm{d}x+\frac{1}{T}\int_2^{\infty}\exp\left(-\frac{x}{T}\right)R(x)\mathrm{d}x+O(1)$$

$$=T\exp\left(-\frac{2}{T}\right)+O\left(\frac{1}{T}\int_2^{\infty}x^{\frac{1}{2}}(\log x)^2\exp\left(-\frac{x}{T}\right)\mathrm{d}x\right)+O(1)$$

$$=T+O\left(T^{\frac{1}{2}}\int_0^{\infty}x^{\frac{1}{2}}(\log x+\log T)^2\exp(-x)\mathrm{d}x\right)+O(1)$$

$$=T+O\left(T^{\frac{1}{2}}\log^2 T\right).$$

This completes the proof of Lemma 3.8.

4. Conclusions

When $a=\frac{1}{T},T\ge 50,b=2\pi$, n is the positive integer; by Lemma 2.1, we have

$$\frac{1}{2\pi i}\int_{(2)}\Gamma(s)(a+ib)^{-s}n^{-s}\mathrm{d}s=\exp(-an-ibn)=\exp\left(-\frac{n}{T}\right).$$

By Lemma 2.2, we have

$$-\sum_{n=2}^{\infty}\Lambda(n)\exp\left(-\frac{n}{T}\right)=\frac{1}{2\pi i}\int_{(2)}\Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}\,\mathrm{d}s.$$

By Lemma 2.2 and RH, the above formula is

$$=-(a+ib)^{-1}+\sum_{-\infty<\gamma<+\infty}\Gamma\left(\frac{1}{2}+i\gamma\right)(a+ib)^{-\frac{1}{2}-i\gamma}+\frac{\zeta'}{\zeta}(0)+\frac{1}{2\pi i}\int_{\left(-\frac{3}{4}\right)}\Gamma(s)\frac{\zeta'}{\zeta}(s)(a+ib)^{-s}\,\mathrm{d}s.$$

By Lemma 3.3 and Lemma 3.7, the above formula is $\ll \log^2 T$.

By Lemma 3.8, we get a contradiction; therefore the RH is incorrect.

References

[1] Montgomery, H.L. and Vaughan, R.C. (2006) Multiplicative Number Theory I. Classical Theory. Cambridge University Press, Cambridge. http://dx.doi.org/10.1017/CBO9780511618314

[2] Titchmarsh, E.C. (1988) The Theory of the Riemann Zeta Function. Oxford University Press, Oxford.

[3] Davenport, H. (1967) Multiplicative Number Theory. Springer Verlag, Berlin.