Applied Mathematics, 2016, 7, 335-345
Published Online March 2016 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2016.74031

Graph-Directed Coalescence Hidden Variable Fractal Interpolation Functions

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Received 20 January 2016; accepted 7 March 2016; published 10 March 2016

Abstract

Fractal interpolation function (FIF) is a special type of continuous function which interpolates certain data set and the attractor of the Iterated Function System (IFS) corresponding to a data set is the graph of the FIF. Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) is both self-affine and non self-affine in nature depending on the free variables and constrained free variables for a generalized IFS. In this article, graph directed iterated function system for a finite number of generalized data sets is considered and it is shown that the projection of the attractors on $\mathbb{R}^2$ is the graph of the CHFIFs interpolating the corresponding data sets.

Keywords

Iterated Function System, Graph-Directed Iterated Function System, Fractal Interpolation Functions, Coalescence Hidden Variable FIFs

1. Introduction

The concept of fractal interpolation function (FIF) based on an iterated function system (IFS) as a fixed point of Hutchinson's operator is introduced by Barnsley [1] [2]. The attractor of the IFS is the graph of a fractal function interpolating certain data set. These FIFs are generally self-affine in nature. The idea has been extended to a generalized data set in $\mathbb{R}^3$ such that the projection of the graph of the corresponding FIF onto $\mathbb{R}^2$ provides a non self-affine interpolation function namely Hidden variable FIFs for a given data set $\{(x_n, y_n): n = 0,1,\cdots,N\}$ [3]. Chand and Kapoor [4], introduced the concept of Coalescence Hidden Variable FIFs which are both self-affine and non self-affine for generalized IFS. The extra degree of freedom is useful to adjust the shape and fractal dimension of the interpolation functions. For Coalescence Hidden Variable Fractal Interpolation Surfaces one can see [5] [6]. In [7], Barnsley *et al*. proved existence of a differentiable FIF. The continuous but nowhere differentiable fractal function namely α-fractal interpolation function f^{α} is intro-

How to cite this paper: Akhtar, Md.N. and Prasad, M.G.P. (2016) Graph-Directed Coalescence Hidden Variable Fractal Interpolation Functions. *Applied Mathematics*, **7**, 335-345. http://dx.doi.org/10.4236/am.2016.74031

duced by Navascues as perturbation of a continuous function f on a compact interval I of $\mathbb{R}$ [8]. Interested reader can see for the theory and application of α-fractal interpolation function f^{α} which has been extensively explored by Navascues [9]-[12].

In [13], Deniz *et al*. considered graph-directed iterated function system (GDIFS) for finite number of data sets and proved the existence of fractal functions interpolating corresponding data sets with graphs as the attractors of the GDIFS.

In the present work, generalized GDIFS for generalized interpolation data sets in $\mathbb{R}^3$ is considered. Corresponding to the data sets, it is shown that there exist CHFIFs whose graphs are the projections of the attractors of the GDIFS on $\mathbb{R}^2$.

2. Preliminaries

2.1. Iterated Function System

Let $\mathcal{X} \subset \mathbb{R}^n$ and $(\mathcal{X}, d_{\mathcal{X}})$ be a complete metric space. Also assume,
$\mathcal{H}(\mathcal{X}) = \{S \subset \mathcal{X}; S \neq \varnothing, S \text{ is compact in } \mathcal{X}\}$ with the Hausdorff metric $d_{\mathcal{H}}(A,B)$ defined as
$d_{\mathcal{H}}(A,B) = \max\{d_{\mathcal{X}}(A,B), d_{\mathcal{X}}(B,A)\}$, where $d_{\mathcal{X}}(A,B) = \max_{x\in A}\min_{y\in B} d_{\mathcal{X}}(A,B)$ for any two sets A, B in $\mathcal{H}(\mathcal{X})$. The completeness of the metric space $(\mathcal{X}, d_{\mathcal{X}})$ imply that $(\mathcal{H}, d_{\mathcal{H}})$ is complete. For $i = 1, 2, \cdots, N$, let $w_i : \mathcal{X} \to \mathcal{X}$ be continuous maps. Then $\{\mathcal{X}; w_i : i = 1, 2, \cdots, N\}$ is called an iterated function system (IFS). If the maps w_i's are contractions, the set valued Hutchinson operator $W : \mathcal{H}(\mathcal{X}) \to \mathcal{H}(\mathcal{X})$ defined by $W(B) = \bigcup_{i=1}^{N} w_i(B)$, where $w_i(B) := \{w_i(b) : b \in B\}$ is also contraction. The Banach fixed point theorem ensures that there exists a unique set $G \in \mathcal{H}(\mathcal{X})$ such that $G = W(G) = \bigcup_{i=1}^{N} w_i(G)$. The set G is called the attractor associated with the IFS $\{\mathcal{X}; w_i : i = 1, 2, \cdots, N\}$.

2.2. Fractal Interpolation Function

Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \cdots, N\}$ be given, where $\Delta : x_0 < x_1 < \cdots < x_N$ is a partition of the closed interval $I = [x_0, x_N]$ and $y_i \in [g_1, g_2] \subset \mathbb{R}$, $i = 0, 1, \cdots, N$. Set $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, N$ and $K = I \times [g_1, g_2]$. Let $L_i : I \to I_i, i = 1, 2, \cdots, N$, be contraction homeomorphisms such that

$$L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i, \tag{1}$$

$$|L_i(c_1) - L_i(c_2)| \leq d\,|c_1 - c_2| \text{ for all } c_1 \text{ and } c_2 \text{ in } I,$$

for some $0 \leq d < 1$. Furthermore, let $H_i : K \to \mathbb{R}$, $i = 1, 2, \cdots, N$ be given continuous functions such that

$$H_i(x_0, y_0) = y_{i-1}, \quad H_i(x_N, y_N) = y_i, \tag{2}$$

$$|H_i(x, \xi_1) - H_i(x, \xi_2)| \leq |\alpha_i||\xi_1 - \xi_2| \tag{3}$$

for all $x \in I$ and for all ξ_1 and ξ_2 in $[g_1, g_2]$, for some $\alpha_i \in (-1, 1)$, $i = 1, 2, \cdots, N$. Define mappings $W_i : K \to I_i \times \mathbb{R}$, $i = 1, 2, \cdots, N$ by

$$W_i(x, y) = (L_i(x), H_i(x, y)) \text{ for all } (x, y) \in K.$$

Then,

$$\{K; W_i(x, y) : i = 1, 2, \cdots, N\}$$

constitutes an IFS. Barnsley [1] proved that the IFS $K; W_i : i = 1, 2, \ldots, N$ defined above has a unique attractor G where G is the graph of a continuous function $\{f : I \to \mathbb{R}\}$ which obeys $f(x_i) = y_i$ for $i = 0, 1, \cdots, N$. This function f is called a fractal interpolation function (FIF) or simply fractal function and it is the unique function satisfying the following fixed point equation

$$f(x) = H_i\left(L_i^{-1}(x), f\left(L_i^{-1}(x)\right)\right) \text{ for all } x \in I_i, i = 1, 2, \cdots, N.$$

The widely studied FIFs so far are defined by the iterated mappings

$$L_i(x) = a_i\, x + d_i, \quad H_i(x, y) = \alpha_i\, y + q_i(x), \quad i = 1, 2, \cdots, N, \tag{4}$$

where the real constants a_i and d_i are determined by the condition (1) as

$$a_i = \frac{(x_i - x_{i-1})}{(x_N - x_0)} \text{ and } d_i = \frac{(x_N x_{i-1} - x_0 x_i)}{(x_N - x_0)}$$

and $q_i(x)$'s are suitable continuous functions such that the conditions (2) and (3) hold. For each i, α_i is a free parameter with $|\alpha_i| < 1$ and is called a vertical scaling factor of the transformation W_i. Then the vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is called the scale vector of the IFS. If $q_i(x)$ is taken as linear then the corresponding FIF is known as affine FIF (AFIF).

2.3. Coalescence FIF

To construct a Coalescence Hidden-variable Fractal Interpolation Function, a set of real parameters z_i for $i = 1, 2, \cdots, N$ are introduced and the generalized interpolation data $\{(x_i, y_i, z_i) \in \mathbb{R}^3 : i = 0, 1, \cdots, N\}$ is considered. Then define the maps $w_i : I \times \mathbb{R}^2 \to I_i \times \mathbb{R}^2, i = 1, 2, \cdots, N$ by

$$w_i(x, y, z) = (L_i(x), F_i(x, y, z))$$

where $L_i : I \to I_i, i = 1, 2, \cdots, N$ are given in (4) and the functions $F_i : I \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_i(x, y, z) = (F_i^1(x, y, z), F_i^2(x, y, z)) = (\alpha_i y + \beta_i z + c_i x + d_i, \gamma_i z + e_i x + f_i)$ satisfy the join-up conditions

$$F_i(x_0, y_0, z_0) = (y_{i-1}, z_{i-1}) \text{ and } F_i(x_N, y_N, z_N) = (y_i, z_i).$$

Here α_i, γ_i are free variables with $|\alpha_i| < 1$, $|\gamma_i| < 1$ and β_i are constrained variables such that $|\beta_i| + |\gamma_i| < 1$. Then the generalized IFS

$$\{I \times \mathbb{R}^2; w_i(x, y, z) : i = 1, 2, \cdots, N\}$$

has an attractor G such that $G = \bigcup_{i=1}^N w_i(G) = \bigcup_{i=1}^N \{w_i(x, y, z) : (x, y, z) \in G\}$. The attractor G is the graph of a vector valued function $f : I \to \mathbb{R}^2$ such that $f(x_i) = (y_i, z_i)$ for $i = 0, 1, \cdots, N$ and $G = \{(x, f(x)) : x \in I, f(x) = (y(x), z(x))\}$. If $f = (f_1, f_2)$, then the projection of the attractor G on $\mathbb{R}^2$ is the graph of the function f_1 which satisfies $f_1(x_i) = y_i$ and is of the form

$$f_1(L_i(x)) = F_i^1(x, f_1(x), f_2(x)) = \alpha_i f_1(x) + \beta_i f_2(x) + c_i x + d_i, x \in I$$

also known as CHFIF corresponding to the data $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \cdots, N\}$ [4].

2.4. Graph-Directed Iterated Function Systems

Let $G = (V, E)$ be a directed graph where V denote the set of vertices and E is the set of edges. For all $u, v \in V$, let E^{uv} denote the set of edges from u to v with elements $e_i^{uv}, i = 1, 2, \cdots, K^{uv}$ where K^{uv} denotes the number of elements of E^{uv}. An iterated function system realizing the graph G is given by a collection of metric spaces $(X^v, \rho^v), v \in V$ with contraction mappings $w_i^{uv} : X^v \to X^u$ corresponding to the edge e_i^{uv} in the opposite direction of e_i^{uv}. An attractor (or invariant list) for such an iterated function system is a list of nonempty compact sets $A^u \subset X^u$ such that for all $u \in V$,

$$A^u = \bigcup_{v \in V} \bigcup_{i=1}^{K^{uv}} w_i^{uv}(A^v).$$

Then, $(X^u; w_i^{uv})$ is the graph directed iterated function system (GDIFS) realizing the graph G [14] [15].

Example 1. *An example of GDIFS may be seen in* [13] [16].

3. Graph Directed Coalescence FIF

In this section, for a finite number of data sets, generalized graph-directed iterated function system (GDIFS) is defined so that projection of each attractor on $\mathbb{R}^2$ is the graph of a CHFIF which interpolates the corresponding data set and calls it as graph-directed coalescence hidden-variable fractal interpolation function (GDCHFIF). For simplicity, only two sets of data are considered. Let the two data sets be

$$D^1 = \{(x_0^1, y_0^1), (x_1^1, y_1^1), \cdots, (x_N^1, y_N^1)\}$$

$$D^2 = \left\{\left(x_0^2, y_0^2\right), \left(x_1^2, y_1^2\right), \cdots, \left(x_M^2, y_M^2\right)\right\}$$

where $N, M \geq 2$ with

$$\frac{x_i^1 - x_{i-1}^1}{x_M^2 - x_0^2} < 1 \text{ and } \frac{x_j^2 - x_{j-1}^2}{x_N^1 - x_0^1} < 1 \tag{5}$$

for all $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, M$. By introducing two sets of real parameters z_i^1, z_j^2 for $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, M$, consider the two generalized data sets

$$\mathcal{D}^1 = \left\{\left(x_0^1, y_0^1, z_0^1\right), \left(x_1^1, y_1^1, z_1^1\right), \cdots, \left(x_N^1, y_N^1, z_N^1\right)\right\}$$

$$\mathcal{D}^2 = \left\{\left(x_0^2, y_0^2, z_0^2\right), \left(x_1^2, y_1^2, z_1^2\right), \cdots, \left(x_M^2, y_M^2, z_M^2\right)\right\}$$

corresponding to D^1 and D^2 respectively. Also consider the directed graph $G = (V, E)$ with $V = \{1, 2\}$ such that

$$K^{11} + K^{12} = N \text{ and } K^{21} + K^{22} = M .$$

To construct a generalized GDIFS associated with the data $\mathcal{D}^r, (r = 1, 2)$ and realize the graph G, consider the functions $w_n^{rs} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$w_n^{rs}(x, y, z) = \left(L_n^{rs}(x), F_n^{rs}(x, y, z)\right), n = 1, 2, \cdots, K^{rs} \tag{6}$$

such that

- $\begin{cases} w_n^{11}\left(x_0^1, y_0^1, z_0^1\right) = \left(x_{n-1}^1, y_{n-1}^1, z_{n-1}^1\right) \\ w_n^{11}\left(x_N^1, y_N^1, z_N^1\right) = \left(x_n^1, y_n^1, z_n^1\right) \end{cases}$ for $n = 1, 2, \cdots, K^{11}$
- $\begin{cases} w_{n-K^{11}}^{12}\left(x_0^2, y_0^2, z_0^2\right) = \left(x_{n-1}^1, y_{n-1}^1, z_{n-1}^1\right) \\ w_{n-K^{11}}^{12}\left(x_M^2, y_M^2, z_M^2\right) = \left(x_n^1, y_n^1, z_n^1\right) \end{cases}$ for $n = K^{11} + 1, \cdots, K^{11} + K^{12} = N$
- $\begin{cases} w_n^{21}\left(x_0^1, y_0^1, z_0^1\right) = \left(x_{n-1}^2, y_{n-1}^2, z_{n-1}^2\right) \\ w_n^{21}\left(x_N^1, y_N^1, z_N^1\right) = \left(x_n^2, y_n^2, z_n^2\right) \end{cases}$ for $n = 1, 2, \cdots, K^{21}$
- $\begin{cases} w_{n-K^{21}}^{22}\left(x_0^2, y_0^2, z_0^2\right) = \left(x_{n-1}^2, y_{n-1}^2, z_{n-1}^2\right) \\ w_{n-K^{21}}^{22}\left(x_M^2, y_M^2, z_M^2\right) = \left(x_n^2, y_n^2, z_n^2\right) \end{cases}$ for $n = K^{21} + 1, \cdots, K^{21} + K^{22} = M$

From each of the above conditions, the following can be derived respectively.

$$\begin{cases} a_n^{11} x_0^1 + b_n^{11} = x_{n-1}^1 \\ a_n^{11} x_N^1 + b_n^{11} = x_n^1 \\ c_n^{11} x_0^1 + \alpha_n^{11} y_0^1 + \beta_n^{11} z_0^1 + d_n^{11} = y_{n-1}^1 \\ c_n^{11} x_N^1 + \alpha_n^{11} y_N^1 + \beta_n^{11} z_N^1 + d_n^{11} = y_n^1 \\ e_n^{11} x_0^1 + \gamma_n^{11} z_0^1 + f_n^{11} = z_{n-1}^1 \\ e_n^{11} x_N^1 + \gamma_n^{11} z_N^1 + f_n^{11} = z_n^1 \end{cases} \quad \text{for } n = 1, 2, \cdots, K^{11} \tag{7}$$

$$\begin{cases} a_{n-K^{11}}^{12} x_0^2 + b_{n-K^{11}}^{12} = x_{n-1}^1 \\ a_{n-K^{11}}^{12} x_M^2 + b_{n-K^{11}}^{12} = x_n^1 \\ c_{n-K^{11}}^{12} x_0^2 + \alpha_{n-K^{11}}^{12} y_0^2 + \beta_{n-K^{11}}^{12} z_0^2 + d_{n-K^{11}}^{12} = y_{n-1}^1 \\ c_{n-K^{11}}^{12} x_M^2 + \alpha_{n-K^{11}}^{12} y_M^2 + \beta_{n-K^{11}}^{12} z_M^2 + d_{n-K^{11}}^{12} = y_n^1 \\ e_{n-K^{11}}^{12} x_0^2 + \gamma_{n-K^{11}}^{12} z_0^2 + f_{n-K^{11}}^{12} = z_{n-1}^1 \\ e_{n-K^{11}}^{12} x_M^2 + \gamma_{n-K^{11}}^{12} z_M^2 + f_{n-K^{11}}^{12} = z_n^1 \end{cases} \quad \text{for } n = K^{11} + 1, \cdots, N \tag{8}$$

$$\begin{cases} a_n^{21}x_0^1 + b_n^{21} = x_{n-1}^2 \\ a_n^{21}x_N^1 + b_n^{21} = x_n^2 \\ c_n^{21}x_0^1 + \alpha_n^{21}y_0^1 + \beta_n^{21}z_0^1 + d_n^{21} = y_{n-1}^2 \\ c_n^{21}x_N^1 + \alpha_n^{21}y_N^1 + \beta_n^{21}z_N^1 + d_n^{21} = y_n^2 \\ e_n^{21}x_0^1 + \gamma_n^{21}z_0^1 + f_n^{21} = z_{n-1}^2 \\ e_n^{21}x_N^1 + \gamma_n^{21}z_N^1 + f_n^{21} = z_n^2 \end{cases} \quad \text{for } n = 1, 2, \cdots, K^{21} \tag{9}$$

$$\begin{cases} a_{n-K^{21}}^{22}x_0^2 + b_{n-K^{21}}^{22} = x_{n-1}^2 \\ a_{n-K^{21}}^{22}x_M^2 + b_{n-K^{21}}^{22} = x_n^2 \\ c_{n-K^{21}}^{22}x_0^2 + \alpha_{n-K^{21}}^{22}y_0^2 + \beta_{n-K^{21}}^{22}z_0^2 + d_{n-K^{21}}^{22} = y_{n-1}^2 \\ c_{n-K^{21}}^{22}x_M^2 + \alpha_{n-K^{21}}^{22}y_M^2 + \beta_{n-K^{21}}^{22}z_M^2 + d_{n-K^{21}}^{22} = y_n^2 \\ e_{n-K^{21}}^{22}x_0^2 + \gamma_{n-K^{21}}^{22}z_0^2 + f_{n-K^{21}}^{22} = z_{n-1}^2 \\ e_{n-K^{21}}^{22}x_M^2 + \gamma_{n-K^{21}}^{22}z_M^2 + f_{n-K^{21}}^{22} = z_n^2 \end{cases} \quad \text{for } n = K^{21} + 1, \cdots, M. \tag{10}$$

From the linear system of Equations (7)-(10) the constants a_i^{rs}, b_i^{rs}, c_i^{rs}, d_i^{rs}, e_i^{rs} and f_i^{rs} for $r, s \in \{1, 2\}$, $i = 1, 2, \cdots, K^{rs}$ are determined as follows:

$$a_n^{11} = \frac{x_n^1 - x_{n-1}^1}{x_N^1 - x_0^1} \qquad a_n^{12} = \frac{x_n^1 - x_{n-1}^1}{x_M^2 - x_0^2}$$

$$b_n^{11} = \frac{x_N^1 x_{n-1}^1 - x_0^1 x_n^1}{x_N^1 - x_0^1} \qquad b_n^{12} = \frac{x_M^2 x_{n-1}^1 - x_0^2 x_n^1}{x_M^2 - x_0^2}$$

$$c_n^{11} = \frac{y_n^1 - y_{n-1}^1 - \alpha_n^{11}\left(y_N^1 - y_0^1\right) - \beta_n^{11}\left(z_N^1 - z_0^1\right)}{x_N^1 - x_0^1} \qquad c_n^{12} = \frac{y_n^1 - y_{n-1}^1 - \alpha_n^{12}\left(y_M^2 - y_0^2\right) - \beta_n^{12}\left(z_M^2 - z_0^2\right)}{x_M^2 - x_0^2}$$

$$d_n^{11} = \frac{x_N^1 y_{n-1}^1 - x_0^1 y_n^1 - \alpha_n^{11}\left(x_N^1 y_0^1 - x_0^1 y_N^1\right) - \beta_n^{11}\left(x_N^1 z_0^1 - x_0^1 z_N^1\right)}{x_N^1 - x_0^1} \qquad d_n^{12} = \frac{x_M^2 y_{n-1}^1 - x_0^2 y_n^1 - \alpha_n^{12}\left(x_M^2 y_0^2 - x_0^2 y_M^2\right) - \beta_n^{12}\left(x_M^2 z_0^2 - x_0^2 z_M^2\right)}{x_M^2 - x_0^2}$$

$$e_n^{11} = \frac{z_n^1 - z_{n-1}^1 - \gamma_n^{11}\left(z_N^1 - z_0^1\right)}{x_N^1 - x_0^1} \qquad e_n^{12} = \frac{z_n^1 - z_{n-1}^1 - \gamma_n^{12}\left(z_M^2 - z_0^2\right)}{x_M^2 - x_0^2}$$

$$f_n^{11} = \frac{x_N^1 z_{n-1}^1 - x_0^1 z_n^1 - \gamma_n^{11}\left(x_N^1 z_0^1 - x_0^1 z_N^1\right)}{x_N^1 - x_0^1} \qquad f_n^{12} = \frac{x_M^2 z_{n-1}^1 - x_0^2 z_n^1 - \gamma_n^{12}\left(x_M^2 z_0^2 - x_0^2 z_M^2\right)}{x_M^2 - x_0^2}$$

$$a_n^{21} = \frac{x_n^2 - x_{n-1}^2}{x_N^1 - x_0^1} \qquad a_n^{22} = \frac{x_n^2 - x_{n-1}^2}{x_M^2 - x_0^2}$$

$$b_n^{21} = \frac{x_N^1 x_{n-1}^2 - x_0^1 x_n^2}{x_N^1 - x_0^1} \qquad b_n^{22} = \frac{x_M^2 x_{n-1}^2 - x_0^2 x_n^2}{x_M^2 - x_0^2}$$

$$c_n^{21} = \frac{y_n^2 - y_{n-1}^2 - \alpha_n^{21}\left(y_N^1 - y_0^1\right) - \beta_n^{21}\left(z_N^1 - z_0^1\right)}{x_N^1 - x_0^1} \qquad c_n^{22} = \frac{y_n^2 - y_{n-1}^2 - \alpha_n^{22}\left(y_M^2 - y_0^2\right) - \beta_n^{22}\left(z_M^2 - z_0^2\right)}{x_M^2 - x_0^2}$$

$$d_n^{21} = \frac{x_N^1 y_{n-1}^2 - x_0^1 y_n^2 - \alpha_n^{21}\left(x_N^1 y_0^1 - x_0^1 y_N^1\right) - \beta_n^{21}\left(x_N^1 z_0^1 - x_0^1 z_N^1\right)}{x_N^1 - x_0^1} \qquad d_n^{22} = \frac{x_M^2 y_{n-1}^2 - x_0^2 y_n^2 - \alpha_n^{22}\left(x_M^2 y_0^2 - x_0^2 y_M^2\right) - \beta_n^{22}\left(x_M^2 z_0^2 - x_0^2 z_M^2\right)}{x_M^2 - x_0^2}$$

$$e_n^{21} = \frac{z_n^2 - z_{n-1}^2 - \gamma_n^{21}\left(z_N^1 - z_0^1\right)}{x_N^1 - x_0^1} \qquad e_n^{22} = \frac{z_n^2 - z_{n-1}^2 - \gamma_n^{22}\left(z_M^2 - z_0^2\right)}{x_M^2 - x_0^2}$$

$$f_n^{21} = \frac{x_N^1 z_{n-1}^2 - x_0^1 z_n^2 - \gamma_n^{21}\left(x_N^1 z_0^1 - x_0^1 z_N^1\right)}{x_N^1 - x_0^1} \qquad f_n^{22} = \frac{x_M^2 z_{n-1}^2 - x_0^2 z_n^2 - \gamma_n^{22}\left(x_M^2 z_0^2 - x_0^2 z_M^2\right)}{x_M^2 - x_0^2}$$

The following theorem shows that each map w_n^{rs} is contraction with respect to metric equivalent to the Euclidean metric and ensures the existence of attractors of generalized GDIFS.

Theorem 2. *Let* $\left\{\mathbb{R}^3; w_n^{rs}, n = 1, 2, \cdots, K^{rs}\right\}$ *be the generalized GDIFS defined in* (6) *realizing the graph and associated with the data sets* $\mathcal{D}^r, (r = 1, 2)$ *which satisfy* (5). *If* $\left|\alpha_n^{rs}\right| < 1$, $\left|\gamma_n^{rs}\right| < 1$ *and* β_n^{rs} *are chosen such that* $\left|\beta_n^{rs}\right| + \left|\gamma_n^{rs}\right| < 1$ *for all* $r, s \in \{1, 2\}$ *and* $n = 1, 2, \cdots, K^{rs}$. *Then there exists a metric* δ *on* $\mathbb{R}^3$ *equivalent to the Euclidean metric such that the GDIFS is hyperbolic with respect to* δ. *In particular, there exist non empty compact sets* G^r *such that*

$$G^r = \bigcup_{s=1}^{2} \bigcup_{n=1}^{K^{rs}} w_n^{rs}\left(G^s\right).$$

Proof. Proof follows in the similar lines of Theorem 2.1.1 of [17] and using the above condition (5). □

Following is the main result regarding existence of coalescence Hidden-variable FIFs for generalized GDIFS.

Theorem 3. *Let* $G^r, r \in V$ *be the attractors of the generalized GDIFS as in Theorem* 2. *Then* $G^r, r \in V$ *is the graph of a vector valued continuous function* $f^r : I^r \to \mathbb{R}^2$ *such that for* $r \in V$, $f^r\left(x_n^r\right) = \left(y_n^r, z_n^r\right)$ *for all* $n = 1, 2, \cdots, N^r$. *If* $f^r = \left(f_1^r, f_2^r\right)$ *then the projection of the attractors* $G^r, r \in V$ *on* $\mathbb{R}^2$ *is the graph of the continuous function* $f_1^r : I^r \to \mathbb{R}$ *known as CHFIF such that for* $r \in V$, $f_1^r\left(x_n^r\right) = \left(y_n^r\right)$. *That is* $G^r\big|_{\mathbb{R}^2} = \left\{\left(x, f_1^r(x)\right) : x \in I^r\right\}$.

Proof. Consider the vector valued function spaces

$$\mathcal{F} = \left\{f : \left[x_0^1, x_N^1\right] \to \mathbb{R}^2 \text{ continuous such that } f\left(x_0^1\right) = \left(y_0^1, z_0^1\right), f\left(x_N^1\right) = \left(y_N^1, z_N^1\right)\right\}$$

$$\mathcal{H} = \left\{h : \left[x_0^2, x_M^2\right] \to \mathbb{R}^2 \text{ continuous such that } h\left(x_0^2\right) = \left(y_0^2, z_0^2\right), h\left(x_M^2\right) = \left(y_M^2, z_M^2\right)\right\}$$

with metrics

$$d_{\mathcal{F}}\left(f_1, f_2\right) = \sup_{x \in \left[x_0^1, x_N^1\right]} \left\| f_1(x) - f_2(x) \right\|$$

$$d_{\mathcal{H}}\left(h_1, h_2\right) = \sup_{x \in \left[x_0^2, x_M^2\right]} \left\| h_1(x) - h_2(x) \right\|$$

respectively, where $\|.\|$ denotes a norm on $\mathbb{R}^2$. Since $\left(\mathcal{F}, d_{\mathcal{F}}\right)$ and $\left(\mathcal{H}, d_{\mathcal{H}}\right)$ are complete metric spaces, $\left(\mathcal{F} \times \mathcal{H}, d\right)$ is also a complete metric space where

$$d\left(\left(f_1, h_1\right), \left(f_2, h_2\right)\right) = \max\left\{d_{\mathcal{F}}\left(f_1, f_2\right), d_{\mathcal{H}}\left(h_1, h_2\right)\right\}.$$

Following are the affine maps,

$$I_n : \left[x_0^1, x_N^1\right] \to \left[x_{n-1}^1, x_n^1\right], \; I_n(x) = a_n^{11} x + b_n^{11} \text{ for } n = 1, 2, \cdots, K^{11}$$

$$I_n : \left[x_0^2, x_M^2\right] \to \left[x_{n-1}^1, x_n^1\right],$$

$$I_n(x) = a_{n-K^{11}}^{12} x + b_{n-K^{11}}^{12} \text{ for } n = K^{11} + 1, \cdots, N$$

$$J_n : \left[x_0^1, x_N^1\right] \to \left[x_{n-1}^2, x_n^2\right], \; J_n(x) = a_n^{21} x + b_n^{21} \text{ for } n = 1, 2, \cdots, K^{21}$$

$$J_n : \left[x_0^2, x_M^2\right] \to \left[x_{n-1}^2, x_n^2\right],$$

$$J_n(x) = a_{n-K^{21}}^{22} x + b_{n-K^{21}}^{22} \text{ for } n = K^{21} + 1, \cdots, M.$$

Now define the mapping

$$T:\mathcal{F}\times\mathcal{H}\rightarrow\mathcal{F}\times\mathcal{H}$$
$$T(f,h)(x,y)=\left(\tilde{f}(x),\tilde{h}(y)\right)$$

where for $x\in\left[x_{n-1}^1,x_n^1\right]$,

$$\tilde{f}(x)=\begin{cases}\left(c_n^{11}I_n^{-1}(x)+\alpha_n^{11}y_f^1\left(I_n^{-1}(x)\right)+\beta_n^{11}z_f^1\left(I_n^{-1}(x)\right)+d_n^{11},\right.\\ \quad\left.\gamma_n^{11}z_f^1\left(I_n^{-1}(x)\right)+e_n^{11}I_n^{-1}(x)+f_n^{11}\right) & \text{for } n=1,2,\cdots,K^{11}\\ \left(c_{n-K^{11}}^{12}I_n^{-1}(x)+\alpha_{n-K^{11}}^{12}y_h^2\left(I_n^{-1}(x)\right)+\beta_{n-K^{11}}^{12}z_h^2\left(I_n^{-1}(x)\right)+d_{n-K^{11}}^{12},\right.\\ \quad\left.\gamma_{n-K^{11}}^{12}z_h^2\left(I_n^{-1}(x)\right)+e_{n-K^{11}}^{12}I_n^{-1}(x)+f_{n-K^{11}}^{12}\right) & \text{for } n=K^{11}+1,\cdots,N\end{cases}$$

and for $x\in\left[x_{m-1}^2,x_m^2\right]$,

$$\tilde{h}(x)=\begin{cases}\left(c_m^{21}J_m^{-1}(x)+\alpha_m^{21}y_f^1\left(J_m^{-1}(x)\right)+\beta_m^{21}z_f^1\left(J_m^{-1}(x)\right)+d_m^{21},\right.\\ \quad\left.\gamma_m^{21}z_f^1\left(J_m^{-1}(x)\right)+e_m^{21}J_m^{-1}(x)+f_m^{21}\right) & \text{for } m=1,\cdots,K^{21}\\ \left(c_{m-K^{21}}^{22}J_m^{-1}(x)+\alpha_{m-K^{21}}^{22}y_h^2\left(J_m^{-1}(x)\right)+\beta_{m-K^{21}}^{22}z_h^2\left(J_m^{-1}(x)\right)+d_{m-K^{21}}^{22},\right.\\ \quad\left.\gamma_{m-K^{21}}^{22}z_h^2\left(J_m^{-1}(x)\right)+e_{m-K^{21}}^{22}J_m^{-1}(x)+f_{m-K^{21}}^{22}\right) & \text{for } m=K^{21}+1,\cdots,M.\end{cases}$$

Now using Equations (7)-(10) it is clear that,

$$\tilde{f}\left(x_0^1\right)=F_1^{11}\left(I_n^{-1}(x),y_f^1\left(I_n^{-1}(x)\right),z_f^1\left(I_n^{-1}(x)\right)\right)=\left(y_0^1,z_0^1\right)$$
$$\tilde{f}\left(x_N^1\right)=F_N^{12}\left(I_n^{-1}(x),y_h^2\left(I_n^{-1}(x)\right),z_h^2\left(I_n^{-1}(x)\right)\right)=\left(y_N^1,z_N^1\right).$$

Similarly, $\tilde{h}\left(x_0^2\right)=\left(y_0^2,z_0^2\right)$, $\tilde{h}\left(x_M^2\right)=\left(y_M^2,z_M^2\right)$. It proves that T maps $\mathcal{F}\times\mathcal{H}$ into itself. Since for each $n=1,2,\cdots,N$, $I_n^{-1}(x)$ is continuous and therefore, $\tilde{f}$ is continuous on each subintervals $\left[x_{n-1}^1,x_n^1\right]$.

For $n=1,2,\cdots,K^{11}$, using (7) it follows that $\tilde{f}\left(x_n^{1-}\right)=\tilde{f}\left(x_n^{1+}\right)=\left(y_n^1,z_n^1\right)$.

For $n=K^{11}+1,\cdots,N-1$, using (8) it follows that $\tilde{f}\left(x_n^{1-}\right)=\tilde{f}\left(x_n^{1+}\right)=\left(y_n^1,z_n^1\right)$.

For $n=K^{11}$, using (7) and (8) it follows that $\tilde{f}\left(x_n^{1-}\right)=\tilde{f}\left(x_n^{1+}\right)=\left(y_n^1,z_n^1\right)$ since $I_n^{-1}\left(x_n^1\right)=x_N^1$ and $I_{n+1}^{-1}\left(x_n^1\right)=x_0^2$.

Hence $\tilde{f}$ is continuous on *I*. Similarly it can be shown that $\tilde{h}$ is continuous on *J*. Consequently *T* is continuous.

To show that *T* is a contraction map on $\mathcal{F}\times\mathcal{H}$, let $T\left(f_1,f_2\right)=\left(\tilde{f}_1,\tilde{f}_2\right)$ and $T\left(h_1,h_2\right)=\left(\tilde{h}_1,\tilde{h}_2\right)$. Now,

$$\begin{aligned}&\sup_{x\in\left[x_0^1,x_{K^{11}}^1\right]}\left\{\left\|\tilde{f}_1(x)-\tilde{f}_2(x)\right\|\right\}\\ &=\max_{\substack{n=1,2,\cdots,K^{11}\\ x\in\left[x_{n-1}^1,x_n^1\right]}}\left\{\left\|\alpha_n^{11}\left(y_{f_1}^1(I_n^{-1}(x))-y_{f_2}^1\left(I_n^{-1}(x)\right)\right.\right.\right.\\ &\quad\left.\left.\left.+\beta_n^{11}\left(z_{f_1}^1\left(I_n^{-1}(x)\right)-z_{f_2}^1\left(I_n^{-1}(x)\right)\right),\gamma_n^{11}\left(z_{f_1}^1\left(I_n^{-1}(x)\right)-z_{f_2}^1\left(I_n^{-1}(x)\right)\right)\right\|\right\}\\ &\le\delta^{11}\max_{\substack{n=1,2,\cdots,K^{11}\\ x\in\left[x_{n-1}^1,x_n^1\right]}}\left\{y_{f_1}^1\left(I_n^{-1}(x)\right)-y_{f_2}^1\left(I_n^{-1}(x)\right)+z_{f_1}^1\left(I_n^{-1}(x)\right)-z_{f_2}^1\left(I_n^{-1}(x)\right),\right.\\ &\quad\left.z_{f_1}^1\left(I_n^{-1}(x)\right)-z_{f_2}^1\left(I_n^{-1}(x)\right)\right\}\\ &\le\delta^{11}d_{\mathcal{F}}\left(f_1,f_2\right),\end{aligned}$$

$$\begin{aligned}
&\sup_{x\in\left[x^1_{K^{11}},x^1_N\right]}\left\{\left\|\tilde f_1(x)-\tilde f_2(x)\right\|\right\}\\
&=\max_{\substack{n=K^{11}+1,\cdots,N\\ x\in\left[x^1_{n-1},x^1_n\right]}}\left\{\left\|\alpha^{12}_{n-K^{11}}(y^2_{h_1}\left(I_n^{-1}(x)\right)-y^2_{h_2}\left(I_n^{-1}(x)\right)\right.\right.\\
&\quad\left.\left.+\beta^{12}_{n-K^{11}}\left(z^2_{h_1}\left(I_n^{-1}(x)\right)-z^2_{h_2}\left(I_n^{-1}(x)\right)\right),\gamma^{12}_{n-K^{11}}\left(z^2_{h_1}\left(I_n^{-1}(x)\right)-z^2_{h_2}\left(I_n^{-1}(x)\right)\right)\right\|\right\}\\
&\le\delta^{12}\max_{\substack{n=K^{11}+1,\cdots,N\\ x\in\left[x^1_{n-1},x^1_n\right]}}\left\{y^2_{h_1}\left(I_n^{-1}(x)\right)-y^2_{h_2}\left(I_n^{-1}(x)\right)+z^2_{h_1}\left(I_n^{-1}(x)\right)-z^2_{h_2}\left(I_n^{-1}(x)\right),\right.\\
&\quad\left.z^2_{h_1}\left(I_n^{-1}(x)\right)-z^2_{h_2}\left(I_n^{-1}(x)\right)\right\}\\
&\le\delta^{12}d_{\mathcal{H}}(h_1,h_2),
\end{aligned}$$

where $\delta^{11}=\max_{n=1,2,\cdots,K^{11}}\left\{\left|\alpha^{11}_n\right|,\left|\beta^{11}_n\right|,\left|\gamma^{11}_n\right|\right\}<1$ and $\delta^{12}=\max_{n=K^{11}+1,\cdots,N}\left\{\left|\alpha^{12}_n\right|,\left|\beta^{12}_n\right|,\left|\gamma^{12}_n\right|\right\}<1$. Therefore

$$d_{\mathcal{F}}\left(\tilde f_1,\tilde f_2\right)\le\max\left\{\delta^{11},\delta^{12}\right\}\max\left\{d_{\mathcal{F}}(f_1,f_2),d_{\mathcal{H}}(h_1,h_2)\right\}.$$

Similarly, it follows that

$$d_{\mathcal{H}}\left(\tilde h_1,\tilde h_2\right)\le\max\left\{\delta^{21},\delta^{22}\right\}\max\left\{d_{\mathcal{F}}(f_1,f_2),d_{\mathcal{H}}(h_1,h_2)\right\},$$

where $\delta^{21}=\max_{n=1,2,\cdots,K^{21}}\left\{\left|\alpha^{21}_n\right|,\left|\beta^{21}_n\right|,\left|\gamma^{21}_n\right|\right\}<1$ and $\delta^{22}=\max_{n=K^{21}+1,\cdots,M}\left\{\left|\alpha^{22}_n\right|,\left|\beta^{22}_n\right|,\left|\gamma^{22}_n\right|\right\}<1$. Then

$$d\left(T(f_1,h_1),T(f_2,h_2)\right)=\max\left\{d_{\mathcal{F}}\left(\tilde f_1,\tilde f_2\right),d_{\mathcal{H}}\left(\tilde h_1,\tilde h_2\right)\right\}\le\delta\max\left\{d_{\mathcal{F}}(f_1,f_2),d_{\mathcal{H}}(h_1,h_2)\right\}$$

where $\delta=\max\left\{\delta^{11},\delta^{12},\delta^{21},\delta^{22}\right\}<1$ and hence T is a contraction mapping. By Banach fixed point theorem, T possesses a unique fixed point, say (f_0,h_0).

Now, for $n=1,2,\cdots,K^{11}$,

$$\begin{aligned}
f_0\left(x^1_n\right)=&\left(c^{11}_{n+1}I^{-1}_{n+1}\left(x^1_n\right)+\alpha^{11}_{n+1}y^1_{f_0}\left(I^{-1}_{n+1}\left(x^1_n\right)\right)+\beta^{11}_{n+1}z^1_{f_0}\left(I^{-1}_{n+1}\left(x^1_n\right)\right)+d^{11}_{n+1},\right.\\
&\left.\gamma^{11}_{n+1}z^1_{f_0}\left(I^{-1}_{n+1}\left(x^1_n\right)\right)+e^{11}_{n+1}I^{-1}_{n+1}\left(x^1_n\right)+f^{11}_{n+1}\right)=\left(y^1_n,z^1_n\right).
\end{aligned}$$

For $n=K^{11}+1,\cdots,N-1$,

$$\begin{aligned}
f_0\left(x^1_n\right)=&\left(c^{12}_{n+1-K^{11}}I^{-1}_{n+1}\left(x^1_n\right)+\alpha^{12}_{n+1-K^{11}}y^2_{h_0}\left(I^{-1}_{n+1}\left(x^1_n\right)\right)+\beta^{12}_{n+1-K^{11}}z^2_{h_0}\left(I^{-1}_{n+1}\left(x^1_n\right)\right)+d^{12}_{n+1-K^{11}},\right.\\
&\left.\gamma^{12}_{n+1-K^{11}}z^2_{h_0}\left(I^{-1}_{n+1}\left(x^1_n\right)\right)+e^{12}_{n+1-K^{11}}I^{-1}_{n+1}\left(x^1_n\right)+f^{12}_{n+1-K^{11}}\right)=\left(y^1_n,z^1_n\right)
\end{aligned}$$

This shows that f_0 is the function which interpolates the data $\left\{\left(x^1_n,y^1_n,z^1_n\right):n=0,1,\cdots,N\right\}$. Similarly, it can be shown that g_0 is the function which interpolates the data $\left\{\left(x^2_n,y^2_n,z^2_n\right):n=0,1,\cdots,M\right\}$. For $x\in\left[x^1_0,x^1_N\right]$ and $x\in\left[x^2_0,x^2_M\right]$,

$$f_0\left(I_n(x)\right)=\left(c^{11}_nx+\alpha^{11}_ny^1_{f_0}(x)+\beta^{11}_nz^1_{f_0}(x)+d^{11}_n,\gamma^{11}_nz^1_{f_0}(x)+e^{11}_nx+f^{11}_n\right)\quad\text{for } n=1,2,\cdots,K^{11}$$

$$f_0\left(I_n(x)\right)=\left(c^{12}_nx+\alpha^{12}_ny^2_{h_0}(x)+\beta^{12}_nz^2_{h_0}(x)+d^{12}_n,\gamma^{12}_nz^2_{h_0}(x)+e^{12}_nx+f^{12}_n\right)\quad\text{for } n=1,2,\cdots,K^{12}$$

and

$$h_0\left(J_n(x)\right)=\left(c^{21}_nx+\alpha^{21}_ny^1_{f_0}(x)+\beta^{21}_nz^1_{f_0}(x)+d^{21}_n,\gamma^{21}_nz^1_{f_0}(x)+e^{21}_nx+f^{21}_n\right)\quad\text{for } n=1,2,\cdots,K^{21}$$

$$h_0\left(J_n(x)\right)=\left(c^{22}_nx+\alpha^{22}_ny^2_{h_0}(x)+\beta^{22}_nz^2_{h_0}(x)+d^{22}_n,\gamma^{22}_nz^2_{h_0}(x)+e^{22}_nx+f^{22}_n\right)\quad\text{for } n=1,2,\cdots,K^{22}.$$

If F and H are the graphs of f_0 and h_0 respectively, then

$$F=\bigcup_{i=1}^{K^{11}} w_i^{11}(F)\bigcup\bigcup_{i=1}^{K^{12}} w_i^{12}(H)$$

$$H=\bigcup_{i=1}^{K^{21}} w_i^{21}(F)\bigcup\bigcup_{i=1}^{K^{22}} w_i^{22}(H).$$

The uniqueness of the attractor implies that $F=G^1$ and $H=G^2$. That is $G^1=\{(x,f_0(x)):x\in I\}$ and $G^2=\{(x,h_0(x)):x\in J\}$. Denoting $f_0=(f_1^1,f_2^1)$ and $h_0=(f_1^2,f_2^2)$, result follows. □

Example 4. *Consider the data sets as*

$$D^1=\{(0,5),(1,4),(2,1),(3,1),(4,4),(5,5)\}$$

$$D^2=\{(0,1),(1,2),(2,3),(3,2),(4,1)\}$$

realizing the graph with $K^{11}=3$, $K^{12}=2$, $K^{21}=1$, $K^{22}=3$ as in **Figure 1**. Take the first set of generalized data

$$\mathcal{D}^1=\{(0,5,5),(1,4,4),(2,1,1),(3,1,1),(4,4,4),(5,5,5)\}$$

and

$$\mathcal{D}^2=\{(0,1,1),(1,2,2),(2,3,3),(3,2,2),(4,1,1)\}$$

corresponding to D^1 and D^2 respectively. Here $y_n=z_n$ for both the generalized data sets. Choose $\alpha_n^{rs}=1/3$, $\beta_n^{rs}=1/3$, $\gamma_n^{rs}=1/3$ for all $r,s\in\{1,2\}$ and $n=1,2,\cdots,K^{rs}$. Then **Figure 2** is the attractors of the corresponding generalized GDIFS.

Keeping the free variables and constrained variables same, **Figure 3** is the attractors of the generalized GDIFS associated with the second set of generalized data

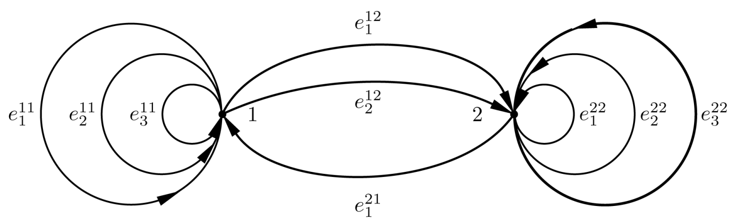

Figure 1. Directed graph for Example 4.

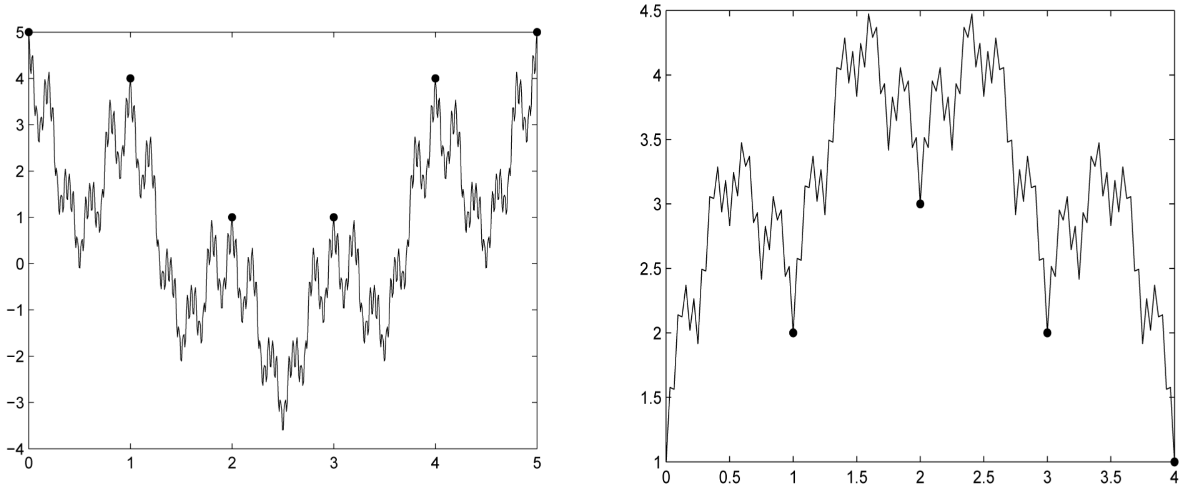

Figure 2. Attractors for the first set of generalized data.

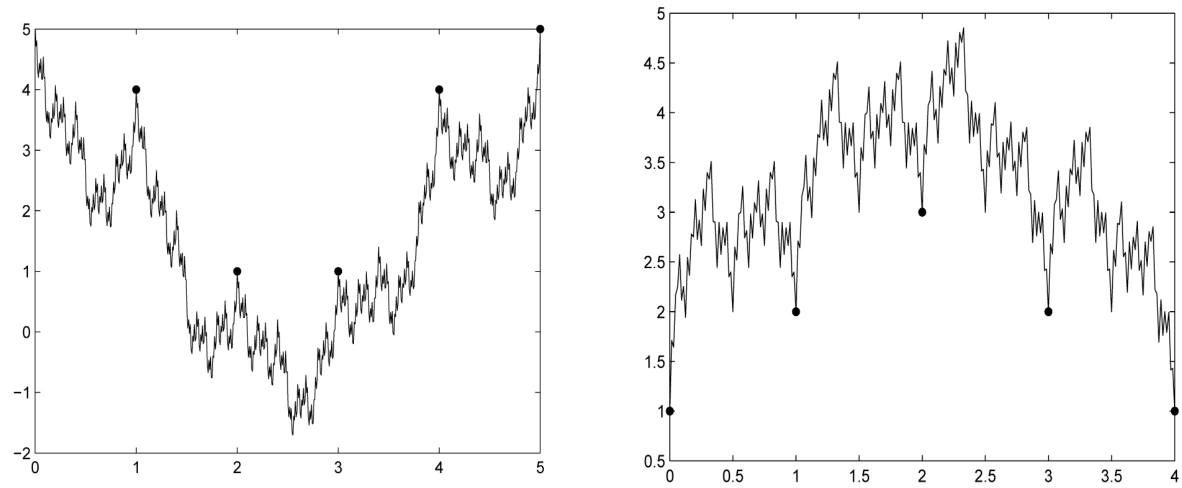

Figure 3. Attractors for the second set of generalized data.

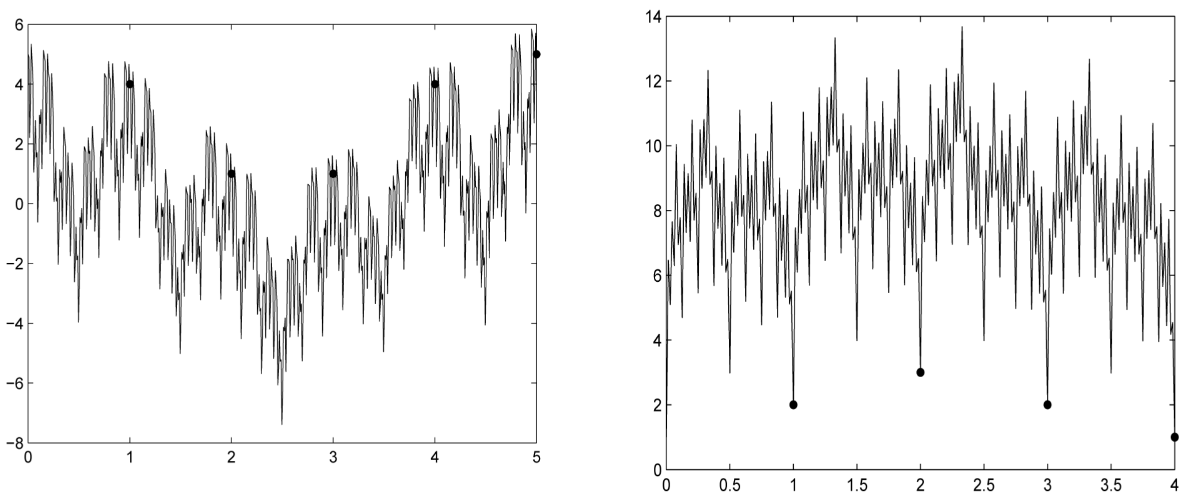

Figure 4. Attractors for the third set of generalized data.

Table 1. The generalized GDIFS with the free variables and constraints variables.

α	α_1^{11}	α_2^{11}	α_3^{11}	α_1^{12}	α_2^{12}	α_1^{21}	α_1^{22}	α_2^{22}	α_3^{22}
	0.8	0.7	0.8	0.7	0.8	0.99	0.99	0.99	0.99
β	β_1^{11}	β_2^{11}	β_3^{11}	β_1^{12}	β_2^{12}	β_1^{21}	β_1^{22}	β_2^{22}	β_3^{22}
	−0.3	−0.4	−0.2	−0.3	−0.4	0.99	0.99	0.99	0.99
γ	γ_1^{11}	γ_2^{11}	γ_3^{11}	γ_1^{12}	γ_2^{12}	γ_1^{21}	γ_1^{22}	γ_2^{22}	γ_3^{22}
	0.5	0.3	0.6	0.5	0.3	0.005	0.005	0.005	0.005

$$\mathcal{D}^1 = \{(0,5,3),(1,4,2),(2,1,5),(3,1,2),(4,4,1),(5,5,4)\}$$

$$\mathcal{D}^2 = \{(0,1,2),(1,2,5),(2,3,1),(3,2,3),(4,1,1)\}.$$

Take the third set of generalized data

$$\mathcal{D}^1 = \{(0,5,3),(1,4,2),(2,1,5),(3,1,2),(4,4,1),(5,5,4)\}$$

and

$$\mathcal{D}^2 = \{(0,1,2),(1,5,5),(2,3,1),(3,2,3),(4,4,1)\}$$

corresponding to D^1 and D^2 respectively. For the generalized GDIFS with the free variables and constraints variables given in following **Table 1**, the attractors are given in **Figure 4**.

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