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Axes of Möbius Transformations in H_3*

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Abstract

This paper gives the relationship between the positions of axes of the two nonparabolic elements that generate a discrete group and the nature including the translation lengths along the axes and the rotation angles. We mainly research the intersecting position and the coplanar (but disjoint) position.

Keywords: Geodesic, Discrete, Axis

1. Introduction

Hyperbolic 3-space is the set

$$H^3 = \{(x_1, x_2, x_3) \in R^3 : x_3 > 0\}$$

endowed with the complete Riemannian metric $ds = |dx|/x_3$ of constant curvature equal to –1. A *Kleinian group* G is a discrete nonelementary subgroup of $Isom^+(H^3)$, where $Isom^+(H^3)$ is the group of orientation preserving isometries.

Each Möbius transformation of $\bar{C} = \partial H^3$ extends uniquely via the *Poincare*' extension [1] to an orientation-preserving isometry of hyperbolic 3-space H^3. In this way we identify Kleinian groups with discrete Möbius groups.

Let M denote the group of all Möbius transformations of the extended complex plane $\bar{C} = C \cup \{\infty\}$. We associate with each Möbius transformation

$$f = \frac{az+b}{cz+d} \in M, ad - bc = 1$$

the matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,C)$$

And set $tr(f) = tr(\boldsymbol{A})$, where $tr(\boldsymbol{A}) = a + d$ denotes the trace of the matrix $\boldsymbol{A}$. Next, for each f and g in M we let $[f,g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\gamma(f,g) = tr(fgf^{-1}g^{-1}) - 2$$

$$\beta(f) = tr^2(f) - 4, \beta(g) = tr^2(g) - 4$$

the parameters of $\langle f, g\rangle$. These parameters are independent of the choice of matrix representations for f and g in $SL(2,C)$, and they determine $\langle f, g\rangle$ uniquely up to conjugacy whenever $\gamma(f,g) \neq 0$.

The elements of f of M, other than the identity, fall into three types.

1) Elliptic: $\beta(f) \in [-4, 0)$ and f is conjugate to $z \mapsto \mu z$ where $|\mu| = 1$.

2) Loxodromic: $\beta(f) \notin [-4, 0]$ and f is conjugate to $z \mapsto \mu z$ where $|\mu| = 1$; f is hyperbolic if, in addition, $\mu > 0$.

3) Parabolic: $\beta(f) = 0$ and f is conjugate to $z \mapsto \mu z$.

If $f \in M$ is nonparabolic, then f fixes two points of $\bar{C}$ and the closed hyperbolic line joining these two fixed points is called the axis of f, denoted by $ax(f)$. In this case, f translates along $ax(f)$ by an amount $\tau(f) \geq 0$, the translation length of f, f rotates about $ax(f)$ by an angle $\theta(f) \in (-\pi, \pi]$, and

$$\beta(f) = 4\sin^2\left(\frac{\tau(f) + i\theta(f)}{2}\right)$$

In [4], F.W.Gehring and G. J. Martin have shown :

Theorem 1.1: [4] If $\langle f, g\rangle$ is discrete, if f and g are loxodromics with $\beta(f) = \beta(g)$, and if $ax(f)$ and $ax(g)$ intersect at an angle φ where $0 < \varphi < \pi$, then

$$\sinh(\tau(f))\sin(\varphi) \geq \lambda$$

where $0.122 \leq \lambda \leq 0.435$. In particular,

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$$\tau(f) \geq \mu$$

where $0.122 \leq \mu \leq 0.492$. The exponent of $\sin(\varphi)$ cannot be replaced by a constant greater than 1.

In this paper, we will discuss the situation when $ax(f)$ and $ax(g)$ coplanar but disjoint. In [4], F. W. Gehring and G. J. Martin have analyzed the situation when *f* is loxodromic and *g* is loxodromic or elliptic. In the following, we will consider the condition when the two generators are elliptics.

2. Preliminary Results

Lemma 2.1: [1] Let f and g be Möbius transformations, neither the identity. Then f and g are conjugate if and only if $tr^2(f) = tr^2(g)$.

Lemma 2.2: [4] If $\langle f, g\rangle$ is a *Kleinian group*, if f is elliptic of order $n \geq 3$, and if g is not of order 2, then

$$|\gamma(f,g)| \geq a(n)$$

where

$$a(n) = \begin{cases} 2\cos(2\pi/7) - 1 & \text{if } n = 3 \\ 2\cos(2\pi/5) & \text{if } n = 4,5 \\ 2\cos(2\pi/6) & \text{if } n = 6 \\ 2\cos(2\pi/n) - 1 & \text{if } n \geq 7 \end{cases}$$

Lemma 2.3: [3] Suppose that f and g in M have disjoint pairs of fixed points in $\overline{C}$ and α is the hyperbolic line in H^3 which is orthogonal to the axes of f and g. Then

$$\frac{4\gamma(f,g)}{\beta(f)\beta(g)} = \sinh^2(\delta \pm i\varphi)$$

where $\delta = \delta(f,g) = \rho(axis(f), axis(g))$ and φ is the angle between the sphere or hyperplanes which contain $ax(f) \cup \alpha$ and $ax(g) \cup \alpha$ respectively.

Lemma 2.4: [4] For each loxodromic Möbius transformation f there exists an integer $m \geq 1$ such that

$$|\beta(f^m)| \leq \frac{4\pi}{\sqrt{3}} \sinh(\tau(f))$$

The coefficient of $\sinh(\tau(f))$ cannot be replaced by smaller constant.

3. Main Results

Theorem 3.1: If $\langle f, g\rangle$ is discrete, if f and g are elliptics with orders *m*, *n* respectively where *m*, *n* ≥ 3, then

1) If $ax(f)$ and $ax(g)$ intersect at an angle φ where $0 < \varphi < \pi$, then

$$\sin(\pi/n)\sin(\pi/m)\sin(\varphi) \geq \frac{\sqrt{a(3)}}{2}$$

2) If $ax(f)$ and $ax(g)$ are coplanar but disjoint, then

$$\sin(\pi/n)\sin(\pi/m)\sin(\delta) \geq \frac{\sqrt{a(3)}}{2}$$

and the inequality is sharp.

Proof. Let δ denote the hyperbolic distance between $ax(f)$ and $ax(g)$. Let φ denote the the angle between the sphere or hyperplanes which contain $ax(f) \cup \alpha$ and $ax(g) \cup \alpha$ respectively. If α is the hyperbolic line in H^3 that is orthogonal to $ax(f)$ and $ax(g)$, then

$$\frac{4\gamma(f,g)}{\beta(f)\beta(g)} = \sinh^2(\delta \pm i\varphi)$$

by Lemma 2.3. If $ax(f)$ and $ax(g)$ intersect at an angle φ, then

$$\frac{4\gamma(f,g)}{\beta(f)\beta(g)} = -\sin^2(\varphi)$$

We may assume without loss of generality that f, g are primitive elliptics. From Lemma 2.2 we can obtain $|\gamma(f,g)| \geq a(3)$, so

$$\begin{aligned} &16\sin^2(\pi/m)\sin^2(\pi/n)\sin^2(\varphi) \\ &= \beta(f)\beta(g)\sin^2(\varphi) = |4\gamma(f,g)| \\ &\geq 4a(3) \end{aligned}$$

that is

$$\sin(\pi/n)\sin(\pi/m)\sin(\varphi) \geq \frac{\sqrt{a(3)}}{2}$$

In the same way, if *ax*(*f*) and *ax*(*g*) are coplanar but disjoint, then

$$\begin{aligned} &16\sin^2(\pi/m)\sin^2(\pi/n)\sin^2(\delta) \\ &= \beta(f)\beta(g)\sinh^2(\delta) = |4\gamma(f,g)| \\ &\geq 4a(3) \end{aligned}$$

by $\frac{4\gamma(f,g)}{\beta(f)\beta(g)} = \sinh^2(\delta)$ To show that the inequality is sharp, we let $\langle f, g\rangle$ denote the (2,3,7) triangle group where f and g are primitive with

$f^3 = g^7 = (fg)^2 = I$. Then

$$\gamma(f,g) = tr([f,g]) - 2 = tr^2 f + tr^2 g - 4$$
$$= \beta(f) + \beta(g) + 4 = 2\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{2\pi}{3}\right)$$
$$= tr^2 f + tr^2 g - 4 = a(3) \quad \square$$

Remark: In [4], according to Lemma 2.3, F. W. Gehring and G. J. Martin considered the situation when $\delta = 0$. They discuss the relationship between the angle φ, translation length of f and g or rotation angle when f is loxodromic and g is loxodromic or elliptic. Theorem3.1 show the condition when f and g are ellipitics.

Corollary 3.1: If $\langle f,g\rangle$ is discrete, if f and g are ellipitics with $\beta(f) = \beta(g)$, $\gamma(f,g) \neq 0$ and if $ax(f)$ and $ax(g)$ intersect at an angle φ, where $0 < \varphi \leq \frac{\pi}{2}$. If the order of f is k with $k \geq 3$, then

$$\sin^2\left(\frac{\pi}{k}\right)\sin(\varphi) \geq \frac{\sqrt{a(3)}}{2}$$

In particular, if $ax(f)$ and $ax(g)$ meet at right angles and the order of f is k, then

$$3 \leq k \leq 6$$

Proof. $\sin^2\left(\frac{\pi}{k}\right)\sin(\varphi) \geq \frac{\sqrt{a(3)}}{2}$ can easily seen from the former theorem. If $ax(f)$ and $ax(g)$ meet at right angles, then

$$\sin^2\left(\frac{\pi}{k}\right) \geq \frac{\sqrt{a(3)}}{2} = 0.248\cdots$$

As k is an integer, so

$$3 \leq k \leq 6 \quad \square$$

In the following, we will consider the thing when $\varphi = 0$.

Theorem 3.2: If $\langle f,g\rangle$ is discrete, if f and g are loxodromics with $\beta(f) = \beta(g)$ $ax(f)$ and if $ax(f)$ and $ax(g)$ coplanar but disjoint, let $\tau(f)$ be the translation length of f, δ be the distance between the $ax(f)$ and $ax(g)$, then

$$\sinh(\tau(f))\sinh(\delta) \geq \frac{\sqrt{3d}}{2\pi}$$

where $d = 2\left(1 - \cos\left(\frac{\pi}{7}\right)\right)$.

Proof. By Lemma 2.4, we can choose an integer number $m \geq 1$ such that

$$\left|\beta\left(f^m\right)\right| \leq \frac{4\pi}{\sqrt{3}}\sinh(\tau(f))$$

Then $\langle f^m, g^m\rangle$ is a discrete nonelementary group with $\beta(f^m) = \beta(g^m)$.

By Lemma 2.2 and Lemma 2.3, we can obtain

$$\frac{4\pi}{\sqrt{3}}\sinh(\tau(f))\sinh(\delta)$$
$$\geq \sqrt{\left|\beta(f^m)\beta(g^m)\right|\sinh^2(\delta)}$$
$$= \sqrt{4\left|\gamma(f^m, g^m)\right|}$$
$$\geq 2\sqrt{d}$$

then

$$\sinh(\tau(f))\sinh(\delta) \geq \frac{\sqrt{3d}}{2\pi} \quad \square$$

As for Theorem 3.5 and Theorem 3.15 in [4], we can obtain related results in similar way when $ax(f)$ and $ax(g)$ coplanar but disjoint.

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5. References

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