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Positive Solutions for Systems of Coupled Fractional Boundary Value Problems

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Abstract

We investigate the existence and nonexistence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with coupled integral boundary conditions which contain some positive constants.

Keywords

Riemann-Liouville Fractional Differential Equations, Coupled Integral Boundary Conditions, Positive Solutions

1. Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [1]-[6]). For some recent developments on the topic, which can be seen in [7]-[19] and the references therein.

In this paper, we consider the system of nonlinear ordinary fractional differential equations

$$(S)\quad \begin{cases} D_{0+}^{\alpha}u(t)+a(t)f\left(v(t)\right)=0,\ t\in(0,1), \\ D_{0+}^{\beta}v(t)+b(t)g\left(u(t)\right)=0,\ t\in(0,1), \end{cases}$$

with the coupled integral boundary conditions

$$(BC)\quad \begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ u(1)=\int_0^1 v(s)\,\mathrm{d}H(s)+a_0, \\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0,\ v(1)=\int_0^1 u(s)\,\mathrm{d}K(s)+b_0, \end{cases}$$

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where $n-1<\alpha\le n$, $m-1<\beta\le m$, $n,m\in\mathbb{N}$, $n,m\ge 3$, D_{0+}^{α} and D_{0+}^{β} denote the Riemann-Liouville derivatives of orders α and β, respectively, the integrals from (BC) are Riemann-Stieltjes integrals, a_0 and b_0 are positive constants.

Under some assumptions on the functions *f* and *g*, we shall prove the existence of positive solutions of problem (*S*)-(*BC*). By a positive solution of (*S*)-(*BC*), we mean a pair of functions $(u,v)\in C([0,1];\mathbb{R}_+)\times C([0,1];\mathbb{R}_+)$ satisfying (*S*) and (*BC*) with $u(t)>0$, $v(t)>0$ for all $t\in(0,1]$. We shall also give sufficient conditions for the nonexistence of positive solutions for this problem. Some systems of fractional equations with parameters subject to coupled integral boundary conditions were studied in [20] by using the Guo-Krasnosel'skii fixed point theorem. We also mentioned the paper [21], where we investigated the existence and multiplicity of positive solutions for the system $D_{0+}^{\alpha}u(t)+f(t,v(t))=0, t\in(0,1)$, $D_{0+}^{\beta}v(t)+g(t,u(t))=0, t\in(0,1)$, with the integral boundary conditions (*BC*) with $a_0=b_0=0$ by using some theorems from the fixed point index theory and the Guo-Krasnosel'skii fixed point theorem. In [21], the nonlinearities *f* and *g* may be nonsingular or singular in $t=0$ and/or $t=1$. Some systems of Riemann-Liouville fractional equations with or without parameters subject to uncoupled boundary conditions are studied in the papers [22]-[25], and the book [26].

In Section 2, we present some auxiliary results which investigate a system of Riemann-Liouville fractional equations subject to coupled integral boundary conditions. In Section 3, we prove our main results, and an example which supports the obtained results is finally presented in Section 4. In the proof of our existence result, we shall use the Schauder fixed point theorem which we present now.

Theorem 1. *Let X be a Banach space and* $Y\subset X$ *a nonempty, bounded, convex and closed subset. If the operator* $A:Y\to Y$ *is completely continuous, then A has at least one fixed point.*

2. Auxiliary Results

We present here the definitions of the fractional integral and Riemann-Liouville fractional derivative of a function, and some auxiliary results from [20] and [22] that will be used to prove our main theorems.

Definition 2.1: *The (left-sided) fractional integral of order* $\alpha>0$ *of a function* $f:(0,\infty)\to\mathbb{R}$ *is given by*

$$\left(I_{0+}^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,\mathrm{d}s,\ t>0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, *where* $\Gamma(\alpha)$ *is the Euler gamma function defined by* $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}\mathrm{e}^{-t}\mathrm{d}t$, $\alpha>0$.

Definition 2.2: *The Riemann-Liouville fractional derivative of order* $\alpha\ge 0$ *for a function* $f:(0,\infty)\to\mathbb{R}$ *is given by*

$$\left(D_{0+}^{\alpha}f\right)(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\left(I_{0+}^{n-\alpha}f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t\frac{f(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,\ t>0,$$

where $n=[\![\alpha]\!]+1$, *provided that the right-hand side is pointwise defined on* $(0,\infty)$.

The notation $[\![\alpha]\!]$ stands for the largest integer not greater than α. If $\alpha=m\in\mathbb{N}$ then $D_{0+}^{m}f(t)=f^{(m)}(t)$ for $t>0$, and if $\alpha=0$ then $D_{0+}^{0}f(t)=f(t)$ for $t>0$.

We consider now the fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t)+x(t)=0,\ t\in(0,1),\ n-1<\alpha\le n,\\ D_{0+}^{\beta}v(t)+y(t)=0,\ t\in(0,1),\ m-1<\beta\le m,\end{cases}\tag{1}$$

with the coupled integral boundary conditions

$$\begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ u(1)=\int_0^1 v(s)\mathrm{d}H(s),\\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0,\ v(1)=\int_0^1 u(s)\mathrm{d}K(s),\end{cases}\tag{2}$$

where $n,m\in\mathbb{N}$, $n,m\geq 3$ and $H,K:[0,1]\to\mathbb{R}$ are functions of bounded variation.

Lemma 1. ([20]) *If* $H,K:[0,1]\to\mathbb{R}$ *are functions of bounded variations,* $\Delta=1-\left(\int_0^1\tau^{\alpha-1}\mathrm{d}K(\tau)\right)\left(\int_0^1\tau^{\beta-1}\mathrm{d}H(\tau)\right)\neq 0$ *and* $x,y\in C(0,1)\cap L^1(0,1)$, *then the unique solution of problem* (1)-(2) *is given by*

$$\begin{cases} u(t)=\int_0^1 G_1(t,s)x(s)\mathrm{d}s+\int_0^1 G_2(t,s)y(s)\mathrm{d}s, \\ v(t)=\int_0^1 G_3(t,s)y(s)\mathrm{d}s+\int_0^1 G_4(t,s)x(s)\mathrm{d}s,\ t\in[0,1], \end{cases} \tag{3}$$

where

$$\begin{aligned} G_1(t,s)&=g_1(t,s)+\frac{t^{\alpha-1}}{\Delta}\left(\int_0^1\tau^{\beta-1}\mathrm{d}H(\tau)\right)\left(\int_0^1 g_1(\tau,s)\mathrm{d}K(\tau)\right),\\ G_2(t,s)&=\frac{t^{\alpha-1}}{\Delta}\int_0^1 g_2(\tau,s)\mathrm{d}H(\tau),\\ G_3(t,s)&=g_2(t,s)+\frac{t^{\beta-1}}{\Delta}\left(\int_0^1\tau^{\alpha-1}\mathrm{d}K(\tau)\right)\left(\int_0^1 g_2(\tau,s)\mathrm{d}H(\tau)\right),\\ G_4(t,s)&=\frac{t^{\beta-1}}{\Delta}\int_0^1 g_1(\tau,s)\mathrm{d}K(\tau),\ \forall t,s\in[0,1], \end{aligned} \tag{4}$$

and

$$\begin{cases} g_1(t,s)=\dfrac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1},\ 0\leq s\leq t\leq 1,\\ t^{\alpha-1}(1-s)^{\alpha-1},\ 0\leq t\leq s\leq 1,\end{cases}\\ g_2(t,s)=\dfrac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1},\ 0\leq s\leq t\leq 1,\\ t^{\beta-1}(1-s)^{\beta-1},\ 0\leq t\leq s\leq 1.\end{cases} \end{cases} \tag{5}$$

Lemma 2. ([22]) *The functions* g_1 *and* g_2 *given by* (5) *have the properties*

a) $g_1,g_2:[0,1]\times[0,1]\to\mathbb{R}_+$ *are continuous functions and* $g_1(t,s)>0$, $g_2(t,s)>0$ *for all* $(t,s)\in(0,1)\times(0,1)$;

b) $g_1(t,s)\leq g_1(\theta_1(s),s), g_2(t,s)\leq g_2(\theta_2(s),s)$, *for all* $(t,s)\in[0,1]\times[0,1]$;

c) *For any* $c\in(0,1/2)$, *we have*

$$\min_{t\in[c,1-c]} g_1(t,s)\geq\gamma_1 g_1(\theta_1(s),s),\ \min_{t\in[c,1-c]} g_2(t,s)\geq\gamma_2 g_2(\theta_2(s),s),$$

for all $s\in[0,1]$, *where* $\gamma_1=c^{\alpha-1}$, $\gamma_2=c^{\beta-1}$, $\theta_1(s)=\begin{cases}\dfrac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}}, & s\in(0,1],\\ \dfrac{\alpha-2}{\alpha-1}, & s=0,\end{cases}$ *and*

$$\theta_2(s)=\begin{cases}\dfrac{s}{1-(1-s)^{\frac{\beta-1}{\beta-2}}}, & s\in(0,1],\\ \dfrac{\beta-2}{\beta-1}, & s=0.\end{cases}$$

Lemma 3. ([20]) *If* $H,K:[0,1]\to\mathbb{R}$ *are nondecreasing functions, and* $\Delta>0$, *then* G_i, $i=1,\cdots,4$, *given by* (4) *are continuous functions on* $[0,1]\times[0,1]$ *and satisfy* $G_i(t,s)\geq 0$ *for all* $(t,s)\in[0,1]\times[0,1]$, $i=1,\cdots,4$. *Moreover, if* $x,y\in C(0,1)\cap L^1(0,1)$ *satisfy* $x(t)\geq 0$, $y(t)\geq 0$ *for all* $t\in(0,1)$, *then the solution* (u,v) *of* *problem* (1)-(2) *satisfies* $u(t)\ge 0, v(t)\ge 0$ *for all* $t\in[0,1]$.

Lemma 4. ([20]) *Assume that* $H,K:[0,1]\to\mathbb{R}$ *are nondecreasing functions and* $\Delta>0$. *Then the functions* $G_i, i=1,\cdots,4,$ *satisfy the inequalities*

a_1) $G_1(t,s)\le J_1(s), \forall(t,s)\in[0,1]\times[0,1]$, *where*

$$J_1(s)=g_1(\theta_1(s),s)+\frac{1}{\Delta}\left(\int_0^1\tau^{\beta-1}\mathrm{d}H(\tau)\right)\left(\int_0^1 g_1(\tau,s)\mathrm{d}K(\tau)\right);$$

a_2) *For every* $c\in(0,1/2)$, *we have*

$$\min_{t\in[c,1-c]}G_1(t,s)\ge\gamma_1 J_1(s)\ge\gamma_1 G_1(t',s), \forall t',s\in[0,1];$$

b_1) $G_2(t,s)\le J_2(s), \forall(t,s)\in[0,1]\times[0,1]$, *where* $J_2(s)=\frac{1}{\Delta}\int_0^1 g_2(\tau,s)\mathrm{d}H(\tau)$;

b_2) *For every* $c\in(0,1/2)$, *we have*

$$\min_{t\in[c,1-c]}G_2(t,s)\ge\gamma_1 J_2(s)\ge\gamma_1 G_2(t',s), \forall t',s\in[0,1];$$

c_1) $G_3(t,s)\le J_3(s), \forall(t,s)\in[0,1]\times[0,1]$, *where*

$$J_3(s)=g_2(\theta_2(s),s)+\frac{1}{\Delta}\left(\int_0^1\tau^{\alpha-1}\mathrm{d}K(\tau)\right)\left(\int_0^1 g_2(\tau,s)\mathrm{d}H(\tau)\right);$$

c_2) *For every* $c\in(0,1/2)$, *we have*

$$\min_{t\in[c,1-c]}G_3(t,s)\ge\gamma_2 J_3(s)\ge\gamma_2 G_3(t',s), \forall t',s\in[0,1];$$

d_1) $G_4(t,s)\le J_4(s), \forall(t,s)\in[0,1]\times[0,1]$, *where* $J_4 s=\frac{1}{\Delta}\int_0^1 g_1(\tau,s)\mathrm{d}K(\tau)$;

d_2) *For every* $c\in(0,1/2)$, *we have*

$$\min_{t\in[c,1-c]}G_4(t,s)\ge\gamma_2 J_4(s)\ge\gamma_2 G_4(t',s), \forall t',s\in[0,1].$$

Lemma 5. ([20]) *Assume that* $H,K:[0,1]\to\mathbb{R}$ *are nondecreasing functions,* $\Delta>0$, $c\in(0,1/2)$ *and* $x,y\in C(0,1)\cap L^1(0,1)$, $x(t)\ge 0$, $y(t)\ge 0$ *for all* $t\in(0,1)$. *Then the solution* $(u(t),v(t))$, $t\in[0,1]$ *of problem* (1)-(2) (*given by* (3)) *satisfies the inequalities*

$$\inf_{t\in[c,1-c]}u(t)\ge\gamma_1\sup_{t'\in[0,1]}u(t'), \quad \inf_{t\in[c,1-c]}v(t)\ge\gamma_2\sup_{t'\in[0,1]}v(t').$$

3. Main Results

We present first the assumptions that we shall use in the sequel.

(J_1) $H,K:[0,1]\to\mathbb{R}$ are nondecreasing functions and $\Delta=1-\left(\int_0^1\tau^{\alpha-1}\mathrm{d}K(\tau)\right)\left(\int_0^1\tau^{\beta-1}\mathrm{d}H(\tau)\right)>0$.

(J_2) The functions $a,b:[0,1]\to[0,\infty)$ are continuous and there exist $t_1,t_2\in(0,1)$ such that $a(t_1)>0$, $b(t_2)>0$.

(J_3) $f,g:[0,\infty)\to[0,\infty)$ are continuous functions and there exists $c_0>0$ such that

$$f(u)<\frac{c_0}{L}, \quad g(u)<\frac{c_0}{L} \quad \text{for all} \quad u\in[0,c_0],$$

where $L=\max\left\{\int_0^1 a(s)J_1(s)\mathrm{d}s+\int_0^1 b(s)J_2(s)\mathrm{d}s, \int_0^1 b(s)J_3(s)\mathrm{d}s+\int_0^1 a(s)J_4(s)\mathrm{d}s\right\}$ and $J_i, i=1,\cdots,4$ are defined in Lemma 4.

(J_4) are continuous functions and satisfy the conditions

$$\lim_{u\to\infty} f(u)/u = \infty, \lim_{u\to\infty} g(u)/u = \infty.$$

By assumption (J_2) we deduce that $\int_0^1 a(s)J_1(s)\mathrm{d}s > 0$, $\int_0^1 b(s)J_2(s)\mathrm{d}s > 0$, $\int_0^1 b(s)J_3(s)ds > 0$ and $\int_0^1 a(s)J_4(s)ds > 0$, that is, the constant L from (J_3) is positive.

Our first theorem is the following existence result for problem (S)-(BC).

Theorem 2. *Assume that assumptions* (J_1)-(J_3) *hold. Then problem* (S)-(BC) *has at least one positive solution for* $a_0 > 0$ *and* $b_0 > 0$ *sufficiently small.*

Proof. We consider the system of ordinary fractional differential equations

$$\begin{cases} D_{0+}^{\alpha} h(t) = 0, t \in (0,1), \\ D_{0+}^{\beta} k(t) = 0, t \in (0,1), \end{cases} \tag{6}$$

with the coupled integral boundary conditions

$$\begin{cases} h(0) = h'(0) = \cdots = h^{(n-2)}(0) = 0, h(1) = \int_0^1 k(s)\mathrm{d}H(s) + a_0, \\ k(0) = k'(0) = \cdots = k^{(m-2)}(0) = 0, k(1) = \int_0^1 h(s)\mathrm{d}K(s) + b_0, \end{cases} \tag{7}$$

with $a_0 > 0$ and $b_0 > 0$.

The above problem (6)-(7) has the solution

$$\begin{aligned} h(t) &= \frac{t^{\alpha-1}}{\Delta}\left(a_0 + b_0 \int_0^1 s^{\beta-1}\mathrm{d}H(s)\right), t \in [0,1], \\ k(t) &= \frac{t^{\beta-1}}{\Delta}\left(b_0 + a_0 \int_0^1 s^{\alpha-1}\mathrm{d}K(s)\right), t \in [0,1], \end{aligned} \tag{8}$$

where Δ is defined in (J_1). By assumption (J_1) we obtain $h(t) > 0$ and $k(t) > 0$ for all $t \in (0,1]$.

We define the functions $x(t)$ and $y(t)$, $t \in [0,1]$ by

$$x(t) = u(t) - h(t),\ y(t) = v(t) - k(t),\ \forall t \in [0,1],$$

where (u,v) is a solution of (S)-(BC). Then (S)-(BC) can be equivalently written as

$$\begin{cases} D_{0+}^{\alpha} x(t) + a(t) f\big(y(t) + k(t)\big) = 0, t \in (0,1), \\ D_{0+}^{\beta} y(t) + b(t) g\big(x(t) + h(t)\big) = 0, t \in (0,1), \end{cases} \tag{9}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, x(1) = \int_0^1 y(s)\mathrm{d}H(s), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0, y(1) = \int_0^1 x(s)\mathrm{d}K(s). \end{cases} \tag{10}$$

Using the Green's functions G_i, $i = 1,\cdots,4$, from Lemma 1, a pair (x,y) is a solution of problem (9)-(10) if and only if (x,y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t) = \int_0^1 G_1(t,s)a(s) f\big(y(s)+k(s)\big)\mathrm{d}s + \int_0^1 G_2(t,s)b(s) g\big(x(s)+h(s)\big)\mathrm{d}s, t \in [0,1], \\ y(t) = \int_0^1 G_3(t,s)b(s) g\big(x(s)+h(s)\big)\mathrm{d}s + \int_0^1 G_4(t,s)a(s) f\big(y(s)+k(s)\big)\mathrm{d}s, t \in [0,1], \end{cases} \tag{11}$$

where $h(t)$ and $k(t)$, $t \in [0,1]$ are given in (8).

We consider the Banach space $X = C([0,1])$ with the supremum norm $\|\cdot\|$, the space $Y = X \times X$ with the norm $\|(x,y)\|_Y = \|x\| + \|y\|$, and we define the set

$$E = \left\{x \in C([0,1]), 0 \le x(t) \le c_0, \forall t \in [0,1]\right\} \subset X.$$

We also define the operators $S_1, S_2 : E \times E \to X$ and $S : E \times E \to Y$ by

$$S_1(x,y)(t)=\int_0^1 G_1(t,s)a(s)f\left(y(s)+k(s)\right)\mathrm{d}s+\int_0^1 G_2(t,s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s,$$
$$S_2(x,y)(t)=\int_0^1 G_3(t,s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s+\int_0^1 G_4(t,s)a(s)f\left(y(s)+k(s)\right)\mathrm{d}s,$$

for all $t\in[0,1]$, and $S(x,y)=\left(S_1(x,y),S_2(x,y)\right),(x,y)\in E\times E$.

For sufficiently small $a_0>0$ and $b_0>0$, by (J3), we deduce

$$f\left(y(t)+k(t)\right)\le\frac{c_0}{L},\ g\left(x(t)+h(t)\right)\le\frac{c_0}{L},\ \forall t\in[0,1],\ \forall x,y\in E.$$

Then, by using Lemma 3, we obtain $S_1(x,y)(t)\ge 0$, $S_2(x,y)(t)\ge 0$ for all $t\in[0,1]$ and $(x,y)\in E\times E$. By Lemma 4, for all $(x,y)\in E\times E$, we have

$$\begin{aligned}S_1(x,y)(t)&\le\int_0^1 J_1(s)a(s)f\left(y(s)+k(s)\right)\mathrm{d}s+\int_0^1 J_2(s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s\\&\le\frac{c_0}{L}\left(\int_0^1 a(s)J_1(s)\mathrm{d}s+\int_0^1 b(s)J_2(s)\mathrm{d}s\right)\le c_0,\ \forall t\in[0,1],\end{aligned}$$

and

$$\begin{aligned}S_2(x,y)(t)&\le\int_0^1 J_3(s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s+\int_0^1 J_4(s)a(s)f\left(y(s)+k(s)\right)\mathrm{d}s\\&\le\frac{c_0}{L}\left(\int_0^1 b(s)J_3(s)\mathrm{d}s+\int_0^1 a(s)J_4(s)\mathrm{d}s\right)\le c_0,\ \forall t\in[0,1].\end{aligned}$$

Therefore $S(E\times E)\subset E\times E$.

Using standard arguments, we deduce that S is completely continuous. By Theorem 1, we conclude that S has a fixed point $(x,y)\in E\times E$, which represents a solution for problem (9)-(10). This shows that our problem (S)-(BC) has a positive solution (u,v) with $u=x+h, v=y+k$ for sufficiently small $a_0>0$ and $b_0>0$.

In what follows, we present sufficient conditions for the nonexistence of positive solutions of (S)-(BC).

Theorem 3. *Assume that assumptions* (J_1), (J_2) *and* (J_4) *hold. Then problem* (S)-(BC) *has no positive solution for* a_0 *and* b_0 *sufficiently large.*

Proof. We suppose that (u,v) is a positive solution of (S)-(BC). Then (x,y) with $x=u-h$, $y=v-k$ is a solution for problem (9)-(10), where (h,k) is the solution of problem (6)-(7) (given by (8)). By (J_2) there exists $c\in(0,1/2)$ such that $t_1,t_2\in(c,1-c)$, and then $\int_c^{1-c}a(s)J_1(s)\mathrm{d}s>0$, $\int_c^{1-c}b(s)J_2(s)\mathrm{d}s>0$, $\int_c^{1-c}b(s)J_3(s)\mathrm{d}s>0$, $\int_c^{1-c}a(s)J_4(s)\mathrm{d}s>0$. Now by using Lemma 3, we have $x(t)\ge 0$, $y(t)\ge 0$ for all $t\in[0,1]$, and by Lemma 5 we obtain $\inf_{t\in[c,1-c]}x(t)\ge\gamma_1\|x\|$ and $\inf_{t\in[c,1-c]}y(t)\ge\gamma_2\|y\|$.

Using now (8), we deduce that $\inf_{t\in[c,1-c]}h(t)=\gamma_1\|h\|$ and $\inf_{t\in[c,1-c]}k(t)=\gamma_2\|k\|$. Therefore, we obtain $\inf_{t\in[c,1-c]}\left(x(t)+h(t)\right)\ge\gamma_1\|x+h\|$ and $\inf_{t\in[c,1-c]}\left(y(t)+k(t)\right)\ge\gamma_2\|y+k\|$.

We now consider $R=\left(\gamma_1^2\int_c^{1-c}b(s)J_2(s)\mathrm{d}s\right)^{-1}>0$. By using ($J_4$), for R defined above, we conclude that there exists $M>0$ such that $f(u)>2Ru$, $g(u)>2Ru$ for all $u\ge M$. We consider $a_0>0$ and $b_0>0$ sufficiently large such that $\inf_{t\in[c,1-c]}\left(x(t)+h(t)\right)\ge M$ and $\inf_{t\in[c,1-c]}\left(y(t)+k(t)\right)\ge M$. By ($J_2$), (9), (10) and the above inequalities, we deduce that $\|x\|>0$ and $\|y\|>0$.

Now by using Lemma 4 and the above considerations, we have

$$\begin{aligned}x(c)&=\int_0^1 G_1(c,s)a(s)f(y(s)+k(s)\mathrm{d}s+\int_0^1 G_2(c,s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s\\&\ge\gamma_1\int_0^1 J_2(s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s\ge\gamma_1\int_c^{1-c}J_2(s)b(s)g\left(x(s)+h(s)\right)\mathrm{d}s\\&\ge 2R\gamma_1\int_c^{1-c}J_2(s)b(s)\left(x(s)+h(s)\right)\mathrm{d}s\ge 2R\gamma_1\int_c^{1-c}J_2(s)b(s)\inf_{\tau\in[c,1-c]}\left(x(\tau)+h(\tau)\right)\mathrm{d}s\\&\ge 2R\gamma_1^2\int_c^{1-c}J_2(s)b(s)\|x+h\|\mathrm{d}s=2\|x+h\|\ge 2\|x\|.\end{aligned}$$

Therefore, we obtain $\|x\| \le \frac{1}{2} x(c) \le \frac{1}{2}\|x\|$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem (S)-(BC) has no positive solution.

4. An Example

We consider $a(t) = 1$, $b(t) = 1$ for all $t \in [0,1]$, $\alpha = 7/3$ $(n = 3)$, $\beta = 5/2$ $(m = 3)$, $H(t) = t^2$, for all $t \in [0,1]$, $K(t) = \begin{cases} 0, \ t \in [0,1/3), \\ 1, \ t \in [1/3, 2/3), \\ 3/2, t \in [2/3, 1]. \end{cases}$ then $\int_0^1 v(s)\,\mathrm{d}H(s) = 2\int_0^1 s v(s)\,\mathrm{d}s$ and $\int_0^1 u(s)\,\mathrm{d}K(s) = u\left(\frac{1}{3}\right) + \frac{1}{2} u\left(\frac{2}{3}\right)$.

We also consider the functions $f, g : [0,\infty) \to [0,\infty)$, $f(x) = \tilde{a}x^2$, $g(x) = \tilde{b}x^3$, for all $x \in [0,\infty)$, with $\tilde{a}, \tilde{b} > 0$. We have $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} g(x)/x = \infty$.

Therefore, we consider the system of fractional differential equations

$$(S_0) \quad \begin{cases} D_{0+}^{7/3} u(t) + \tilde{a} v^2(t) = 0, \ t \in (0,1), \\ D_{0+}^{5/2} v(t) + \tilde{b} u^3(t) = 0, \ t \in (0,1), \end{cases}$$

with the boundary conditions

$$(BC_0) \quad \begin{cases} u(0) = u'(0) = 0, \ u(1) = 2\int_0^1 s v(s)\,\mathrm{d}s + a_0, \\ v(0) = v'(0) = 0, \ v(1) = u\left(\frac{1}{3}\right) + \frac{1}{2} u\left(\frac{2}{3}\right) + b_0. \end{cases}$$

Then we obtain

$$\Delta = 1 - \left(\int_0^1 \tau^{4/3}\,\mathrm{d}K(\tau)\right)\left(\int_0^1 \tau^{3/2}\,\mathrm{d}H(\tau)\right) = 1 - \left(\left(\frac{1}{3}\right)^{4/3} + \frac{1}{2}\left(\frac{2}{3}\right)^{4/3}\right)\left(2\int_0^1 \tau^{5/2}\,\mathrm{d}\tau\right) = \frac{21\sqrt[3]{3} - 4 - 4\sqrt[3]{2}}{21\sqrt[3]{3}} \approx 0.70153491 > 0$$

We also deduce

$$g_1(t,s) = \frac{1}{\Gamma(7/3)} \begin{cases} t^{4/3}(1-s)^{4/3} - (t-s)^{4/3}, \ 0 \le s \le t \le 1, \\ t^{4/3}(1-s)^{4/3}, \ 0 \le t \le s \le 1, \end{cases}$$

$$g_2(t,s) = \frac{4}{3\sqrt{\pi}} \begin{cases} t^{3/2}(1-s)^{3/2} - (t-s)^{3/2}, \ 0 \le s \le t \le 1, \\ t^{3/2}(1-s)^{3/2}, \ 0 \le t \le s \le 1, \end{cases}$$

$\theta_1(s) = \dfrac{1}{4 - 6s + 4s^2 - s^3}$, $\theta_2(s) = \dfrac{1}{3 - 3s + s^2}$ for all $s \in [0,1]$. For the functions J_i, $i = 1, \cdots, 4$, we obtain

$$J_1(s) = \begin{cases} \frac{1}{\Gamma(7/3)}\left\{\frac{s(1-s)^{4/3}}{(4-6s+4s^2-s^3)^{1/3}} + \frac{2}{21\sqrt[3]{3}\Delta}\left[2(1-s)^{4/3} - 2(1-3s)^{4/3} + (2-2s)^{4/3} - (2-3s)^{4/3}\right]\right\}, \ 0 \le s < 1/3, \\ \frac{1}{\Gamma(7/3)}\left\{\frac{s(1-s)^{4/3}}{(4-6s+4s^2-s^3)^{1/3}} + \frac{2}{21\sqrt[3]{3}\Delta}\left[2(1-s)^{4/3} + (2-2s)^{4/3} - (2-3s)^{4/3}\right]\right\}, 1/3 \le s < 2/3, \\ \frac{1}{\Gamma(7/3)}\left\{\frac{s(1-s)^{4/3}}{(4-6s+4s^2-s^3)^{1/3}} + \frac{2}{21\sqrt[3]{3}\Delta}\left[2(1-s)^{4/3} + (2-2s)^{4/3}\right]\right\}, \ 2/3 \le s \le 1, \end{cases}$$

$$J_2(s) = \frac{16}{3\sqrt{\pi}\Delta}\left\{\frac{1}{7}(1-s)^{3/2} - \frac{1}{7}(1-s)^{7/2} - \frac{1}{5}s(1-s)^{5/2}\right\}, \ s \in [0,1],$$

$$J_3(s)=\frac{4}{3\sqrt{\pi}}\left\{\frac{s(1-s)^{3/2}}{\left(3-3s+s^2\right)^{1/2}}+\frac{4\left(1+\sqrt[3]{2}\right)}{3\sqrt[3]{3}\Delta}\left[\frac{1}{7}(1-s)^{3/2}-\frac{1}{7}(1-s)^{7/2}-\frac{1}{5}s(1-s)^{5/2}\right]\right\},\quad s\in[0,1]$$

$$J_4(s)=\begin{cases}\dfrac{1}{6\sqrt[3]{3}\Delta\Gamma(7/3)}\left[2(1-s)^{4/3}-2(1-3s)^{4/3}+(2-2s)^{4/3}-(2-3s)^{4/3}\right], 0\le s<1/3,\\ \dfrac{1}{6\sqrt[3]{3}\Delta\Gamma(7/3)}\left[2(1-s)^{4/3}+(2-2s)^{4/3}-(2-3s)^{4/3}\right], 1/3\le s<2/3,\\ \dfrac{1}{6\sqrt[3]{3}\Delta\Gamma(7/3)}\left[2(1-s)^{4/3}+(2-2s)^{4/3}\right], 2/3\le s\le 1.\end{cases}$$

Then we deduce that assumptions (J_1), (J_2) and (J_4) are satisfied. In addition, by using the above functions J_i, $i=1,\cdots,4$, we obtain $A:=\int_0^1 J_1(s)\,\mathrm{d}s\approx 0.15972386$, $B:=\int_0^1 J_2(s)\,\mathrm{d}s\approx 0.05446581$, $C:=\int_0^1 J_3(s)\,\mathrm{d}s\approx 0.09198682$, $D:=\int_0^1 J_4(s)\,\mathrm{d}s\approx 0.12885992$, and then $L=\max\{A+B,C+D\}\approx 0.22084674$. We choose $c_0=1$ and if we select $\tilde{a}<\frac{1}{L},\tilde{b}<\frac{1}{L}$, then we conclude that $f(x)<\frac{1}{L}$, $g(x)<\frac{1}{L}$ for all $x\in[0,1]$. For example, if $\tilde{a}\le 4.52$ and $\tilde{b}\le 4.52$, then the above conditions for *f* and *g* are satisfied. So, assumption (J_3) is also satisfied. By Theorems 2 and 3 we deduce that problem (S_0)-(BC_0) has at least one positive solution for sufficiently small $a_0>0$ and $b_0>0$, and no positive solution for sufficiently large a_0 and b_0.

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