

On Optimal Ordering of Service Parameters of a Coxian Queueing Model with Three Phases

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Abstract

We analyze a Coxian stochastic queueing model with three phases. The Kolmogorov equations of this model are constructed, and limit probabilities and the stationary probabilities of customer numbers in the system are found. The performance measures of this model are obtained and in addition the optimal order of service parameters is given with a theorem by obtaining the loss probabilities of customers in the system. That is, putting the greatest service parameter at first phase and the second greatest service parameter at second phase and the smallest service parameter at third phase makes the loss probability and means waiting time minimum. We also give the loss probability in terms of mean waiting time in the system. α_j is the transition probability from j -th phase to $(j+1)$ -th phase ($j = 1, 2$). In this manner while $\alpha_1 = 0$ and $\alpha_2 \geq 0$ this system turns into $M | M | 1 | 0$ queueing model and while $\alpha_1 \geq 0, \alpha_2 = 0$ the system turns into Cox(2) queueing model. In addition, loss probabilities are graphically given in a 3D graph for corresponding system parameters and phase transient probabilities. Finally it is shown with a numeric example that this theorem holds.

Keywords

Stochastic Coxian Queueing Model, Loss Probability, Limiting Distribution, Optimization, Kolmogorov Equations

1. Introduction

Phase-type queueing models are one of the essential parts of the stochastic queueing models. There is an urgent

need to construct phase-type distributions for complex representations of queueing models. The recent works being done in this field are: D. R. Cox shows how any distribution having a rational Laplace transform can be represented by a sequence of exponential phases [1]. S. Asmussen, O. Nerman and M. Olsson gave a paper on fitting phase-type distributions with the EM Algorithm [2]. Q. -M. He and H. Zhang presented an algorithm for computing minimal ordered Coxian representations of phase-type distributions whose Laplace-Stieltjes transform had only real poles [3]. The optimal ordering of the tandem server with two stages is given by [4]. R. Marie studied on calculating equilibrium probabilities for Coxian queueing systems in [5]. X. A. Papaconstantinou analyzed the stationary $E_k/C_2/s$ queueing system in [6]. P. M. Snyder and W. J. Stewart considered two approaches to the numerical solution of single node queueing models with phase-type [7]. In [8], an exact analysis of a fork/join station in a closed queueing network with inputs from servers with two-phase Coxian service distributions is represented. Q. -M. He and H. Zhang studied the approximation of matrix-exponential distributions by Coxian distributions in [9]. M. Fackrell gave a survey of where the phase-type distributions were used in the healthcare industry and purposed some ideas on how they were further utilized [10]. A. B. Zadeh studied a batch arrival queue system with Coxian-2 server vacations and admissibility restricted in [11]. V. Sağlam et al. give a paper on optimization of a Coxian queueing model with two phases in [12].

There is not enough work on the studies of optimizing the orders of service parameters for Coxian queueing model so far. Considering this fact in this paper we analyze a Coxian stochastic queueing model with three phases, and the Kolmogorov equations of this model are constructed, limit probabilities and the stationary probabilities of customer numbers in the system are found. The performance measures of this model are obtained and in addition the optimal order of service parameters is given by a theorem by obtaining the loss probabilities of customers in the system. We also give the loss probability in terms of mean waiting time in the system. Finally it is shown with a numeric example that this theorem holds.

2. Stochastic Model

We have obtained stochastic equation systems of a Coxian queueing model with three servers in which the stream is Poisson with λ parameter. The service time of any customer at server i ($i = 1, 2, 3$) is exponential with parameter μ_i . Two or more customers can not have service in the system at the same time. Let ξ_t be the state of server 1, η_t be the state of server 2 and ζ_t be the state of server 3 at any t time. α_j is the transition probability from j -th phase to $(j+1)$ -th phase ($j = 1, 2$) and $1 - \alpha_i$ be the loss probability of the system. This stochastic queueing model is illustrated in **Figure 1**. Limit probabilities, differential and difference equations of this system given later.

Limit Probabilities

Here $\{(\xi_t, \eta_t, \zeta_t), t \geq 0\}$ is a three-dimensional Markov chain with continuous parameter and state space is $\Omega = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$$p_{n_1, n_2, n_3}(t) = Prob\{\xi_t = n_1, \eta_t = n_2, \zeta_t = n_3\}, \forall (n_1, n_2, n_3) \in \Omega \tag{1}$$

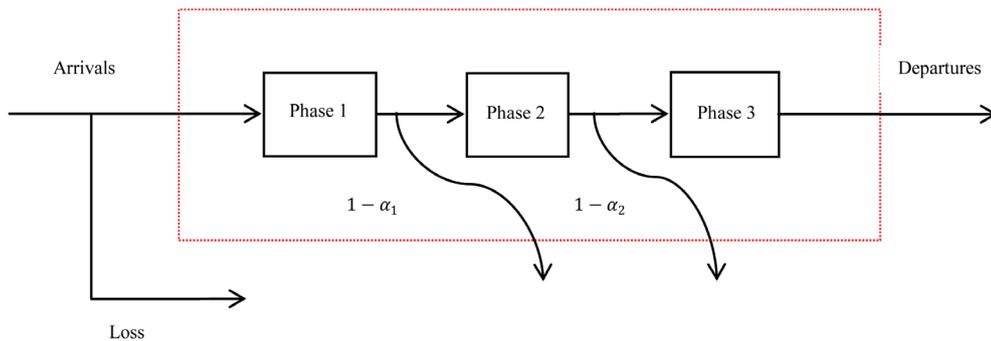


Figure 1. A three phase coxian queueing model.

Kolmogorov differential equation for these probabilities is obtained. The probabilities of the process $\{(\xi_t, \eta_t, \zeta_t), t \geq 0\}$ will be found for $(t, t+h)$, namely

$$\begin{aligned} p_{000}(t+h) &= (1-\lambda h + o(h))p_{000}(t) + (1-\alpha_1)(\mu_1 h + o(h))p_{100}(t) \\ &\quad + (1-\alpha_2)(\mu_2 h + o(h))p_{010}(t) + (\mu_3 h + o(h))p_{001}(t) + o(h) \\ p_{100}(t+h) &= (1-\mu_1 h + o(h))p_{100}(t) + (\lambda h + o(h))p_{000}(t) + o(h) \\ p_{010}(t+h) &= (1-\mu_2 h + o(h))p_{010}(t) + \alpha_1(\mu_1 h + o(h))p_{100}(t) + o(h) \\ p_{001}(t+h) &= (1-\mu_3 h + o(h))p_{001}(t) + \alpha_2(\mu_2 h + o(h))p_{010}(t) + o(h) \end{aligned} \quad (2)$$

We write Equation (2) as follows as $h \rightarrow 0$

$$\begin{aligned} p'_{000}(t) &= -\lambda p_{000}(t) + (1-\alpha_1)\mu_1 p_{100}(t) + (1-\alpha_2)\mu_2 p_{010}(t) + \mu_3 p_{001}(t) \\ p'_{100}(t) &= -\mu_1 p_{100}(t) + \lambda p_{000}(t) \\ p'_{010}(t) &= -\mu_2 p_{010}(t) + \alpha_1 \mu_1 p_{100}(t) \\ p'_{001}(t) &= -\mu_3 p_{001}(t) + \alpha_2 \mu_2 p_{010}(t) \end{aligned} \quad (3)$$

Furthermore, it is supposed that limiting distribution of $p_{n_1, n_2, n_3}(t)$ are exist as follow:

$$\lim_{t \rightarrow \infty} p_{n_1, n_2, n_3}(t) = p_{n_1, n_2, n_3}, \quad \lim_{t \rightarrow \infty} p'_{n_1, n_2, n_3}(t) = 0. \quad (4)$$

Steady-state equations for $\{(\xi_t, \eta_t, \zeta_t), t \geq 0\}$ are obtained as following:

$$\begin{aligned} 0 &= -\lambda p_{000} + (1-\alpha_1)\mu_1 p_{100} + (1-\alpha_2)\mu_2 p_{010} + \mu_3 p_{001} \\ 0 &= -\mu_1 p_{100} + \lambda p_{000} \\ 0 &= -\mu_2 p_{010} + \alpha_1 \mu_1 p_{100} \\ 0 &= -\mu_3 p_{001} + \alpha_2 \mu_2 p_{010} \end{aligned} \quad (5)$$

$$\sum_{(n_1, n_2, n_3) \in \Omega} p_{n_1, n_2, n_3} = 1 \quad (6)$$

We define $\rho_i = \lambda/\mu_i, (i=1, 2, 3)$. If we solve Equation (5) under condition (6), we obtain the following three dimension probability function:

$$p_{n_1 n_2 n_3} = \begin{cases} \frac{1}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}, & (n_1, n_2, n_3) = (0, 0, 0) \\ \frac{\rho_1}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}, & (n_1, n_2, n_3) = (1, 0, 0) \\ \frac{\alpha_1 \rho_2}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}, & (n_1, n_2, n_3) = (0, 1, 0) \\ \frac{\alpha_1 \alpha_2 \rho_3}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}, & (n_1, n_2, n_3) = (0, 0, 1) \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

3. Obtaining the Measures of Performance

Let N be the random variable that describes the number of customers in the system. The mean number of costumers:

$$E[N] = \sum_{(n_1, n_2, n_3) \in \Omega} (n_1 + n_2 + n_3) p_{n_1 n_2 n_3} = \frac{\rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3} \quad (8)$$

$$\begin{aligned}
\text{Var}[N] &= E[N^2] - E^2[N] \\
&= \frac{\rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3} - \left(\frac{\rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3} \right)^2 \\
&= \frac{\rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}{(1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3)^2}
\end{aligned} \tag{9}$$

3.1. The Coxian Queue Using Laplace Transform

Let W be the random variable that describes waiting time of customers in the system. Laplace transform of W

$$\mathcal{L}_W(s) = \frac{(1-\alpha_1)\mu_1}{\mu_1+s} + \frac{\alpha_1\mu_1}{\mu_1+s} \frac{(1-\alpha_2)\mu_2}{\mu_2+s} + \frac{\alpha_1\mu_1}{\mu_1+s} \frac{\alpha_2\mu_2}{\mu_2+s} \frac{\mu_3}{\mu_3+s}. \tag{10}$$

Mean waiting time in system of a customer for Cox(3) is found by formula (10)

$$E[W] = - \left. \frac{d\mathcal{L}_W(s)}{ds} \right|_{s=0} = \frac{1}{\mu_1} + \frac{\alpha_1}{\mu_2} + \frac{\alpha_1\alpha_2}{\mu_3} \tag{11}$$

3.2. The optimization of Measures of Performance

Loss probability

Let P_{loss} be the loss probability of customer in the system. In this regards, since there is no queue in the system, loss probability is calculated as following:

$$P_{\text{loss}} = p_{100} + p_{010} + p_{001}. \tag{12}$$

3.3. Optimal Order of Servers

We can put three different service parameters to three stages μ in 3! different position. In this case there are 6 different loss probabilities.

The following theorem is given on minimization of loss probability.

Theorem 1. Putting the greatest service parameter at first phase and the second greatest service parameter at second phase and the smallest service parameter at third phase makes the loss probability minimum. That is,

$$P_{\text{loss}}^{(1)} \leq P_{\text{loss}}^{(i)}, \quad i = 2, \dots, 6 \tag{13}$$

Proof. Let's suppose $\mu_1 \geq \mu_2 \geq \mu_3$. In this case we have,

$$\begin{aligned}
&\left(\frac{1}{\mu_2} - \frac{1}{\mu_3} \right) \leq 0 \\
&\alpha_1(1-\alpha_2) \left(\frac{1}{\mu_2} - \frac{1}{\mu_3} \right) \leq 0 \\
&\frac{\alpha_1}{\mu_2} + \frac{\alpha_1\alpha_2}{\mu_3} \leq \frac{\alpha_1}{\mu_3} + \frac{\alpha_1\alpha_2}{\mu_2} \\
&\alpha_1\rho_2 + \alpha_1\alpha_2\rho_3 \leq \alpha_1\rho_3 + \alpha_1\alpha_2\rho_2 \\
&\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3 \leq \rho_1 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_2 \\
&1 + \frac{1}{\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3} \geq 1 + \frac{1}{\rho_1 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_2} \\
&\frac{1 + \rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3}{\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3} \geq \frac{1 + \rho_1 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_2}{\rho_1 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_2}
\end{aligned}$$

$$P_{\text{loss}}^{(1)} \leq P_{\text{loss}}^{(2)}.$$

Similarly,

$$(1-\alpha_1)\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) \leq 0$$

$$\frac{1}{\mu_1} + \frac{\alpha_1}{\mu_2} \leq \frac{1}{\mu_2} + \frac{\alpha_1}{\mu_1}$$

$$\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3 \leq \rho_2 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_3$$

$$1 + \frac{1}{\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3} \geq 1 + \frac{1}{\rho_2 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_3}$$

$$\frac{1 + \rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3}{\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3} \geq \frac{1 + \rho_2 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_3}{\rho_2 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_3}$$

$$P_{\text{loss}}^{(1)} \leq P_{\text{loss}}^{(3)}.$$

Since,

$$\alpha_1(1-\alpha_2)\left(\frac{1}{\mu_1} - \frac{1}{\mu_3}\right) + (1-\alpha_1)\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) \leq 0,$$

we obtain

$$\frac{1}{\mu_1} + \frac{\alpha_1}{\mu_2} + \frac{\alpha_1\alpha_2}{\mu_3} \leq \frac{1}{\mu_2} + \frac{\alpha_1}{\mu_3} + \frac{\alpha_1\alpha_2}{\mu_1}$$

$$\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3 \leq \rho_2 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_1$$

$$\frac{1 + \rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3}{\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3} \geq \frac{1 + \rho_2 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_1}{\rho_2 + \alpha_1\rho_3 + \alpha_1\alpha_2\rho_1}$$

$$P_{\text{loss}}^{(1)} \leq P_{\text{loss}}^{(4)}.$$

$$(1-\alpha_1)\left(\frac{1}{\mu_1} - \frac{1}{\mu_3}\right) + \alpha_1(1-\alpha_2)\left(\frac{1}{\mu_2} - \frac{1}{\mu_3}\right) \leq 0$$

$$\frac{1}{\mu_1} + \frac{\alpha_1}{\mu_2} + \frac{\alpha_1\alpha_2}{\mu_3} \leq \frac{1}{\mu_3} + \frac{\alpha_1}{\mu_1} + \frac{\alpha_1\alpha_2}{\mu_2}$$

$$\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3 \leq \rho_3 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_2$$

$$\frac{1 + \rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3}{\rho_1 + \alpha_1\rho_2 + \alpha_1\alpha_2\rho_3} \geq \frac{1 + \rho_3 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_2}{\rho_3 + \alpha_1\rho_1 + \alpha_1\alpha_2\rho_2}$$

$$P_{\text{loss}}^{(1)} \leq P_{\text{loss}}^{(5)}.$$

Finally,

$$(1-\alpha_1\alpha_2)\left(\frac{1}{\mu_1} - \frac{1}{\mu_3}\right) \leq 0$$

$$\frac{1}{\mu_1} + \frac{\alpha_1\alpha_2}{\mu_3} \leq \frac{1}{\mu_3} + \frac{\alpha_1\alpha_2}{\mu_1}$$

$$\begin{aligned} \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3 &\leq \rho_3 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_1 \\ \frac{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}{\rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3} &\geq \frac{1 + \rho_3 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_1}{\rho_3 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_1} \\ P_{\text{loss}}^{(1)} &\leq P_{\text{loss}}^{(6)} \end{aligned}$$

Corollary. Since,

$$P_{\text{loss}} = \frac{1}{1 + \frac{1}{\lambda E[W]}} \tag{14}$$

the minimum value which makes P_{loss} minimum also makes $E[W]$ minimum.

4. Numerical Example

In this section the loss probabilities are calculated for some values of system probabilities and α_1, α_2 probabilities. The calculated loss probabilities are given in **Table 1**. For the values $\lambda = 2.2$, $\mu_1 = 8.6$; $\mu_2 = 6.8$, $\mu_3 = 4.4$ and for various values of α_1 and α_2 it is seen in **Table 1** that $P_{\text{loss}}^{(1)}$ has its minimum value, this shows that Theorem1 holds.

Under condition given in Theorem1, for the values $\lambda = 2.2$, $\mu_1 = 8.6$; $\mu_2 = 6.8$, $\mu_3 = 4.4$ and for all values of α_1 and α_2 in domain set, the loss probabilities are calculated in **Table 2** and graphically given in 3D **Figure 2** in two different view angle. $P_{\text{loss}}^{(1)}$ is indicated by green surface in this figure. As it is seen in this graph, $P_{\text{loss}}^{(1)}$ is minimum for all values of α_1 and α_2 . For a customer to have service at each stage it must be $\alpha_1 = \alpha_2 = 1$ or it must be $\alpha_1 = 0$ for the customer to leave the system after first stage.

5. Conclusion

By constructing this stochastic queueing model, transient probabilities are obtained. Depending on these probabilities, mean number of customer in the system, the mean waiting time in this system by Laplace transform and the loss probability of any customer are given. It is shown by Theorem 1 that putting the greatest service parameter at first phase and the second greatest service parameter at second phase and the smallest service parameter at third phase makes the loss probability minimum. For the values $\lambda = 2.2$, $\mu_1 = 8.6$; $\mu_2 = 6.8$, $\mu_3 = 4.4$ and

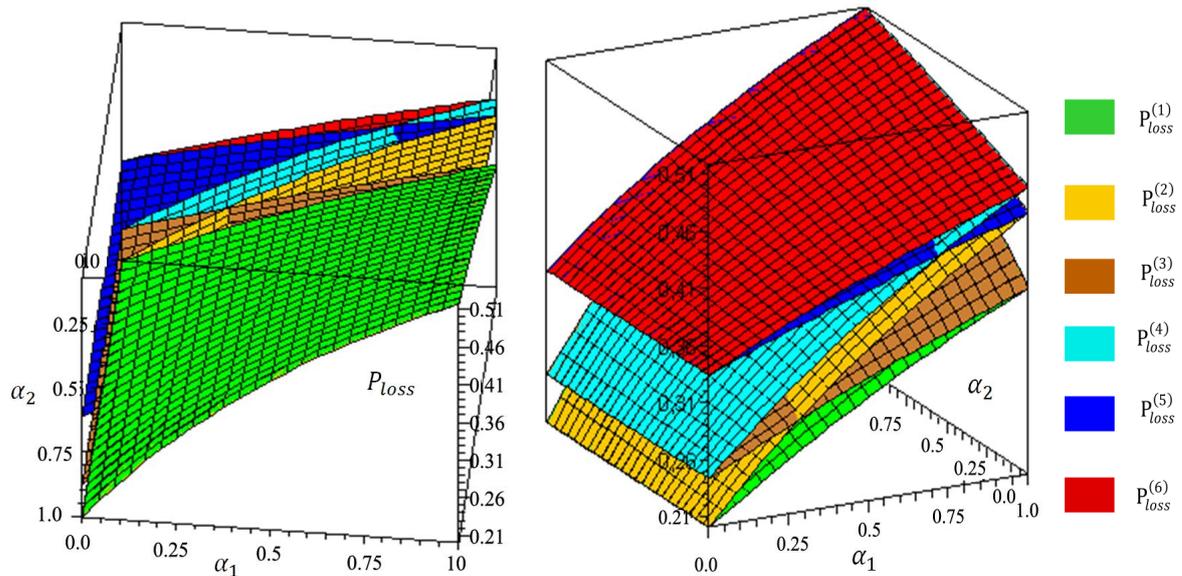


Figure 2. The loss probabilities for α_1 and α_2 and system parameters.

Table 1. Placing the service parameters to phases and corresponding loss probabilities.

Placing service parameters to phases			Loss probabilities
Phase 1	Phase 2	Phase 3	
μ_1	μ_2	μ_3	$P_{\text{loss}}^{(1)} = \frac{\rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}{1 + \rho_1 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_3}$
μ_1	μ_3	μ_2	$P_{\text{loss}}^{(2)} = \frac{\rho_1 + \alpha_1 \rho_3 + \alpha_1 \alpha_2 \rho_2}{1 + \rho_1 + \alpha_1 \rho_3 + \alpha_1 \alpha_2 \rho_2}$
μ_2	μ_1	μ_3	$P_{\text{loss}}^{(3)} = \frac{\rho_2 + \alpha_1 \rho_1 + \alpha_1 \alpha_2 \rho_3}{1 + \rho_2 + \alpha_1 \rho_1 + \alpha_1 \alpha_2 \rho_3}$
μ_2	μ_3	μ_1	$P_{\text{loss}}^{(4)} = \frac{\rho_2 + \alpha_1 \rho_3 + \alpha_1 \alpha_2 \rho_1}{1 + \rho_2 + \alpha_1 \rho_3 + \alpha_1 \alpha_2 \rho_1}$
μ_3	μ_1	μ_2	$P_{\text{loss}}^{(5)} = \frac{\rho_3 + \alpha_1 \rho_1 + \alpha_1 \alpha_2 \rho_2}{1 + \rho_3 + \alpha_1 \rho_1 + \alpha_1 \alpha_2 \rho_2}$
μ_3	μ_2	μ_1	$P_{\text{loss}}^{(6)} = \frac{\rho_3 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_1}{1 + \rho_3 + \alpha_1 \rho_2 + \alpha_1 \alpha_2 \rho_1}$

Table 2. For $\lambda = 2.2$, $\mu_1 = 8.6$; $\mu_2 = 6.8$, $\mu_3 = 4.4$.

α_1	α_2	$P_{\text{loss}}^{(1)}$	$P_{\text{loss}}^{(2)}$	$P_{\text{loss}}^{(3)}$	$P_{\text{loss}}^{(4)}$	$P_{\text{loss}}^{(5)}$	$P_{\text{loss}}^{(6)}$
0	0	0.203703	0.203704	0.244444	0.24444444	0.33333333	0.33333333
0.2	0	0.242722	0.262436	0.272564	0.29752066	0.35532233	0.36090225
0.4	0.8	0.352845	0.358704	0.369425	0.37709835	0.4137837	0.41563943
0.5	0.5	0.351734	0.36976	0.365658	0.3893066	0.41479035	0.42053111
0.8	0.4	0.402855	0.431606	0.407646	0.44610302	0.44695789	0.45672369
1	0	0.366825	0.430464	0.366825	0.4516129	0.43046357	0.4516129
1	1	0.519078	0.519079	0.519079	0.51907894	0.51907894	0.51907894

for various values of α_1 and α_2 it is seen in **Table 1** that $P_{\text{loss}}^{(1)}$ has its minimum value, this shows that Theorem 1 holds. In the case $\alpha_1 = \alpha_2 = 1$, the loss probabilities are all equal to each other. This is seen in both **Table 1** and Graph 1. While $\alpha_1 = 0$ and $\alpha_2 \geq 0$ this system turns into $M|M|1|0|$ queueing model and while $\alpha_1 \geq 0, \alpha_2 = 0$ the system turns into Cox(2) queueing model. For further studies, higher moments of mean waiting time in the system can be obtained and by using these moments some various statistical measures can be calculated such as variance, skewness, kurtosis and coefficient of variation. Also this model can be expanded to k -phases.

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