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A Characterization of Complex Projective Spaces by Sections of Line Bundles

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Abstract

Let M be a n-dimensional compact irreducible complex space with a line bundle L. It is shown that if M is completely intersected with respect to L and $\dim H^0(M, L) = n + 1$, then M is biholomorphic to a complex projective space P^n of dimension n.

Keywords

Complex Space, Projective Space, Line Bundle, Complete Intersected

1. Introduction

Kobayashi and Ochiai [1] have given Characterizations of the complex projective spaces. Kobayashi-Ochiai Theorem [1] has been applied to obtain many important characterizations of the projective spaces, such as the proof of Frankel conjectures [2], the proof of Hartshorne conjecture [3], and many others [4]-[7]. In this note, we want to give a characterization of the complex projective spaces via sections of line bundles.

Results which can be found in [1] [8] and [9] are used freely often without explicit references. Let M be a complex space with a line bundle L. ϑ is the sheaf of germs of sheaf of holomorphic functions, $\vartheta(L)$ is the sheaf of germs of holomorphic sections of a line bundle L. $H^0(M,L)$ means $H^0(M,\vartheta(L))$.

2. Characterization of the Projective Spaces

In this paper, a characterization of the projective space will be given.

Definition. Let M be a compact complex space with a line bundle L. M is said to be completely intersected with respect to a line bundle L, provided that complex subspace $V(\varphi_1,\cdots,\varphi_k)$ is irreducible for any linearly independent elements of $\varphi_1,\cdots,\varphi_k$ of $H^0(M,L)$, where each φ_i is irreducible, and $V(\varphi_1,\cdots,\varphi_k)$ is the

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common zeros of $\varphi_1,\cdots,\varphi_k$.

From the Lemma 1.1 [1] and the proof of theorem 16.2.1 [8], we have

Lemma 1. Let V be a compact irreducible complex space. Let F and L be line bundles over V. Let φ be an irreducible section of L and put $S=V(\varphi)=\{x\in V;\varphi(x)=0\}$. The following sequence of sheaf homomorphisms is exact:

$$0\to\vartheta(F)\xrightarrow{\alpha}\vartheta(F\otimes L)\xrightarrow{\beta}\vartheta_S(F\otimes L)\to 0$$

where μ is the multiplication by φ, $\vartheta_S(F\otimes L)$ is the sheaf defined by $\vartheta_S(F\otimes L)|_S=\vartheta(F\otimes L)|_S$ and $\vartheta_S(F\otimes L)|_{V-S}=0,\beta$ is the restriction map.

Lemma 2. Let M be a n-dimensional compact complex space with a line bundle L. Let $\varphi_1,\cdots,\varphi_k$ be linear independent elements of $H^0(M,L)$, such that each φ_i is irreducible. If M is completely intersected with respect to L, then there is an exact sequence:

$$0\to(\varphi_1,\cdots,\varphi_k)\to H^0(M,L)\xrightarrow{\beta}H^0(V_{n-k},L)$$

where $(\varphi_1,\cdots,\varphi_k)$ is the subspace of $H^0(M,L)$ spanned by the sections $\varphi_1,\cdots,\varphi_k$ $(k\le n)$ and β is the restriction map.

Proof. The proof is by induction on k. The case $k=0$ is trivial. Since M is completely intersected with respect to L, $V_{n-k+1}=V(\varphi_1,\cdots,\varphi_{k-1})$ and $V_{n-k}=V(\varphi_1,\cdots,\varphi_k)$ are irreducible. Assume the lemma for $k-1$, we have the exact sequence:

$$0\to(\varphi_1,\cdots,\varphi_{k-1})\to H^0(M,L)\xrightarrow{\beta}H^0(V_{n-k+1},L)$$

If $V(\varphi_k)\subset V_{n-k+1}$, then $\varphi_k\in(\varphi_1,\cdots,\varphi_{k-1})$ from which it follows that $\varphi_1,\cdots,\varphi_k$ are linearly dependent, a contradiction. Thus, φ_k is nontrivial on V_{n-k+1}; it follows that $V_{n-k}=V_{n-k+1}\cup V(\varphi_k)$ defined as the set of zeros of = φ_k on V_{n-k+1} is an irreducible divisor.

We apply Lemma l to $V=V_{n-k+1}$, $F=\vartheta$, $\varphi=\varphi_k$. Then, $S=V_{n-k}$. The exact sequence in Lemma 1 induces the following exact sequence

$$0\to H^0(V_{n-k+1},\vartheta)\xrightarrow{\alpha}H^0(V_{n-k+1},L)\xrightarrow{\beta}H^0(V_{n-k},L)$$

This means that the kernel of the restriction map β is spanned by the restriction of φ_k to V_{n-k+1}. Combining this with the lemma for $k-1$, we obtain the lemma for k.

Now we give the main result of this paper.

Theorem. Let M be a n-dimensional compact irreducible complex space with a line bundle L. If $\dim H^0(M,L)=n+1$ and M is completely intersected with respect to L, then M is biholomorphic to a complex space P^n of dimension n.

Proof. Since $\dim H^0(M,L)=n+1$, we can choose linearly independent sections $\varphi_1,\cdots,\varphi_{n+1}$ from $H^0(M,L)$ with each φ_i irreducible. Put $D_i=V(\varphi_i)=\{x\in M|\varphi_i(x)=0\},i=1,\cdots,n+1$.

Claim 1. Each D_i is an irreducible divisor of M.

First of all, D_i is nonempty. Indeed, if $D_i=\phi$ for some i, then $\varphi_i(x)\ne 0$ for all $x\in M$. Define map $h:M\times C\to L$ by $h(x,c)=c\varphi_i(x)$. It is clear that h is isomorphic, that is, L is a trivial line bundle over M. It follows that $\dim H^0(M,L)=1$, which is contradictory to the hypothesis that $\dim H^0(M,L)=n+1\ge 2$. Thus, each $D_i=\phi$. Since φ_i is irreducible, D_i is irreducible and $\dim D_i=n-1$ by [Corollary 14, Ch II, 3] and [Theorem 11, Ch III, 3]. Hence, D_i is an irreducible divisor.

Let $V_{n-i}=V(\varphi_1,\cdots,\varphi_i)$ be the common zeros of $\varphi_1,\cdots,\varphi_i$, then $V_{n-i}=D_1\cap\cdots\cap D_i$. By the hypothesis, M is completely intersected with respect to L; each V_{n-i} is irreducible.

Claim 2. $\dim V_{n-i}=n-i,i=1,\cdots,n$.

For $i=1$, $V_{i-1}=D_1$ is an irreducible divisor by Claim 1, thus $\dim V_{n-1}=n-1$. Assume that $\dim V_{n-i+1}=n-i+1$. If $V_{n-i+1}=V(\varphi_1,\cdots,\varphi_{i-1})\subset D_i$, then $\varphi_i\in(\varphi_1,\cdots,\varphi_{i-1})$ by Lemma 2. This induces that $\varphi_1,\cdots,\varphi_i$ are linearly dependent, a contradiction. Thus, φ_i is nontrivial on the irreducible complex subspace V_{n-i+1}. It follows by [Theorem 14, Ch. III, 3] that $\dim V_{n-i}=\dim V_{n-i+1}\cap D_i=\dim V_{n-i+1}-1=n-i$.

Claim 3. $H^0(M,L)$ is base point free.

By Claim 2, V_{n-i} is an irreducible complex subspace of dimension $n-i$. In particular, V_0 is one point. If φ_{n+1} vanishes at V_0, then $\varphi_1,\cdots,\varphi_{n+1}$ are linearly dependent by Lemma 2. Since $\varphi_1,\cdots,\varphi_{n+1}$ are defined as linearly independent sections in the beginning of this proof, it is a contradiction. Thus, φ_{n+1} does not vanish at V_0. This shows that $\varphi_1,\cdots,\varphi_{n+1}$ have no common zeros, that is, $H^0(M,L)$ is base point free.

Since $\dim H^0(M,L)=n+1$, we may let P^n be the complex projective space of dimension *n* defined as the set of hyperplanes through the origin in $H^0(M,L)$. For any point $x\in M$, put $\psi(x)=\{\varphi\in H^0(M,L)|\varphi(x)=0\}$. Since $H^0(M,L)$ is base point free, $\psi(x)$ is a hyperplane through the origin in $H^0(M,L)$ and so $\psi(x)\in P^n$. We now obtain a holomorphic mapping $\psi:M\to P^n$.

Claim 4. The mapping ψ is bijective.

Giving a point *y* of P^n, it is a hyperplane through the origin in $H^0(M,L)$. Let $t_1,\cdots,t_n\in H^0(M,L)$ be a basis for this hyperplane with each t_i irreducible. Then, *y* is the complex subspace spanned by $t_1,\cdots,t_n$, that is, $(t_1,\cdots,t_n)=y$. Let $T_i=V(t_i)$ be the set of zeros of t_i for $i=1,\cdots,n$. Since *M* is completely intersected with respect to *L*, $V(t_1,\cdots,t_n)$ is irreducible and $\dim V(t_1,\cdots,t_n)=0$ by Claim 2, that is, $V(t_1,\cdots,t_n)$ is a point. It follows that there exists a point *x* of *M*, such that $t_1(x)=\cdots=t_n(x)=0$.

Thus, $t_1,\cdots,t_n\in\psi(x)=\{\varphi\in H^0(M,L)|\varphi(x)=0\}$. By Lemma 2, we have the exact sequence

$$0\to(t_1,\cdots,t_n)\to H^0(M,L)\xrightarrow{\beta}H^0(V(t_1,\cdots,t_n)L).$$

For any $\varphi\in\psi(x)$, we have $\varphi(x)=0$. Since $V(t_1,\cdots,t_n)=\{x\}$ and the above sequence is exact, it follows that $\varphi\in(t_1,\cdots,t_n)$ and so $\psi(x)=(t_1,\cdots,t_n)=y$. Thus, ψ is surjective.

On the other hand, let *u* and *v* be any two points of *M*. $\psi(u)$ is a hyperplane through the origin in $H^0(M,L)$. Let $s_1,\cdots,s_n$ be a basis for $\psi(u)$ with each s_i irreducible. By Claim 2, $V(s_1,\cdots,s_n)$ is a single point and this induces that $V(s_1,\cdots,s_n)=\{u\}$. If $\psi(u)=\psi(v)$, then $(s_1,\cdots,s_u)=\psi(v)$. It follows that $v\in V(s_1,\cdots,s_n)=\{u\}$, that is, $v=u$. Thus ψ is injective.

Consequently, we have shown that *M* is biholomorphic to a complex projective space P^n of dimension *n*.

As an application of the theorem above, we give a proof for the famous Kobayashi-Ochiai Theorem [1].

Corollary ([Theorem 1.1 [1]). Let *M* be a *n*-dimensional complex irreducible complex space with an ample line bundle *F*. If $C_1(F)^n[M]=1$ and $\dim H^0(M,F)=n+1$, then *M* is biholomorphic to a complex projective space P^n of dimension *n*.

Proof. By the theorem in this paper, it suffices to show that *M* is completely intersected with respect to *F*. Let $\varphi_1,\cdots,\varphi_k$ be linearly independent elements of $H^0(M,F)$ with each φ_i irreducible. Let $V_{n-k}=V(\varphi_1,\cdots,\varphi_k)$ be the common zeros of $\varphi_1,\cdots,\varphi_k$ on *M*.

Assume that V_{n-k} is reducible, then write $V_{n-k}=V'+V''$ for some nontrivial V' and V''. Because $\varphi_1,\cdots,\varphi_k$ are linearly independent elements of $H^0(M,F)$ with each φ_i irreducible, there are no common zeros of $\varphi_1,\cdots,\varphi_k$ on $C_1(F)^k[M]$. And due to the definition of V_{n-k}, we have

$$\begin{aligned}C_1(F)^n[M]&=C_1(F)^k C_1(F)^{n-k}[M]=C_1(F)^{n-k}[V_{n-k}]\\&=C_1(F)^{n-k}[V'+V'']=C_1(F)^{n-k}[V']+C_1(F)^{n-k}[V'']\end{aligned}$$

By the hypothesis, $C_1(F)^n[M]=1$, and so

$$C_1(F)^{n-k}[V']+C_1(F)^{n-k}[V'']=1.$$

Since *F* is ample, $C_1(F)^{n-k}[V']$ and $C_1(F)^{n-k}[V'']$ are positive integers. Thus, $C_1(F)^n[M]\ge 2$, a contradiction. It follows that V_{n-k} is irreducible. By the theorem above, *M* is biholomorphic to P^n.

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