

The Stochastic Error Rate Estimation of Prediction Distributions

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Abstract

The estimation of claims reserves is usually done by applying techniques called IBNR techniques within a stochastic framework. The main objective of this paper is to predict the partial reserve and to estimate the error rate of prediction distributions by using the stochastic model proposed in [1].

Keywords

Reserving Claims, EDS, Error Rate, Prediction Distributions

1. Introduction

The calculation of the provisions for disaster payments is intended to allow the integral payment of the commitments to the policy-holders and the recipients of the contract. The provisions measure the commitments that the insurer still has to honor. Nevertheless, this countable concept requires a subjacent probabilistic model since it allows one to define the ultimate claim, taking into account the disasters not yet declared, but which have occurred, the disasters not sufficiently funded. Reserves are given by evaluating the provisions for each contract, IBNR (sinister not yet declared) and IBNER (sinister not sufficiently funded). Traditional methods of provisioning (by triangulation) rest on the assumption that the data are homogeneous and in sufficient quantity to ensure a certain stability and a certain credibility. The purpose of this paper is to propose a stochastic extension of the Chain-Ladder model concerning the partial prediction reserve and to estimate the error rate of prediction distributions, which seems to be closer to reality for us than the existing methods of Schnieper [2]; Mack [3]; Liu and Verrall [4]; Verrall and England [5].

Models in which parameters move between a fixed number of regimes with switching controlled by an unobserved stochastic process, are very popular in a great variety of domains (Finance, Biology, Meteorology, Net-

works, etc.). This is notably due to the fact that this additional flexibility allows the model to account for random regime changes in the environment. In this paper we consider the prediction of partial reserve and consider the estimation of error rate of prediction distributions for a model described by a stochastic differential equation (SDE) with Markov regime-switching (MRS), *i.e.*, with parameters controlled by a finite state continuous-time Markov chain (CTMC) [1] and [6]. Such a model was used, for example, in Deshpande and Ghosh (2008) [7], to price options in a regime switching market. In such a setting, the parameter estimation problem posed a real challenge, mainly due to the fact that the paths of the CTMC were unobserved. A standard approach consists in using the celebrated EM algorithm (Dempster, Laird and Rubin, 1977) [8] as proposed, for example in Elliott, Malcolm and Tsoi (2003) [9] and Hamilton (1990) [10], study this problem using a filtering approach.

The rest of the paper is structured as follows. In Section 2, we present the stochastic model for our problem. Section 3 is devoted to predicting the claims reserves variance. We conclude with a summary in the last section.

2. Hypotheses and Description of the Model

We suppose that the available data have a triangular form indexed by the year of accident, i , and the development time, t . Given a triangle, on T years, the goal is to consider models using a minimum of parameters, in order to envisage the best possible amounts of payments of future disasters. We note the evolution of the amounts of payments of the cumulated real disasters obtained by $\{C_t^i; i = 1, 2, \dots, T; t \in [1, T - i + 1]\}$.

C_t^i indicates the evolution of the cumulated real disasters indexed by the year of accident, i , and the time of development, t . We suppose that the increase in the disasters obtained $(C_t^i - C_{t-1}^i)$ is the sum of the disasters not sufficiently funded $(-D_t^i)$ and of the disasters not declared yet claims (N_t^i) . In the paper [1] the following relations between C , N and D is proposed.

$$C_t^i = C_{t-1}^i - D_t^i + N_t^i \quad 1 \leq i \leq n \quad \forall t \in [1, +\infty[. \quad (1)$$

We indicate $\mathcal{H}_s^i = \{D_t^i, N_t^i | i + t \leq s + 1\}$ for all the variables in the triangles D and N observed until the moment s .

To simulate the future claims, it is supposed that the not sufficiently funded claims $(D_t^i)_{t \in [1, +\infty[}$, the stochastic differential equation of diffusion and the not yet incurred claims $(N_t^i)_{t \in [1, +\infty[}$ are governed by the stochastic differential equation of Black and Scholes with jump. This assumption on the probability density function of N_t^i ensures positivity and for D_t^i ensures the membership \mathbb{R} , contrary to what is proposed by Liu and Verrall [4].

Conditionally with \mathcal{H}_t^i , we simulate the distribution of N_t^i as solution of the following stochastic differential equation:

$$dN_t^i = N_{t-}^i \left(\alpha^i(t, X_t, N_{t-}^i) dt + \tau^i(t, X_t, N_{t-}^i) dB_t + \kappa^i d\mathcal{P}_t \right), \quad N_{T-i+1}^i = x, \quad (2)$$

where B_t is a standard Brownian motion on \mathbb{R} ; \mathcal{P}_t is the Poisson process of intensity λ ; X_t is a Markov process at continuous time; $\alpha^i: \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}^+ \rightarrow \mathbb{R}$; $\tau^i: \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}^+ \rightarrow \mathbb{R}$; verifying

$$\int |\alpha^i(t, X_t, N_{t-}^i)| dt < \infty \quad \text{and} \quad \int |\tau^i(t, X_t, N_{t-}^i)|^2 dt < \infty$$

and κ^i is a positive constant.

Conditionally with \mathcal{H}_t^i , we suppose that the evolution of D_t^i is governed by the following diffusion:

$$dD_t^i = \rho^i(t, X_t, D_{t-}^i, N_{t-}^i) dt + \sigma^i(t, X_t, D_{t-}^i, N_{t-}^i) dB_t, \quad D_{T-i+1}^i = y, \quad (3)$$

where B_t is a standard Brownian motion on \mathbb{R} ; X_t is a Markov process at continuous time;

$\mu^i: \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\sigma^i: \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\int |\rho^i(t, X_t, D_{t-}^i, N_{t-}^i)| dt < \infty$ and $\int |\sigma^i(t, X_t, D_{t-}^i, N_{t-}^i)| dB_t < \infty$. The process (X_t) is assumed to be a continuous time Markov process taking values in the set $S = \{i: 1 \leq i \leq M\}$. The transition probabilities of this chain are denoted by p_{ij} , $i, j \in S$ and

the transition rate matrix is $Q_0 = (q_{ij})_{i,j \in S}$ with

$$\xi_i > 0, \quad q_{ij} = \xi_i p_{ij} \text{ if } i \neq j, \text{ and } q_{ii} = -\sum_{j \neq i} q_{ij}, \quad i, j \in S.$$

For any state $i = 1, 2, \dots, M$, consider priors μ_i and σ_i defined as follows

$$\mu_i | \theta \stackrel{\text{ind}}{\sim} \mathcal{N}(\theta, \tau^\mu) \quad (4)$$

$$\theta \sim \mathcal{N}(0, A), \quad A > 0, \quad (5)$$

$$\sigma_i \stackrel{\text{ind}}{\sim} \Gamma(\nu_1, \nu_2) \quad (6)$$

where $\Gamma(\nu_1, \nu_2)$ denotes a Gamma distribution with shape parameter ν_1 and scale parameter ν_2 .

Let us define the log-returns $Y_t = Z_t - Z_{t-1} = \log(N_t/N_{t-1})$ for the Equation (2) and $Y_t = D_t - D_{t-1}$ for the second Equation (3), $t = 1, 2, \dots, n$.

Given a path $X = \{X_s, 0 \leq s \leq n\}$. Let $T_j(t)$ be the time spent by the path X in state j in the time interval $[t-1, t]$. Define

$$\begin{aligned} \mu(t) &:= \sum_{j=1}^M \mu(j) T_j(t); \\ \tau^2(t) &:= \sum_{j=1}^M \tau^2(j) T_j(t); \end{aligned} \quad (7)$$

$$\begin{aligned} \lambda(t) &:= \sum_{j=1}^M \lambda(j) T_j(t). \\ \rho(t) &:= \sum_{j=1}^M \rho(j) T_j(t); \\ \sigma^2(t) &:= \sum_{j=1}^M \sigma^2(j) T_j(t). \end{aligned} \quad (8)$$

Then, conditional on the path X , the solutions of the Equations (2) and (3) are respectively given by :

- the solution of the first Equation (2) is

$$Y_t = Y_0 \exp \left\{ \int_0^t (\tau(s) dB_s + \kappa d\mathcal{P}_s) + \int_0^t \left(\mu(s) - \frac{\tau^2(s)}{2} \right) ds \right\} \times \prod_{0 < s \leq t} (1 + \kappa \Delta \mathcal{P}_s) e^{-\kappa \Delta \mathcal{P}_s} \quad t = 1, 2, \dots, n, \quad (9)$$

and,

- Y_t are i.i.d. $\mathcal{N}(\rho_t, \sigma_t)$, $t = 1, 2, \dots, n$ for the second Equation (3).

3. Predicted Claims Reserves Variance

The main reason for using stochastic models is to estimate the error rate of prediction distributions. It is useful for solvency, capital modelling and measurement of risk. We begin by proving how the predicted error rate can be calculated. The predicted error rate is obtained from the predicted variance of the partial reserve of loss. We remember that $\mathcal{H}_s = \{D_t^i, N_t^i | i \in \{1, \dots, n\}, i+t \leq s+1\}$ the variables in triangles D and N observed until a time s .

Let $0 = t_0 < t_1 < t_2 < \dots < t_j < \dots < t_l = T_F$. The future claims IBNR and IBNER estimators are given respectively by $\hat{W}_k^{N^i}$, $\hat{W}_k^{D^i}$ $i \in \{1, \dots, n\}$ and $k \in \{J+1, \dots, l\}$. Our objective is to predict the partial reserve. We denote the partial reserve by:

$$R^i = \sum_{k=J+1}^l (W_k^{N^i} - W_k^{D^i}), \quad (10)$$

$\hat{R}^i = \sum_{k=J+1}^l (\hat{W}_k^{N^i} - \hat{W}_k^{D^i})$ is its estimator.

The mean square error prediction of $\hat{R}^i | \mathcal{H}_{T_F}$ can be written as follows

$$MSQP(\hat{R}^i | \mathcal{H}_{T_F}) = E \left[(\hat{R}^i - R^i)^2 | \mathcal{H}_{T_F} \right].$$

Theorem 3.1 Let $(W_t^{N^i})_t$ and $(W_t^{D^i})_t$, respectively the future claims IBNR and IBNER of the i^{th} contract, verifying Equations (2) and (3), then the mean square error prediction of partial reserve is

$$\begin{aligned} MSQP(\hat{R}^i) &= \sum_{k=J+1}^l \left(\hat{W}_k^{N^i} - \mu^i(t_k)t_k - \kappa^i \lambda^i(t_k) \right)^2 + \tau^{i2}(t_k)t_k - \lambda^{i2}(t_k)\kappa^{i2} \\ &\quad - 2\mu^i(t_k)\lambda^i(t_k)t_k + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \hat{W}_{k_1}^{N^i} \hat{W}_{k_2}^{N^i} - \hat{W}_{k_2}^{N^i} \left(\mu^i(t_{k_1})t_{k_1} + \kappa^i \lambda^i(t_{k_1}) \right) \\ &\quad - \hat{W}_{k_1}^{N^i} \left(\mu^i(t_{k_2})t_{k_2} + \kappa^i \lambda^i(t_{k_2}) \right) + \tau^i(t_{k_1})\tau^i(t_{k_2})(t_{k_1} \wedge t_{k_2}) \\ &\quad + \kappa^i \lambda^i(t_{k_1})\mu^i(t_{k_2})t_{k_2} + \kappa^i \lambda^i(t_{k_2})\mu^i(t_{k_1})t_{k_1} + \sum_{k=J+1}^l \left(\hat{W}_k^{D^i} - \rho^i(t_k)t_k \right)^2 \\ &\quad + (\sigma^i(t_k))^2 t_k + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \hat{W}_{k_1}^{D^i} \hat{W}_{k_2}^{D^i} - \hat{W}_{k_2}^{D^i} \rho^i(t_{k_1})t_{k_1} \\ &\quad - \hat{W}_{k_1}^{D^i} \rho^i(t_{k_2})t_{k_2} + \rho^i(t_{k_1})\rho^i(t_{k_2})t_{k_1}t_{k_2} + \sigma^i(t_{k_1})\sigma^i(t_{k_2})(t_{k_1} \wedge t_{k_2}) \\ &\quad - 2 \sum_{k=J+1}^l \left(\hat{W}_k^{N^i} - \mu^i(t_k)t_k - \kappa^i \lambda^i(t_k) \right) \sum_{k=J+1}^l \left(\hat{W}_k^{D^i} - \rho^i(t_k)t_k \right). \end{aligned} \quad (11)$$

Proof. The mean square error prediction of the partial reserve can be written as follows

$$\begin{aligned} MSQP(\hat{R}^i) &= E \left[\left(\sum_{k=J+1}^l (\hat{W}_k^{N^i} - \hat{W}_k^{D^i}) - (W_k^{N^i} - W_k^{D^i}) \right)^2 | \mathcal{H} \right] \\ &= E \left[\left(\sum_{k=J+1}^l (\hat{W}_k^{N^i} - W_k^{N^i}) - \sum_{k=J+1}^l (\hat{W}_k^{D^i} - W_k^{D^i}) \right)^2 | \mathcal{H} \right] \\ &= E \left[\left(\sum_{k=J+1}^l (\hat{W}_k^{N^i} - W_k^{N^i}) \right)^2 | \mathcal{H} \right] + E \left[\left(\sum_{k=J+1}^l (\hat{W}_k^{D^i} - W_k^{D^i}) \right)^2 | \mathcal{H} \right] \\ &\quad - 2E \left[\sum_{k=J+1}^l (\hat{W}_k^{N^i} - W_k^{N^i}) | \mathcal{H} \right] E \left[\sum_{k=J+1}^l (\hat{W}_k^{D^i} - W_k^{D^i}) | \mathcal{H} \right]. \end{aligned} \quad (12)$$

Using the fact that

$$\begin{aligned} E \left[W_k^{N^i} | \mathcal{H} \right] &= \mu^i(t_k)t_k + \kappa^i \lambda^i(t_k), \\ \text{var} \left[W_k^{N^i} | \mathcal{H} \right] &= \tau^{i2}(t_k)t_k - \lambda^{i2}(t_k)\kappa^{i2} - 2\mu^i(t_k)\lambda^i(t_k)t_k \end{aligned}$$

and

$$\begin{aligned} E \left[W_{k_1}^{N^i} W_{k_2}^{N^i} | \mathcal{H} \right] &= \tau^i(t_{k_1})\tau^i(t_{k_2})(t_{k_1} \wedge t_{k_2}) + \kappa^i \lambda^i(t_{k_1})\mu^i(t_{k_2})t_{k_2} \\ &\quad + \kappa^i \lambda^i(t_{k_2})\mu^i(t_{k_1})t_{k_1}. \end{aligned}$$

As a first step, we calculate

$$\begin{aligned}
& E \left[\left(\sum_{k=J+1}^l (\widehat{W}_k^{N^i} - W_k^{N^i}) \right)^2 \middle| \mathcal{H} \right] \\
&= E \left[\sum_{k=J+1}^l (\widehat{W}_k^{N^i} - W_k^{N^i})^2 \middle| \mathcal{H} \right] + \sum_{J+1 \leq k_1 \neq k_2 \leq l} E \left[(\widehat{W}_{k_1}^{N^i} - W_{k_1}^{N^i})(\widehat{W}_{k_2}^{N^i} - W_{k_2}^{N^i}) \middle| \mathcal{H} \right] \\
&= \sum_{k=J+1}^l (\widehat{W}_k^{N^i})^2 - 2\widehat{W}_k^{N^i} E \left[W_k^{N^i} \middle| \mathcal{H} \right] + E \left[(W_k^{N^i})^2 \middle| \mathcal{H} \right] \\
&\quad + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \widehat{W}_{k_1}^{N^i} \widehat{W}_{k_2}^{N^i} - \widehat{W}_{k_2}^{N^i} E \left[W_{k_1}^{N^i} \middle| \mathcal{H} \right] - \widehat{W}_{k_1}^{N^i} E \left[W_{k_2}^{N^i} \middle| \mathcal{H} \right] + E \left[W_{k_1}^{N^i} W_{k_2}^{N^i} \middle| \mathcal{H} \right] \\
&= \sum_{k=J+1}^l \left(\widehat{W}_k^{N^i} - E \left[W_k^{N^i} \middle| \mathcal{H} \right] \right)^2 + \text{var} \left[W_k^{N^i} \middle| \mathcal{H} \right] \\
&\quad + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \widehat{W}_{k_1}^{N^i} \widehat{W}_{k_2}^{N^i} - \widehat{W}_{k_2}^{N^i} E \left[W_{k_1}^{N^i} \middle| \mathcal{H} \right] - \widehat{W}_{k_1}^{N^i} E \left[W_{k_2}^{N^i} \middle| \mathcal{H} \right] + E \left[W_{k_1}^{N^i} W_{k_2}^{N^i} \middle| \mathcal{H} \right] \\
&= \sum_{k=J+1}^l \left(\widehat{W}_k^{N^i} - \mu^i(t_k)t_k - \kappa^i \lambda^i(t_k) \right)^2 + \tau^{i2}(t_k)t_k - \lambda^{i2}(t_k)\kappa^{i2} - 2\mu^i(t_k)\lambda^i(t_k)t_k \\
&\quad + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \widehat{W}_{k_1}^{N^i} \widehat{W}_{k_2}^{N^i} - \widehat{W}_{k_2}^{N^i} (\mu^i(t_{k_1})t_{k_1} + \kappa^i \lambda^i(t_{k_1})) - \widehat{W}_{k_1}^{N^i} (\mu^i(t_{k_2})t_{k_2} + \kappa^i \lambda^i(t_{k_2})) \\
&\quad + \tau^i(t_{k_1})\tau^i(t_{k_2})(t_{k_1} \wedge t_{k_2}) + \kappa^i \lambda^i(t_{k_1})\mu^i(t_{k_2})t_{k_2} + \kappa^i \lambda^i(t_{k_2})\mu^i(t_{k_1})t_{k_1}.
\end{aligned}$$

We calculate the second term (12), taking into account

$$E \left[W_k^{D^i} \middle| \mathcal{H} \right] = \rho^i(t_k)t_k, \quad \text{var} \left[W_k^{D^i} \middle| \mathcal{H} \right] = (\sigma^i(t_k))^2 t_k$$

and

$$E \left[W_{k_1}^{D^i} W_{k_2}^{D^i} \middle| \mathcal{H} \right] = \rho^i(t_{k_1})\rho^i(t_{k_2})t_{k_1}t_{k_2} + \sigma^i(t_{k_1})\sigma^i(t_{k_2})(t_{k_1} \wedge t_{k_2}),$$

Then

$$\begin{aligned}
& E \left[\left(\sum_{k=J+1}^l (\widehat{W}_k^{D^i} - W_k^{D^i}) \right)^2 \middle| \mathcal{H} \right] \\
&= \sum_{k=J+1}^l \left(\widehat{W}_k^{D^i} - E \left[W_k^{D^i} \middle| \mathcal{H} \right] \right)^2 + \text{var} \left[W_k^{D^i} \middle| \mathcal{H} \right] \\
&\quad + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \widehat{W}_{k_1}^{D^i} \widehat{W}_{k_2}^{D^i} - \widehat{W}_{k_2}^{D^i} E \left[W_{k_1}^{D^i} \middle| \mathcal{H} \right] - \widehat{W}_{k_1}^{D^i} E \left[W_{k_2}^{D^i} \middle| \mathcal{H} \right] + E \left[W_{k_1}^{D^i} W_{k_2}^{D^i} \middle| \mathcal{H} \right] \\
&= \sum_{k=J+1}^l \left(\widehat{W}_k^{D^i} - \rho^i(t_k)t_k \right)^2 + (\sigma^i(t_k))^2 t_k \\
&\quad + \sum_{J+1 \leq k_1 \neq k_2 \leq l} \left[\widehat{W}_{k_1}^{D^i} \widehat{W}_{k_2}^{D^i} - \widehat{W}_{k_2}^{D^i} \rho^i(t_{k_1})t_{k_1} - \widehat{W}_{k_1}^{D^i} \rho^i(t_{k_2})t_{k_2} + \rho^i(t_{k_1})\rho^i(t_{k_2})t_{k_1}t_{k_2} + \sigma^i(t_{k_1})\sigma^i(t_{k_2})(t_{k_1} \wedge t_{k_2}) \right].
\end{aligned}$$

□

4. Conclusion

We have studied the Bayesian approach for the regime switching geometric Brownian motion proposed by [1]. It has been observed empirically that sinisters fluctuate among periods of high, moderate and low volatilities, so in this paper the estimation of the error rate of prediction distributions is proposed. For the future research, our developments raise an interesting axe when the Markov chain is hidden.

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