Numerical Solution of Green's Function for Solving Inhomogeneous Boundary Value Problems with Trigonometric Functions by New Technique

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Abstract

A numerical technique is presented for solving integration operator of Green's function. The approach is based on Hermite trigonometric scaling function on $[0, 2\pi]$, which is constructed for Hermite interpolation. The operational matrices of derivative for trigonometric scaling function are presented and utilized to reduce the solution of the problem. One test problem is presented and errors plots show the efficiency of the proposed technique for the studied problem.

1. Introduction

In mathematics, a Green's function is the impulse response of an inhomogeneous differential equation defined on a domain, with specified initial conditions or boundary conditions. Via the superposition principle, the convolution of a Green's function with an arbitrary function $f(x)$ on that domain is the solution to the inhomogeneous differential equation for $f(x)$. Green's functions were named after the British mathematician George Green, who first developed the concept in the 1830s. Under many-body theory, the term is also used in physics, specifically in quantum field theory, aerodynamics, aeroacoustics, electrodynamics and statistical field theory, to refer to various types of correlation functions, even those that do not fit the mathematical definition.

For concreteness, we assume that all functions are defined on the interval $[a,b] = [0, 2\pi]$, and we consider second-order ordinary differential operators Λ of the form

$$\begin{cases} \Lambda u = \left[p(x)u' \right]' + q(x)u = \delta(x - \xi), \\ \alpha_1 u(a) + \alpha_2 u(a) = 0, & |\alpha_1| + |\alpha_2| \neq 0; \\ \beta_1 u(b) + \beta_2 u(b) = 0, & |\beta_1| + |\beta_2| \neq 0. \end{cases} \tag{1}$$

where functions $p(x)$ and $q(x)$ are cntinuous for $[a,b]$, and $\delta(x - \xi)$ is dirac delta function. We look for a solution of 1 in the form

$$u(x) = \int_a^b G(x, \xi) f(\xi) \mathrm{d}\xi \tag{2}$$

where $G : [a,b] \times [a,b] \to \mathbb{C}$ is a suitable function, called the Green's function of 1.

In most situations, it is difficult to obtain exact solution of the above integration. Hence, various approximation methods have been proposed and studied. The purpose of the present paper is to develop a trigonometric Hermite wavelet approximation for the computing the Green's function of the problem 2.

Recently, the arisen wavelet Galerkin method has demonstrated its advantages for the treatment of integral operators [1]-[5]. It is discovered in [6] that wavelet represents the singular integral operator. The development of fast methods for integral equations opens new perspectives. The methods like the fast multipole method [7] and the panel clusteing [8] reduce the complexity largely. A difficulty of using wavelet for the representation of integral operators is that quadrature leads to potentially high cost with sparse matrix. This fact particularly encourages us in efforts to devote to some appropriate wavelet bases to simplify the computation expense of the reoresentation matrix, which is important to improve the wavelet method. Nowadays, the trigonometric interpolant wavelet has arisen in the approximation of operators [9]-[11]. Quack [12] has constructed a multiresolution analysis (MRA) of nested subspace of trigonometric Hermite polynomials. The trigonometric Hermite interpolation enables a completely explicit description of the corresponding decomposition and reconstruction coefficients by means of some circular matrices. Chen [13] [14] presented the feasibility of trigonometric wavelet numerical methods for stokes problem and Hadamard integral equation.

The outline of this paper is as follows. In Section 2, we describe the trigonometric scaling function on $[0, 2\pi]$, and in Section 3 we construct the operational matrix of derivative for these function. In Section 4, the proposed method is used to approximate the solution of the problem. As a result, a problem of integration of a matrix is obtained, where by calculating the Green's function of this matrix we get to the solution of the problem. In Section 5, we report our computational results and demonstrate the accuracy of the proposed numerical schemes by presenting numerical examples. Section 6 ends this paper with a brief conclusion.

2. Trigonometric Scaling Function on $[0, 2\pi]$

In this section, we will give a brief introduction of Quak's work on the construction of Hermite interpolatory trigonometric wavelets and their basic properties (see [12]). For all $n \in \mathbb{N}$, the Dirichlet kernel $D_l(x)$ and its conjugate kernel $\widetilde{D}_l(x)$ are defined as

$$D_l(x) = \frac{1}{2} + \sum_{k=1}^{l} \cos(kx) = \begin{cases} \dfrac{\sin\left(\left(l + \dfrac{1}{2}\right)x\right)}{2\sin\left(\dfrac{x}{2}\right)}, & x \notin 2\pi\mathbb{Z}; \\ l + \dfrac{1}{2}, & x \in 2\pi\mathbb{Z}. \end{cases}$$

$$\widetilde{D}_l(x) = \sum_{k=1}^{l} \sin(kx) = \begin{cases} \dfrac{\cos\left(\dfrac{x}{2}\right) - \cos\left(l + \dfrac{1}{2}\right)}{2\sin\left(\dfrac{x}{2}\right)}, & x \notin 2\pi\mathbb{Z}; \\ 0, & x \in 2\pi\mathbb{Z}. \end{cases}$$

Obviously, $D_l(x)$, $\widetilde{D}_l(x) \in T_l$ is the linear space of trigonometric polynomials with degree not exceeding l.

The equally spaced nodes on the interval $[0, 2\pi)$ with a dyadic step are denoted by $t_{j,n} = \dfrac{n\pi}{2^j}$, for any

$j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $n = 0, 1, \cdots, 2^{j+1} - 1$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_0$ is the set of all non-negative integers.

Definition 1 (*Scaling functions*). (*See* [12].) *For all* $j \in \mathbb{N}_0$, *the scaling functions* $\varphi_{j,0}^0(x)$, $\varphi_{j,0}^1(x)$ *where* $\varphi_{j,n}^s(x) = \varphi_{j,0}^s(x - x_{j,n})$ *for* $\forall n = 0, 1, \cdots, 2^{j+1} - 1$ *and* $s = 0, 1$ *are defined as*

$$\varphi_{j,0}^0(x) = \frac{1}{2^{2j+1}} \sum_{l=0}^{2^{j+1}-1} D_l(x),$$

$$\varphi_{j,0}^1(x) = \frac{1}{2^{2j+1}} \left(\widetilde{D}_{2^{j+1}}(x) + \frac{1}{2} \sin(2^{j+1} x) \right).$$

Lemma 1 (*See* [12].) *For* $j \in \mathbb{N}_0$, *we have*

$$\varphi_{j,0}^0(x) = \begin{cases} \dfrac{1}{2^{2j+2}} \dfrac{\sin^2(2^j x)}{\sin^2\left(\dfrac{x}{2}\right)}, & x \notin 2\pi\mathbb{Z}; \\[4mm] 1, & x \in 2\pi\mathbb{Z}. \end{cases}$$

$$\varphi_{j,0}^1(x) = \begin{cases} \dfrac{1}{2^{2j+2}} (1 - \cos(2^{j+1} x)) \cot\left(\dfrac{x}{2}\right), & x \notin 2\pi\mathbb{Z}; \\[4mm] 0, & x \in 2\pi\mathbb{Z}. \end{cases}$$

and their derivations are given by

$$\left(\varphi_{j,0}^0(x)\right)' = \begin{cases} \dfrac{1}{2^{j+2}} \dfrac{\sin(2^{j+1} x)}{\sin^2\left(\dfrac{x}{2}\right)} - \dfrac{1}{2^{2j+2}} \dfrac{\sin^2(2^j x) \cot\left(\dfrac{x}{2}\right)}{\sin\left(\dfrac{x}{2}\right)}, & x \notin 2\pi\mathbb{Z}; \\[4mm] 0 & x \in 2\pi\mathbb{Z}. \end{cases}$$

$$\left(\varphi_{j,0}^1(x)\right)' = \begin{cases} \dfrac{1}{2^{2j+3}} \dfrac{\cos(2^{j+1} x) - 1}{\sin^2\left(\dfrac{x}{2}\right)} + \dfrac{1}{2^{j+1}} \sin(2^{j+1} x) \cot\left(\dfrac{x}{2}\right) & x \notin 2\pi\mathbb{Z}; \\[4mm] 1, & x \in 2\pi\mathbb{Z}. \end{cases}$$

Theorem 2 (*Interpolatory properties of the scaling functions*). (*See* [12].) *For* $j \in \mathbb{N}_0$, *The following interpolatory properties hold for each* $n, k = 0, 1, \cdots, 2^{j+1} - 1$

$$\varphi_{j,n}^0(x_{j,k}) = \delta_{k,n}, \quad \left(\varphi_{j,n}^0(x_{j,k})\right)' = 0, \tag{3}$$

$$\varphi_{j,n}^1(x_{j,k}) = 0, \quad \left(\varphi_{j,n}^1(x_{j,k})\right)' = \delta_{k,n}. \tag{4}$$

From above we can take wavelet functions $\varphi_{j,n}^0(x)$, $\varphi_{j,n}^1(x)$, $n = 0, 1, \cdots, 2^{j+1} - 1$ as scaling functions. Then we have

Definition 3 (*Scaling functions space*). *For all* $j \in \mathbb{N}_0$ *define the wave space* V_j *as follows*

$$V_j = \mathrm{span}\left\{\varphi_{j,n}^0(t), \varphi_{j,n}^1(t), n = 0, 1, \cdots, 2^{j+1} - 1\right\}$$

As a first step of studying the spaces V_j, the following result identifies the trigonometric polynomials which from alternative bases of these spaces.

Theorem 4 *For any* $j \in \mathbb{N}_0$, *we have*

$$V_j = \mathrm{span}\left\{1, \cos(t), \cdots, \cos(2^{j+1} - 1)t, \sin(t), \cdots, \sin(2^{j+1} - 1)t\right\}$$

consequently $\dim V_j = 2^{j+2}$.

Definition 5 *For any* $j \in \mathbb{N}_0$, *the interpolation operator* L_j *mapping any real valued differentiable* 2π-*periodic function* f *into the space* V_j *is defined as*

$$f(x) L_j f(x) \simeq \sum_{k=0}^{2^{j+1}-1} \left[a_k \varphi_{j,k}^0(x) + b_k \varphi_{j,k}^1(x) \right] = C^{\mathrm{T}} \Phi \tag{5}$$

where $a_k = f(x_{j,k})$ $b_k = f'(x_{j,k})$, and C and Φ are vectors with dimension $2^{j+2} \times 1$.

The following properties of the operators L_j are therefore obvious:

$$L_j \in T_{2^{j+1}}$$

$$L_j f(x_{j,k}) = f(x_{j,k}) \quad \text{and} \quad (L_j f)'(x_{j,k}) = f'(x_{j,k}), \quad k \in \mathbb{Z}$$

$$L_j f = f \quad \text{for all} \quad f \in V_j.$$

Theorem 6 *Let* $f(x) \in L_{2\pi}^2$, *and its trigonometric wavelet approximation is* $L_j f$, *then we have*

$$\left\| f(x) - L_j f(x) \right\|_{L_{2\pi}^2} \leq C 2^{-2(J+1)}$$

where C is a positive constant value.

Proof. See [12].

3. The Operational Matrix of Derivative

The differentiation of vector Φ in 5 can be expressed as [15]

$$\Phi' = D_\phi \Phi$$

where D_ϕ is $2^{j+2} \times 2^{j+2}$ operational matrix of derivative for trigonometric scaling function. Suppose

$$\left(\phi_{j,m}^s(x) \right)' = \sum_{k=0}^{2^{j+1}-1} \left[a_{k,m}^s \varphi_{j,k}^0(x) + b_{k,m}^s \varphi_{j,k}^1(x) \right] \tag{6}$$

where $m = 0, 1, \cdots, 2^{j+1} - 1$ and $s = 0, 1$. So the matrix D_ϕ can be respresented as a block matrix as

$$D = \begin{bmatrix} A^0 & B^0 \\ A^1 & B^1 \end{bmatrix}$$

where A^s and B^s are $2^{j+1} \times 2^{j+1}$ matrices. The entries of matrices A^s and B^s may be finding by using 3

$$A^s = \left(a_{k,m}^s \right) = \left(\phi_{j,k}^s \right)'(x_{j,m}), \quad B^s = \left(b_{k,m}^s \right) = \left(\phi_{j,k}^s \right)''(x_{j,m})$$

where A^0 is a $2^{j+1} \times 2^{j+1}$ zero matrix, A^1 is a $2^{j+1} \times 2^{j+1}$ identity matrix. Using $x_{j,m} = \dfrac{m\pi}{2^j}$ we get

$$\phi_{j,k}^s(x_{j,m}) = \phi_{j,0}^s\left(\frac{(m-k)\pi}{2^j} \right) = \phi_{j,0}^s(x_{j,m-k}) \tag{7}$$

Using Equation (7) and $B^s = \left(b_{k,m}^s \right) = \left(\phi_{j,k}^s(x_{j,m}) \right)'' = \left(\phi_{j,0}^s(x_{j,m-k}) \right)''$ $s = 0, 1$ we get

$$B^0 = \left(b_{k,m}^0 \right) = \left(\phi_{j,0}^0 \right)''(x_{j,m-k}) = \begin{cases} \dfrac{1}{2} \dfrac{\cos\left((m-k)2\pi \right)}{\sin^2\left(\dfrac{m-k}{2^{j+1}} \pi \right)}, & k \neq m; \\ \dfrac{-1}{2^{2j+1}} \sum_{p=0}^{2^{j+1}-1} \sum_{k=1}^{p} \dfrac{p(p+1)(2p+1)}{6}, & k = m. \end{cases} \tag{8}$$

and

$$B^1 = \left(b_{k,m}^1\right) = \left(\phi_{j,0}^1\right)'' \left(x_{j,m-k}\right) = \begin{cases} -\cot\left(\dfrac{m-k}{2^{j+1}}\pi\right), & k \neq m; \\ 0, & k = m. \end{cases} \tag{9}$$

for $k, m = 0, 1, \cdots, 2^{j+1} - 1$.

4. Function Approximation

In this section, we give the concrete computational schemes for this integral Equation (2) with the Green's function kernel. The discretization form of (2) is given in the following subsection.

By introducing a basis $\{\phi_{j,k}\}$ for the subspace V_j, the coefficients vector $G(x_j, \xi)$ of the discrete solution $G(x, \xi)$ is defined by

$$GL_j G(x, \xi) = \sum_{k=0}^{2^{j+1}-1} \left[a_k \varphi_{j,k}^0(x) + b_k \varphi_{j,k}^1(x)\right] = C^T \Phi \tag{10}$$

where C is $2^{j+2} \times 1$ unknown vector defined similar to (5). Using (2) we get

$$u(x) = \int_a^b L_j G(x, \xi) f(\xi) d\xi \tag{11}$$

We tries to solve the above function by picking approximate values for N, $W_{N,i}$ and $\xi_{N,i}$. While only defined for the interval $[-1, 1]$, this is a universal function actually, because we can convert the limits of integration for any interval $[a, b]$ to the Legendre-Gauss or $G_N(f)$ interval $[-1, 1]$:

$$\begin{aligned} u(x) &= \int_a^b L_j G(x, \xi) f(\xi) d\xi \\ &= \frac{b-a}{2} \int_{-1}^1 G\left(x, \frac{b-a}{2}\xi + \frac{b+a}{2}\right) f\left(\frac{b-a}{2}\xi + \frac{b+a}{2}\right) d\xi \\ &\simeq \frac{b-a}{2} \sum_{l=1}^N W_{N,l} G\left(x, \frac{b-a}{2}\xi_{N,l} + \frac{b+a}{2}\right) f\left(\frac{b-a}{2}\xi_{N,l} + \frac{b+a}{2}\right) + E_N(G). \end{aligned} \tag{12}$$

The abscissas $\xi_{N,l}$ and weights $W_{N,l}$ to be used have been tabulated and are easily availabe; **Table 1** gives the values up to six points. Also included in the table is the form of the error term $E_N(f)$ that corresponds to $G_N(f)$, and it can be used to determine the accuracy of the Gauss-Legendre integration formula.

Applying Equation (10) in Equation (12) we have

$$u_j(x) = \frac{b-a}{2} \sum_{l=1}^N \sum_{k=0}^{2^{j+1}-1} W_{N,l} f\left(\frac{b-a}{2}\xi_{N,l} + \frac{b+a}{2}\right) \left[a_k \varphi_{j,k}^0(x) + b_k \varphi_{j,k}^1(x)\right] + E \tag{13}$$

By substituting $u(x)$ in (1), we have a linear system. Now for determining unknown coefficients a_k and b_k, we choose collocation method With collocation points as

$$t_i = a + \frac{i(b-a)}{2^{J+2}} = \frac{i\pi}{2^{J+1}}; \quad i = 0, 1, \cdots, 2^{J+2} - 1 \tag{14}$$

$$\begin{cases} \Lambda u = \left[p(t_i)u(t_i)'\right]' + q(t_i)u(t_i) = \delta(t_i - \xi), \\ \alpha_1 u(a) + \alpha_2 u(a) = 0, \quad |\alpha_1| + |\alpha_2| \neq 0; \\ \beta_1 u(b) + \beta_2 u(b) = 0, \quad |\beta_1| + |\beta_2| \neq 0. \end{cases} \tag{15}$$

Thus, we have system of linear equation $A_j X = F_j$ where $A_j = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, N is the points of Gauss-Legendre, and

$$A_s = \left(a_{j,s}\right)_{2^{j+2} \times 2^{j+1}}, \quad s = 1, 2, \quad i = 0, 1, \cdots, 2^{j+2} - 1$$

Table 1. Gauss-legendre abscissas and weights.

			$\int_{-1}^{1} f(\xi)\,d\xi = \sum_{l=1}^{N} W_{N,l} f(\xi_{N,l}) + E_N(f)$
N	Abscissas, $\xi_{N,i}$	Weights, $W_{N,i}$	Truncation error, $E_N(f)$
2	-0.5773502692	1.0000000000	$\dfrac{f^{(4)}(c)}{135}$
	0.5773502692	1.0000000000	
3	±0.7745966692	0.5555555556	$\dfrac{f^{(6)}(c)}{15750}$
	0.0000000000	0.8888888888	
4	±0.8611363116	0.3478548451	$\dfrac{f^{(8)}(c)}{3472875}$
	±0.3399810436	0.6521451549	
5	±0.9061798459	0.2369268851	$\dfrac{f^{(10)}(c)}{1237732650}$
	±0.5384693101	0.4786286705	
	0.0000000000	0.5688888888	
6	±0.9324695142	0.1713244924	$\dfrac{f^{(12)}(c)2^{13}(6!)^4}{(12!)^3 13!}$
	±0.6612093865	0.3607615730	
	±0.2386191861	0.4679139346	

$$\left(a_{j,1}\right) = \frac{b-a}{2}\sum_{l=1}^{N}\sum_{k=0}^{2^{j+1}-1} W_{N,l} f\left(\frac{b-a}{2}\xi_{N,l} + \frac{b+a}{2}\right)\left[\left(p(t_i)\varphi_{j,k}^0(t_i)'\right)' + q(t_i)\varphi_{j,k}^0(t_i)\right]$$

$$\left(a_{j,2}\right) = \frac{b-a}{2}\sum_{l=1}^{N}\sum_{k=0}^{2^{j+1}-1} W_{N,l} f\left(\frac{b-a}{2}\xi_{N,l} + \frac{b+a}{2}\right)\left[\left(p(t_i)\varphi_{j,k}^1(t_i)'\right)' + q(t_i)\varphi_{j,k}^1(t_i)\right]$$

$$X_1 = \left(a_k\right)_{2^{j+1}\times 1}, \quad X_2 = \left(b_k\right)_{2^{j+1}\times 1}$$

$$F_j = \left(f(t_i)\right)_{2^{j+2}\times 2^{j+2}}$$

So, the unknown function $u_j(x)$ can be found. Note that we find the function by MATLAB.

5. Numerical Example

To support our theoretical discussion, we applied the method presented in this paper to several examples. All the generalized Green's function kernels in this numerical example are solved by trigonometric wavelet. Our method compared with exact solution.

Example. Consider the inhomogeneous differential equation with the following coditions:

$$\begin{cases} u'' + u = \sin(x) \\ u(0) = 0 \\ u'(2\pi) = 0 \end{cases} \tag{16}$$

The exact solution is $u(x) = \dfrac{3\sin(x)}{8} - \dfrac{\sin(3x)}{8} - \dfrac{\sin(2x)}{4} - \left(\dfrac{x}{2}\right)\times\cos(x)$. If we solve above problem with

Green's function, we have $u(x) = \int_0^{2\pi} G(x,\xi)\sin(\xi)\,d\xi$. The linear algebraic system is solved by the steepest

descent method and results are shown in **Figure 1**. The relative errors between $u(x)$ and $u_j(x)$ in absulote error are given in **Table 2** and **Table 3**, and different Gauss-Legendre Abscissas and Weights. It is easy to see that our error results are greatly small with low computing cost.

The above example states

1. Our numerical method is also efficient when the wave number J is very large, that is to say, the wave number J can hardly affect the convergence rate,

2. Our numerical method is very fast, for example, the run time is only 2.000 s as $J = 8$, for which the corresponding matrix A_j is $2^{10} \times 2^{10}$.

6. Conclusion

The trigonometric scaling function is used to solve the Green's function of an inhomogeneous differential equation. Some properties of trigonometric scaling function are presented and the operational matrices of derivative for trigonometric scaling function are utilized to reduce the solution of Green's function to the solution of linear
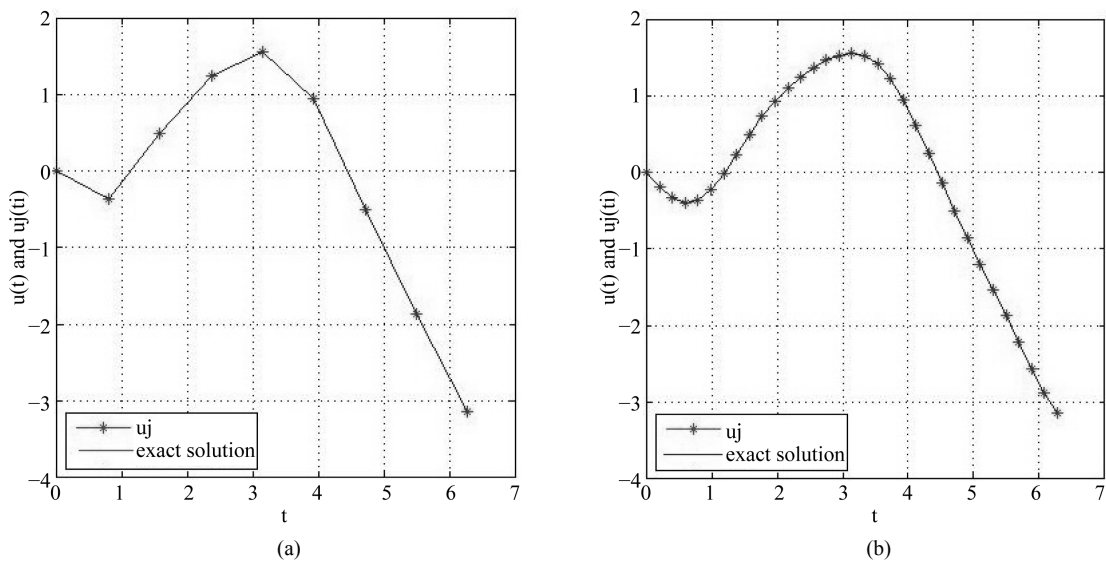
Figure 1. (a) Result for $J = 1$ and $N = 3$; (b) Result for $J = 3$ and $N = 3$.

Table 2. Error analysis and numerical results of example for $J = 1$ and $N = 3$.

Lightaqua t_i	Exact solution	u_j	Absolute error
0	0.00000000000000e − 00	0.00021101700847e − 00	2.1101e − 04
$\frac{\pi}{4}$	−3.50951832051248e − 01	−3.50921684793215e − 01	3.0147e − 04
$\frac{\pi}{2}$	+5.00012518932405e − 01	+5.00218656215217e − 01	2.0613e − 04
$\frac{3\pi}{4}$	+1.25982050034894e − 00	+1.25920847932518e − 00	6.1202e − 04
π	+1.57081208430251e − 00	+1.57080214896312e − 00	9.9353e − 06
$\frac{5\pi}{4}$	+9.61620674920137e − 01	+9.61620105486317e − 01	5.6943e − 06
$\frac{3\pi}{2}$	−5.00012518932405e − 01	−5.00158902156405e − 01	1.4638e − 04
$\frac{7\pi}{4}$	−1.87052054089309e − 00	−1.87024803204009e − 00	2.7250e − 04

Table 3. Error analysis and numerical results of example for $J = 3$ and $N = 3$.

Lightaqua t_i	Exact solution	u_j	Absolute error
0	$0.00000000000000e - 00$	$0.000001700847e - 00$	$1.7008e - 06$
$\frac{\pi}{4}$	$-3.50951832051248e - 01$	$-3.50951027861354e - 01$	$8.042e - 06$
$\frac{\pi}{2}$	$+5.00012518932405e - 01$	$+5.00012586562152e - 01$	$6.7600e - 07$
$\frac{3\pi}{4}$	$+1.25982050034894e - 00$	$+1.25982120453872e - 00$	$7.0400e - 07$
π	$+1.57081208430251e - 00$	$+1.57081217021648e - 00$	$8.600e - 08$
$\frac{5\pi}{4}$	$+9.61620674920137e - 01$	$+9.61620600181084e - 01$	$7.480e - 07$
$\frac{3\pi}{2}$	$-5.00012518932405e - 01$	$-5.00012610124753e - 01$	$9.120e - 07$
$\frac{7\pi}{4}$	$-1.87052054089309e - 00$	$-1.87053053127694e - 00$	$9.991e - 06$

system of equations with sparse matrix of coefficients. Applications of the wavelets allow the creation of more effective and faster algorithms than the ordinary ones. Illustrative examples are included to demonstrate the validity and applicability of the technique. The main advantage of this method is its simplicity and small computation costs.

Acknowledgements

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