Applied Mathematics, 2015, 6, 1-6
Published Online January 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.61001

Generalization of Some Problems with *s*-Separation

Beih El-Sayed El-Desouky, Mohamed Moustafa Gad, Shimaa El-Eraqy

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt
Email: b_desouky@yahoo.com

Received 29 October 2014; revised 26 November 2014; accepted 18 December 2014

Abstract

In this article we apply and discuss El-Desouky technique to derive a generalization of the problem of selecting *k* balls from an *n*-line with no two adjacent balls being *s*-separation. We solve the problem in which the separation of the adjacent elements is not having odd and even separation. Also we enumerate the number of ways of selecting *k* objects from *n*-line objects with no two adjacent being of separations *m*, *m* + 1, ⋯, *pm*, where *p* is positive integer. Moreover we discuss some applications on these problems.

Keywords

Probability Function, *s*-Separation, *s*-Successions, *n*-Line, *n*-Circle

1. Introduction

Kaplansky [1] (see also Riordan ([2] p. 198, lemma) and Moser [3]) studied the problem of selecting k objects from n objects arranged in a line (called n-line) or a circle (called n-circle) with no two selected objects being consecutive. Let $f(x,y)$ and $g(x,y)$ denote the number of ways of such selections for n-line and n-circle respectively. Kaplansky proved that

$$f(n,k)=\begin{cases}\binom{n-k+1}{k}, & 0\le k\le n\\ 0, & \text{otherwise,}\end{cases} \tag{1.1}$$

and

$$g(n,k)=\begin{cases}\frac{n}{k}\binom{n-k-1}{k-1}, & n\ge 2k+1\\ 0, & 1\le n\le k.\end{cases} \tag{1.2}$$

How to cite this paper: El-Desouky, B.S., Gad, M.M. and El-Eraqy, S. (2015) Generalization of Some Problems with *s*-Separation. *Applied Mathematics*, **6**, 1-6. http://dx.doi.org/10.4236/am.2015.61001

El-Desouky [4] studied another related problem with different techniques and proved that

$$l(n,k)=\begin{cases}\sum_{i=0}^{\lambda}\binom{k-1}{i}\binom{n-k+1-i}{i+1}, & \lambda=\min\left(k-1,\left[\frac{n-k}{2}\right]\right),\ 0\le k\le n\\ 0, & \text{otherwise,}\end{cases} \tag{1.3}$$

where $l(n,k)$ is the number of ways of selecting *k* balls from *n* balls arranged in a line with no two adjacent balls being unit separation.

In the following we adopt some conventions: $\left[x^n\right]f(x)$ denotes the coefficient of x^n in the formal power series $f(x)$; $\left[x^n y^m\right]f(x,y)$ denotes the coefficient of $x^n y^m$ in the series $f(x,y)$; $[x]$ is the largest integer less than or equal to *x*, $N=\{0,1,\cdots\}$ and $N_n=\{1,2,3,\cdots\}$.

Also, El-Desouky [5] derived a generalization of the problem given in [4] as follows: let $l_s(n,k)$ denote the number of ways of selecting *k* balls from *n* balls arranged in a line with no two adjacent balls from the *k* selected balls being *s*-separation; two balls have separation *s* if they are separated by exactly *s* balls. Let $d_s(n,k)$ denote the number of ways of selecting *k* balls from *n* balls arranged in a circle with no two adjacent balls from the *k* selected balls being *s*-separation

Let $l_s(n,k)$ be as defined before. Then $l_s(n,k)$ is equal to the number of *k*-subsets of N_n where the difference $s+1$ is not allowed, so

$$l_s(n,k)=\sum_{i=0}^{\nu}(-1)^i\binom{k-1}{i}\binom{n-(s+1)i}{k-i}$$
$$\text{where } \nu=\min\left(k-1,\left[\frac{n-k}{s}\right]\right),\ 0\le k\le n,\ \text{and } s=0,1,\cdots,n-k. \tag{1.4}$$

Let $d_s(n,k)$ be as defined before. Then the difference $s+1$ is not allowed, so

$$d_s(n,k)=\frac{n}{k}\sum_{i=0}^{\beta}(-1)^i\binom{k}{i}\binom{n-(s+1)i-1}{k-i-1},$$
$$\text{where } \beta=\min\left(k,\left[\frac{n-k}{s}\right]\right),\ 0\le k\le n,\ \text{and } s=0,1,\cdots,n-k. \tag{1.5}$$

Let $l_s(n,k,m)$ be the number of ways of selecting *k* balls from *n* balls arranged in a line with exactly *m* adjacent balls being of separation *s* or $(s\text{-successions})$, which gives a generalization of (4.1) in El-Desouky [4]. Thus,

$$l_s(n,k,m)=\sum_{i=m}^{\mu'}\sum_{j=0}^{k-1-i}(-1)^i\binom{k-1}{i}\binom{k-1-i}{j}\binom{n-(s+1)i-sj}{k-m},$$
$$\text{where } \mu'=\min\left(k-1,\left[\frac{n-k+m}{s+1}\right]\right),\ m=0,1,\cdots,k-1,\ s=0,1,\cdots,n-k. \tag{1.6}$$

For more details on such problems, see [3] [6] [7].

2. Main Results

We use El-Desouky technique to solve two problems in the linear case, with new restrictions. That is if the separation of any two adjacent elements from the *k* selected elements being of odd separation and of even separation. Moreover, we enumerate $M_s(n,k;m,pm)$ which denotes the number of ways of selecting *k* objects from *n* objects arrayed in a line where any two adjacent objects from the *k* selected objects are not being of *m*, *m* + 1, ⋯, *pm* separations, where *p* is positive integer.

2.1. No Two Adjacent Being Odd Separation

Let $y_o(n,k)$ denote the number of ways of selecting *k* balls from *n* balls arranged in a line, where the separation of any two adjacent balls from the k selected balls being of odd separation. say s, *i.e.* $s=1,3,5,\cdots$. This means that no two adjacent being of 2, 4, 6, $\cdots$ differences, see Table 1.

So, following Decomposition (2.3.14) see [8] (p. 55), $y_o(n,k)$ is equal to the number of k-subsets of N_n where the differences $s+1$, $s=1,3,5,\cdots$ are not allowed, hence $y_o(n,k)=\left[x^n\right]f(x)$, where

$$
\begin{aligned}
f(x) &= \left(x+x^2+\cdots\right)\left[x+x^2+\cdots-\left(x^2+x^4+\cdots\right)\right]^{k-1}(1+x+\cdots)\\
&= \frac{x}{1-x}\left[\frac{x}{1-x}-\left(x^2+x^4+\cdots\right)\right]^{k-1}\frac{1}{1-x}\\
&= \frac{x}{(1-x)^2}\frac{x^{k-1}}{(1-x)^{k-1}}\left[1-(1-x)\left(x+x^3+\cdots\right)\right]^{k-1}\\
&= x^k(1-x)^{-(k+1)}(1-x)^{-(k+1)},
\end{aligned}
$$

hence

$$
f(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j x^k\binom{k+i}{i}x^i\binom{k+j-2}{j}x^j=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{k+i}{i}\binom{k+j-2}{j}x^{i+j+k}.
$$

Setting $n=i+j+k$ $j=n-i-k$ we have

$$
f(x)=\sum_{n=k}^{\infty}\sum_{i=0}^{n-k}(-1)^{n-i-k}\binom{k+i}{i}\binom{k+n-i-k-2}{n-i-k}x^n=\sum_{n=k}^{\infty}\sum_{i=0}^{n-k}(-1)^{n-i-k}\binom{k+i}{i}\binom{n-i-2}{n-i-k}x^n.
$$

Therefore, the coefficient of x^n gives

$$
y_o(n,k)=\sum_{i=0}^{n-k}(-1)^{n-i-k}\binom{k+i}{i}\binom{n-i-2}{k-2}.
$$

A calculated table for the values of $y_o(n,k)$ is given in Table 1, where $1\le n$, $k\le 10$.

Remark 1. *It is easy to conclude that* $y_o(n,k)$ *satisfies the following recurrence relation*

$$
y_o(n,k)=y_o(n-1,k-1)+y_o(n-2,k),\quad n,\ k\ge 2\ \ and\ \ y_o(n,k)=0\ \ for\ \ k>n \tag{2.1}
$$

with the convention $y_o(n,1)=n$, $n\ge 1$.

Table 1. A calculated table for the values of $y_o(n,k)$.

n \ k	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0
3	3	2	1	0	0	0	0	0	0	0
4	4	4	2	1	0	0	0	0	0	0
5	5	6	5	2	1	0	0	0	0	0
6	6	9	8	6	2	1	0	0	0	0
7	7	12	14	10	7	2	1	0	0	0
8	8	16	20	20	12	8	2	1	0	0
9	9	20	30	30	27	14	9	2	1	0
10	10	25	40	50	42	35	16	10	2	1

2.2. No Two Adjacent Being Even Separation

Let $y_e(n,k)$ denote the number of ways of selecting k balls from n balls arranged in a line, where the separation of any two adjacent balls from the k selected balls are not being of even separation, say s *i.e.* $s = 0, 2, 4, \cdots$. This means that no two adjacent being of 1, 3, 5,$\cdots$ differences.

So, following Decomposition (2.3.14) see [8] (p. 55) then $y_e(n,k)$ is equal to the number of k-subsets of N_n where the differences $s+1$, $s = 0, 2, 4, \cdots$ are not allowed, hence $y_e(n,k) = \left[x^n\right] f(x)$, where

$$\begin{aligned} f(x) &= \left(x + x^2 + \cdots\right)\left[x + x^2 + \cdots - \left(x + x^3 + \cdots\right)\right]^{k-1}(1 + x + \cdots) \\ &= \frac{x}{1-x}\left[\frac{x}{1-x} - \left(x + x^3 + \cdots\right)\right]^{k-1}\frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}\frac{x^{k-1}}{(1-x)^{k-1}}\left[1 - (1-x)\left(1 + x^2 + \cdots\right)\right]^{k-1} \\ &= \frac{x^{2k-1}}{(1-x)^{k+1}(1+x)^{k-1}} \\ &= x^{2k-1}(1-x)^{-(k-1)}(1+x)^{-(k-1)}, \end{aligned}$$

hence

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j x^{2k-1}\binom{k+i}{i}x^i\binom{k+j-2}{j}x^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{k+i}{i}\binom{k+j-2}{j}x^{2k-1+i+j}.$$

Setting $n = 2k-1+i+j$, $j = n-2k+1-i$ we get

$$f(x) = \sum_{j=0}^{\infty}\sum_{i=0}^{n-2k+1}(-1)^{n+1-i}\binom{k+i}{i}\binom{k+n-2k+1-i-2}{n-2k+1-i}x^n = \sum_{j=0}^{\infty}\sum_{i=0}^{n-2k+1}(-1)^{n+1-i}\binom{k+i}{i}\binom{n-k-i-1}{k-2}x^n.$$

Therefore, the coefficient of x^n gives

$$y_e(n,k) = \sum_{i=0}^{n-2k+1}(-1)^{n+1-i}\binom{k+i}{i}\binom{n-k-i-1}{k-2}. \tag{2.2}$$

Moreover in the next subsection, we use our technique to enumerate $M_s(n,k;m,pm)$ the number of ways of selecting k objects from n objects arrayed in a line such that no two adjacent elements have the differences m + 1, m + 2, $\cdots$, pm + 1 *i.e.* no two adjacent element being of m, m + 1, $\cdots$, pm separations, where p is positive integer.

2.3. Explicit Formula for $M_s(n,k;m,pm)$

Let $M_s(n,k;m,pm)$ be the number of ways of selecting k objects from n objects arrayed in a line where any two adjacent objects from the k selected objects are not being of m, m + 1, $\cdots$, pm separations, where p is positive integer, hence $M_s(n,k;m,pm) = \left[x^n\right] f(x)$, where

$$\begin{aligned} f(x) &= \left(x + x^2 + \cdots\right)\left[x + x^2 + \cdots - \left(x^{m+1} + x^{m+2} + \cdots + x^{pm+1}\right)\right]^{k-1}\frac{1}{1-x} \\ &= \frac{x^k}{(1-x)^2}\left[\frac{1-x^m}{1-x} + x^{pm+1}(1-x)^{-1}\right]^{k-1} \\ &= x^k(1-x)^{-(k+1)}\left[1 - x^m\left(1 - x^{pm-m+1}\right)\right]^{k-1} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{i+j}\binom{k-1}{i}\binom{i}{j}x^{j(pm-m+1)+mi}x^l\binom{k+l}{l}x^k. \end{aligned}$$

Setting $n = j(pm-m+1)+mi+l+k$ it is easy to find the coefficient of x^n hence

$$M_s(n,k;m,pm) = \sum_{i=0}^{k-1}\sum_{j=0}^{i}(-1)^{i+j}\binom{k-1}{i}\binom{i}{j}\binom{n-j(pm-m+1)-mi}{k}. \tag{2.3}$$

3. Some Applications

Let n urns be set out along a line, that is, one-dimensional.

Suppose we have m balls of which m_i are of colour c_i, $i=1,2,\cdots,k$ and we assign these balls to urns so that, see Pease [9]:

i) No urn contains more than one ball.

ii) All m_i balls of colour c_i are in consecutive urns, $i=1,2,\cdots,k$.

El-Desouky proved that if the order of colours of the groups is specified, the number of arrangement is just $\binom{n-m+k}{k}$. Hence if the total number of balls $\sum_{i=1}^{k} m_i = 2k-1$, the number of arrangements is $l_o(n,k) = f(n,k) = \binom{n-k+1}{k}$ as a special case of El-Desouky results [5].

It is of practical interest to find the asymptotic behavior of $f(n,k)$ or the probability $p(n,k) = f(n,k)\Big/\binom{n}{k}$ for large n and k.

Let X be a random variable having the probability function $p(n,k)$ then

$$P(X=k) = p(n,k) = \frac{\binom{n-k+1}{k}}{\binom{n}{k}},$$

so

$$\begin{aligned}
\ln P(X=k) &= \ln\left[\left(1-\frac{k-1}{n}\right)\left(1-\frac{k}{n}\right)\cdots\left(1-\frac{2(k-1)}{n}\right)\right] - \ln\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-2}{n}\right)\left(1-\frac{k-1}{n}\right)\right] \\
&\simeq \left[-\frac{k-1}{n}-\frac{k}{n}-\cdots-\frac{2(k-1)}{n}\right] - \left[-\frac{1}{n}-\frac{2}{n}-\cdots-\frac{k-2}{n}-\frac{k-1}{n}\right] \\
&= -\frac{3k(k-1)}{2n}+\frac{k(k-1)}{2n} = -\frac{k(k-1)}{n},
\end{aligned}$$

where we used the first aproximation

$$\ln(1-x) = -x.$$

Therefore,

$$P(X=k) = \mathrm{e}^{-\frac{k(k-1)}{n}}.$$

Putting $Y = \frac{X}{\sqrt{n}}$ we have

$$\begin{aligned}
P(Y=t) &= P\left(\frac{X}{\sqrt{n}}=t\right) = P\left(X=\sqrt{n}t\right) \\
&= \mathrm{e}^{\frac{-\sqrt{n}t\left(\sqrt{n}t-1\right)}{n}}, \text{ hence}
\end{aligned}$$

$$\lim_{n\to\infty} P(Y=t) = \mathrm{e}^{-t^2}.$$

Maosen [10] considered the following problem. Let t be any nonnegative integer.

If we want to select k balls from an n-line or an n-circle under the restriction that any two adjacent selected balls are not t-separated, how many ways are there to do it? He solved these problems by means of a direct structural analysis. For the two kinds of problems, he used $F_t(n,k)$ to denote the number of ways of selecting k balls from n balls arranged in a line with no two adjacent selected balls being t-separation and $G_t(n,k)$ to denote the number of ways of selecting k balls from an n-circle with no two adjacent selected being t-separation. He proved that

$$F_t(n,k) = \sum_{t\ge 0} (-1)^t \binom{k-1}{l} \binom{n-l(t+1)}{k-1}, \tag{3.2}$$

$$G_t(n,k) = \frac{n}{k} \left\{ (-1)^j \binom{k}{j} \binom{n-j(t+1)-1}{k-1-j} + (-1)^k \delta\left[n, k(1+t)\right] \right\}. \tag{3.3}$$

Remark 2. *In fact El-Desouky* [5] *has proved* (3.2) *in* 1988.

References

[1] Kplansky, I. (1943) Solution of the "Problems des Ménages". *Bulletin of the American Mathematical Society*, **49**, 784-785. http://dx.doi.org/10.1090/S0002-9904-1943-08035-4

[2] Riordan, J. (1958) An Introduction to Combinatorial Analysis. Wiley, New York.

[3] Moser, W.O.J. (1986) The Number of Subsets without a Fixed Circular Distance. *Journal of Combinatorial Theory*, *Series A*, **43**, 130-132. http://dx.doi.org/10.1016/0097-3165(86)90030-0

[4] El-Desouky, B.S. (1988) On Selecting k Balls from an n-Line without Unit Separation. *Indian Journal of Pure and Applied Mathematics*, **19**, 145-148.

[5] El-Desouky, B.S. (1988) Selecting k Balls without s-Separation. *The 23rd Annual Conference on Statistics*, *Computer Science*, *Operations Research and Mathematics*, Cairo, December 1988, 40-46.

[6] Mansour, T. and Sun, Y.D. (2008) On Selecting the Number of Combinations without Certain Separations. *European Journal of Combinatorics*, **29**, 1200-1206. http://dx.doi.org/10.1016/j.ejc.2007.06.024

[7] Mansour, T. (2014) Set Partitions with Circular Successions. *European Journal of Combinatorics*, **41**, 207-216. http://dx.doi.org/10.1016/j.ejc.2014.06.008

[8] Gourden, J.P. and Jackson, D.M. (1993) Combinatorial Enumeration. Wiley, New York.

[9] Pease, R.W. (1975) General Solution to the Occupancy Problem with Variably Sized Runs of Adjacent Cells Occupied by Single Balls. *Mathematics Magazine*, **48**, 131-134. http://dx.doi.org/10.2307/2689693

[10] Maosen, J. (1995) On Selecting k Balls from an n-Line or n-Circle without t-Separations. *Northeastern Mathematical Journal*, **11**, 355-364.