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# On the Cauchy Problem for Von Neumann-Landau Wave Equation

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## Abstract

**In present paper we prove the local well-posedness for Von Neumann-Landau wave equation by the T. Kato's method.**

## Keywords

**Von Neumann-Landau Wave Equation, Strichartz Estimate, Cauchy Problem**

## 1. Introduction

For the stationary Von Neumann-Landau wave equation, Chen investigated the Dirichlet problems [1], where the generalized solution is studied by Function-analytic method. The present paper is related to the Cauchy problem: the Von Neumann-Landau wave equation

$$\begin{cases} i\partial_t u = \left(-\Delta_x + \Delta_y\right)u + f\left(u\right) \\ u\left(0,x,y\right) = u_0\left(x,y\right) \end{cases}, \tag{1}$$

where $\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ for $x = \left(x_1,\cdots,x_n\right) \in \mathbb{R}^n$, $u\left(t,x,y\right)$ is an unknown complex valued function on $\mathbb{R}^{1+2n}$ and $f$ is a nonlinear complex valued function.

If the plus "+" is replaced by the minus "−" on right hand in Equation (1), then the resulted equation is the Schrödinger equation. For the Schrödinger equation, the well-posedness problem is investigated for various nonlinear terms $f$. In terms of the nonlinear terms $f$, the problem (1) can be divided into the subcritical case and the critical case for $H^1$ solutions. We are concerned with the subcritical case and obtain a local well-

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posedness result by the T. Kato's method.

The paper is organized as follows. Section 2 contains the list of assumptions on the interaction term $f$ and the main result is presented. Section 3 is concerned with the Strichartz estimates. Finally, in Section 4, the main result is proved.

## 2. Statement of the Main Result

In this section we list the assumptions on the interaction term $f$ and state the main result. Firstly, we recall that the definition of admissible pair [2].

**Definition 2.1.** *Fix $d = 2n$, $n \ge 1$. We say that a pair $(q,r)$ of exponents is admissible if*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \tag{2}$$

*and*

$$2 \le r \le \frac{2d}{d-2} \quad (2 \le r < \infty, \text{ if } d = 2). \tag{3}$$

**Remark 2.1.** *The pairs $(\infty, 2)$ is always admissible, so is the $\left(2, \frac{2d}{d-2}\right)$ if $d > 2$. The two pairs are called the endpoint cases.*

Secondly, let $f \in C(\mathbb{C}, \mathbb{C})$ satisfy

$$f(0) = 0, \tag{4}$$

and

$$|f(u) - f(v)| \le K(M)|u - v|, \tag{5}$$

for all $u, v \in \mathbb{C}$ such that $|u|, |v| \le M$, with

$$K(t) \le C_1(1 + t^\alpha), \quad 0 < \alpha < \frac{4}{d-2}, \tag{6}$$

where $C_1$ is a constant independent of $t$. Set

$$f(u)(x) = f(u(x)), \tag{7}$$

for all measurable function $u$ and a.e. $x \in \mathbb{R}^{1+2n}$.

Finally, let us make the notion of solution more precise.

**Definition 2.2.** *Let $I$ be an interval such that $0 \in I$. We say that $u$ is a strong $H^1$-solution of* (1) *on $I$ if $u \in C\left(I, H^1\left(\mathbb{R}^d\right)\right)$ satisfies the integral equation*

$$u(t) = \mathrm{e}^{-itL}u_0 - i\int_0^t \mathrm{e}^{-i(t-s)L} f(u(s))\,\mathrm{d}s, \tag{8}$$

*for all $t \in I$, where $L =: -\Delta_x + \Delta_y$.*

The main result is the following theorem:

**Theorem 1.** *Suppose $n \ge 1$. Let $f \in C(\mathbb{C}, \mathbb{C})$ satisfy* (4)-(6). *If $f$ (considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$) is of class $C^1$, then the Cauchy problem* (1) *is locally well posed in $H^1(\mathbb{R}^d)$. More specially, the following properties hold*:

(i) *For any $R > 0$ there exists a time $T = T(d, \alpha, R) > 0$ and constant $c = c(d, \alpha)$ such that for each $u_0$ in the ball $B_R := \left\{\varphi \in H^1\left(\mathbb{R}^d\right) : \|\varphi\|_{H^1\left(\mathbb{R}^d\right)} \le R\right\}$ there exists a unique strong $H^1$-solution $u$ to the Equation* (1) *in $C\left([-T, T], H^1\left(\mathbb{R}^d\right)\right)$ such that*

$$\|u\|_{L^\infty\left((-T,T),H^1(\mathbb{R}^d)\right)}+\|u\|_{L^q\left((-T,T),W^{1,r}(\mathbb{R}^d)\right)}\le c\|u_0\|_{H^1(\mathbb{R}^d)},\tag{9}$$

*where* $r=\alpha+2$, *and* $(q,r)$ *is an admissible pair*.

(ii) *The map* $u_0\mapsto u$ *is continuous from* $B_R$ *to* $C\left([-T,T],H^1(\mathbb{R}^d)\right)$;

(iii) *For every* $u_0\in H^1(\mathbb{R}^d)$, *the unique solution* $u$ *is defined on a maximal interval* $(-T_{\min},T_{\max})$, *with* $T_{\max}=T_{\max}(u_0)\in(0,\infty]$ *and* $T_{\min}=T_{\min}(u_0)\in(0,\infty]$;

(iv) *There is the blowup alternative*: *If* $T_{\max}<\infty$, *then* $\|u(t)\|_{H^1(\mathbb{R}^d)}\to+\infty$ *as* $t\nearrow T_{\max}$ (*respectively*, *if* $T_{\min}<\infty$, *then* $\|u(t)\|_{H^1(\mathbb{R}^d)}\to+\infty$ *as* $t\searrow -T_{\min}$).

**Remark 2.2.** *It follows from Strichartz estimates that*

$$u\in L^\gamma_{loc}\left((-T_{\min},T_{\max}),W^{1,\rho}(\mathbb{R}^d)\right),$$

*for any admissible pair* $(\gamma,\rho)$.

**Remark 2.3.** *For the Schrödinger equations, the similar results hold* [2]. *It implies a fact that the ellipticity of the operator* $-\Delta_x-\Delta_y$ *is not the key point in the local well-posedness problem.*

## 3. Strichartz Estimates

In this subsection, we recall that the Strichartz estimates. Let $(\xi,\eta)$ denote a general Fourier variable in $\mathbb{R}^{2n}$, $\xi=(\xi_1,\cdots,\xi_n)$, $\eta=(\eta_1,\cdots,\eta_n)$. Let $L=:-\Delta_x+\Delta_y$, then by Fourier transform(denoting by $\mathcal{F}$ or $\wedge$) we have

$$Lu=\mathcal{F}^{-1}\left(\left(|\xi|^2-|\eta|^2\right)\hat{u}\right),\tag{10}$$

for any $u\in H^2(\mathbb{R}^{2n})$. It is easy to verify that the $L$ is a self-adjoint unbounded operator on $L^2(\mathbb{R}^{2n})$ with the domain $H^2(\mathbb{R}^{2n})$. Then, by Stone theorem we see that $e^{itL}$ is an unitary group on $L^2(\mathbb{R}^{2n})$. Moreover, $\mathrm{e}^{itL}$ can be expressed explicitly by Fourier transform.

$$\mathrm{e}^{itL}\varphi=\mathcal{F}^{-1}\left(\mathrm{e}^{it\left(|\xi|^2-|\eta|^2\right)}\hat{\varphi}\right),\tag{11}$$

for any $\varphi\in L^2(\mathbb{R}^{2n})$. By the direct compute, we conclude

$$\left(\mathrm{e}^{itL}\varphi\right)(x,y)=\frac{1}{(4\pi i|t|)^n}\int_{\mathbb{R}^{2n}}\mathrm{e}^{\frac{-i|x-x'|^2}{4t}}\mathrm{e}^{\frac{i|y-y'|^2}{4t}}\varphi(x',y')\mathrm{d}x'\mathrm{d}y'.\tag{12}$$

The following result is the fundamental estimate for $\mathrm{e}^{itL}$.

**Lemma 1.** *If* $p\in[1,2]$ *and* $t\ne0$, *then* $\mathrm{e}^{itL}$ *maps* $L^p(\mathbb{R}^d)$ *continuously to* $L^{p'}(\mathbb{R}^d)$ *and*

$$\left\|\mathrm{e}^{itL}\varphi\right\|_{L^{p'}(\mathbb{R}^d)}\le(4\pi|t|)^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}\|\varphi\|_{L^p(\mathbb{R}^d)}\tag{13}$$

*where* $p'$ *is the dual exponent of* $p$, *defined by the formula* $\frac{1}{p}+\frac{1}{p'}=1$.

*Proof.* For the proof please see [3] or [4]. □

The following estimates, known as Strichartz estimates, are key points in the method introduced by T. Kato [5].

**Lemma 2.** *Let* $(q,r)$ *and* $(\tilde{q},\tilde{r})$ *be any admissible exponents. Then, we have the homogeneous Strichartz* *estimate*

$$\left\| e^{itL}\varphi \right\|_{L^q\left(\mathbb{R},L^r\left(\mathbb{R}^d\right)\right)} \lesssim_{d,q,r} \left\|\varphi\right\|_{L^2\left(\mathbb{R}^d\right)}, \tag{14}$$

*the dual homogeneous Strichartz estimate*

$$\left\| \int_{\mathbb{R}} e^{-itL}\phi(t)\,dt \right\|_{L^2\left(\mathbb{R}^d\right)} \lesssim_{d,q,r} \left\|\phi\right\|_{L^{q'}\left(\mathbb{R},L^{r'}\left(\mathbb{R}^d\right)\right)}, \tag{15}$$

*and the inhomogeneous Strichartz estimate*

$$\left\| \int_{t_0}^{t} e^{i(t-s)L}\phi(s)\,ds \right\|_{L^q\left(J,L^r\left(\mathbb{R}^d\right)\right)} \lesssim_{d,q,r,\tilde{q},\tilde{r}} \left\|\phi\right\|_{L^{\tilde{q}'}\left(J,L^{\tilde{r}'}\left(\mathbb{R}^d\right)\right)}, \tag{16}$$

*for any interval* $J$ *and real number* $t_0$.

*Proof*. For the proof please see [3] or [4] in the non-endpoint case. On the other hand, the proof in the endpoint case follows from the theorem 1.2 in [6] and the lemma 1 in the present paper. □

## 4. The Proof of Theorem

*Proof*. Let $\chi \in C_0^\infty\left(\mathbb{R}^2\right)$ be such that $\chi(z)=1$ for $|z|\le 1$ and $\chi(z)=0$ for $|z|\ge 2$. Setting

$$f_1(z) = \chi(z) f(z),$$
$$f_2(z) = \left(1-\chi(z)\right) f(z),$$

one easily verifies that for any $z,w\in\mathbb{C}$

$$\begin{aligned} \left| f_1(z) - f_1(w)\right| &\lesssim_\alpha |z-w|, \\ \left| f_2(z) - f_2(w)\right| &\lesssim_\alpha \left(|z|^\alpha + |w|^\alpha\right)|z-w|. \end{aligned} \tag{17}$$

Set $f_l(u)(x) = f_l(u(x))$ for $l=1,2$. Using (17), we deduce from Hölder's inequality that

$$\begin{aligned} \left\| f_1(u) - f_1(v)\right\|_{L^2\left(\mathbb{R}^d\right)} &\lesssim_\alpha \left\|u-v\right\|_{L^2\left(\mathbb{R}^d\right)}, \\ \left\| f_2(u) - f_2(v)\right\|_{L^{r'}\left(\mathbb{R}^d\right)} &\lesssim_\alpha \left(\left\|u\right\|^\alpha_{L^r\left(\mathbb{R}^d\right)} + \left\|v\right\|^\alpha_{L^r\left(\mathbb{R}^d\right)}\right)\left\|u-v\right\|_{L^r\left(\mathbb{R}^d\right)}. \end{aligned} \tag{18}$$

And it follows from Remark 1.3.1 (vii) in [2] that

$$\begin{aligned} \left\|\nabla f_1(u)\right\|_{L^2\left(\mathbb{R}^d\right)} &\lesssim_\alpha \left\|\nabla u\right\|_{L^2\left(\mathbb{R}^d\right)}, \\ \left\|\nabla f_2(u)\right\|_{L^{r'}\left(\mathbb{R}^d\right)} &\lesssim_\alpha \left\|u\right\|^\alpha_{L^r\left(\mathbb{R}^d\right)} \left\|\nabla u\right\|_{L^r\left(\mathbb{R}^d\right)}. \end{aligned} \tag{19}$$

We now proceed in four steps.

Step 1. Proof of (i). Fix $A, T>0$, to be chosen later, and let $r=\alpha+2$, $q$ be such that $(q,r)$ is an admissible pair, and set $I=(-T,T)$. Consider the set

$$E = \left\{ u\in L^\infty\left(I,H^1\left(\mathbb{R}^d\right)\right)\cap L^q\left(I,W^{1,r}\left(\mathbb{R}^d\right)\right) : \left\|u\right\|_{L^\infty\left(I,H^1\left(\mathbb{R}^d\right)\right)} \le A, \left\|u\right\|_{L^q\left(I,W^{1,r}\left(\mathbb{R}^d\right)\right)} \le A \right\}, \tag{20}$$

equipped with the distance

$$d(u,v) = \left\|u-v\right\|_{L^\infty\left(I,L^2\left(\mathbb{R}^d\right)\right)} + \left\|u-v\right\|_{L^q\left(I,L^r\left(\mathbb{R}^d\right)\right)}. \tag{21}$$

We claim that $(E,d)$ is a complete metric space. Indeed, let $\{u_k\}_{k\ge 1}\subset E$ be a Cauchy sequence. Clearly, $\{u_k\}_{k\ge 1}$ is also a Cauchy sequence in $L^\infty\left(I,L^2\left(\mathbb{R}^d\right)\right)$ and $L^q\left(I,L^r\left(\mathbb{R}^d\right)\right)$. In particular, there exists a function $u\in L^{\infty}\left(I,L^{2}\left(\mathbb{R}^{d}\right)\right)\cap L^{q}\left(I,L^{r}\left(\mathbb{R}^{d}\right)\right)$ such that $u_k\to u$ in $L^{\infty}\left(I,L^{2}\left(\mathbb{R}^{d}\right)\right)$ and $L^{q}\left(I,L^{r}\left(\mathbb{R}^{d}\right)\right)$ as $k\to\infty$. Applying theorem 1.2.5 in [2] twice, we conclude that

$$u\in L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)\cap L^{q}\left(I,W^{1,r}\left(\mathbb{R}^{d}\right)\right),$$

and that

$$\|u\|_{L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)}\le\liminf_{k\to\infty}\|u_k\|_{L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)}\le A,$$
$$\|u\|_{L^{q}\left(I,W^{1,r}\left(\mathbb{R}^{d}\right)\right)}\le\liminf_{k\to\infty}\|u_k\|_{L^{q}\left(I,W^{1,r}\left(\mathbb{R}^{d}\right)\right)}\le A;$$

thus, $u_k\to u$ in $E$ as $k\to\infty$.

Taking up any $u,v\in E$. Since $f_1$ is continuous $L^{2}\left(\mathbb{R}^{d}\right)\to L^{2}\left(\mathbb{R}^{d}\right)$, it follows that $f_1(u):I\to L^{2}\left(\mathbb{R}^{d}\right)$ is measurable, and we deduce easily that $f_1(u)\in L^{\infty}\left(I,L^{2}\left(\mathbb{R}^{d}\right)\right)$. Similarly, since $f_2$ is continuous $L^{r}\left(\mathbb{R}^{d}\right)\to L^{r'}\left(\mathbb{R}^{d}\right)$, we see that $f_2(u)\in L^{q}\left(I,L^{r'}\left(\mathbb{R}^{d}\right)\right)$. Using inequalities (18) and (19) and Remark 1.2.2 (iii) in [2], We deduce the following:

$$f_1(u)\in L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right),\ f_2(u)\in L^{q}\left(I,w^{1,r'}\left(\mathbb{R}^{d}\right)\right),$$
$$\left\|f_1(u)\right\|_{L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)}\lesssim_{\alpha}\|u\|_{L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)},$$
$$\left\|f_2(u)\right\|_{L^{q}\left(I,w^{1,r'}\left(\mathbb{R}^{d}\right)\right)}\lesssim_{\alpha}\|u\|^{\alpha}_{L^{\infty}\left(I,L^{r}\left(\mathbb{R}^{d}\right)\right)}\|u\|_{L^{q}\left(I,W^{1,r}\left(\mathbb{R}^{d}\right)\right)},$$

and

$$\left\|f_1(u)-f_1(v)\right\|_{L^{\infty}\left(I,L^{2}\left(\mathbb{R}^{d}\right)\right)}\lesssim_{\alpha}\|u-v\|_{L^{\infty}\left(I,L^{2}\left(\mathbb{R}^{d}\right)\right)},$$
$$\left\|f_2(u)-f_2(v)\right\|_{L^{q}\left(I,L^{r'}\left(\mathbb{R}^{d}\right)\right)}\lesssim_{\alpha}\left(\|u\|^{\alpha}_{L^{\infty}\left(I,L^{r}\left(\mathbb{R}^{d}\right)\right)}+\|v\|^{\alpha}_{L^{\infty}\left(I,L^{r}\left(\mathbb{R}^{d}\right)\right)}\right)\|u-v\|_{L^{q}\left(I,L^{r}\left(\mathbb{R}^{d}\right)\right)}.$$

Using the embedding $H^{1}\left(\mathbb{R}^{d}\right)\hookrightarrow L^{r}\left(\mathbb{R}^{d}\right)$ and Hölder's inequality in time, we deduce from the above estimates that

$$\left\|f_1(u)\right\|_{L^{1}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)}+\left\|f_2(u)\right\|_{L^{q'}\left(I,W^{1,r'}\left(\mathbb{R}^{d}\right)\right)}\lesssim_{d,\alpha}\left(T+T^{\frac{q-q'}{qq'}}\right)\left(1+A^{\alpha}\right)A, \tag{22}$$

and

$$\left\|f_1(u)-f_1(v)\right\|_{L^{1}\left(I,L^{2}\left(\mathbb{R}^{d}\right)\right)}+\left\|f_2(u)-f_2(v)\right\|_{L^{q'}\left(I,L^{r'}\left(\mathbb{R}^{d}\right)\right)}\lesssim_{d,\alpha}\left(T+T^{\frac{q-q'}{qq'}}\right)\left(1+A^{\alpha}\right)d(u,v). \tag{23}$$

Given $u_0\in H^{1}\left(\mathbb{R}^{d}\right)$. For any $u\in E$, let $\mathcal{H}(u)$ be defined by

$$\mathcal{H}(u)(t)=\mathrm{e}^{-itL}u_0-i\int_0^t\mathrm{e}^{-i(t-s)L}f\left(u(s)\right)\mathrm{d}s. \tag{24}$$

It follows from (22) and Strichartz estimates (lemma 2) that

$$\mathcal{H}(u)\in C\left([-T,T],H^{1}\left(\mathbb{R}^{d}\right)\right)\cap L^{q}\left((-T,T),W^{1,r}\left(\mathbb{R}^{d}\right)\right), \tag{25}$$

and

$$\|\mathcal{H}(u)\|_{L^{\infty}\left(I,H^{1}\left(\mathbb{R}^{d}\right)\right)}+\|\mathcal{H}(u)\|_{L^{q}\left(I,W^{1,r}\left(\mathbb{R}^{d}\right)\right)}\le C_1(d,\alpha)\left[\|u_0\|_{H^{1}\left(\mathbb{R}^{d}\right)}+\left(T+T^{\frac{q-q'}{qq'}}\right)\left(1+A^{\alpha}\right)A\right]. \tag{26}$$

Also, we deduce from (23) that

$$\left\|\mathcal{H}(u)-\mathcal{H}(v)\right\|_{L^{\infty}\left(I,L^2\left(\mathbb{R}^d\right)\right)}+\left\|\mathcal{H}(u)-\mathcal{H}(v)\right\|_{L^{q}\left(I,L^r\left(\mathbb{R}^d\right)\right)}\le C_1(d,\alpha)\left(T+T^{\frac{q-q'}{qq'}}\right)\left(1+A^{\alpha}\right)d(u,v). \tag{27}$$

Finally, note that $q>q'$. We now proceed as follows. For any $R\ge\|u_0\|_{H^1\left(\mathbb{R}^d\right)}$, we set $A=2C_1(d,\alpha)R$, and we let $T=T(d,\alpha,R)$ be the unique positive number so that

$$C_1(d,\alpha)\left(T+T^{\frac{q-q'}{qq'}}\right)\left(1+A^{\alpha}\right)=\frac{1}{2}. \tag{28}$$

It then follows from (26) and (28) that for any $u_0\in B_R$

$$\left\|\mathcal{H}(u)\right\|_{L^{\infty}\left(I,H^1\left(\mathbb{R}^d\right)\right)}+\left\|\mathcal{H}(u)\right\|_{L^{q}\left(I,W^{1,r}\left(\mathbb{R}^d\right)\right)}\le C_1(d,\alpha)\|u_0\|_{H^1\left(\mathbb{R}^d\right)}+\frac{1}{2}A\le C_1(d,\alpha)R+\frac{1}{2}A=A. \tag{29}$$

Thus, $\mathcal{H}(u)\in E$ and by (27) we obtain

$$d\left(\mathcal{H}(u),\mathcal{H}(v)\right)\le\frac{1}{2}d(u,v). \tag{30}$$

In particular, $\mathcal{H}$ is a strict contraction on $E$. By Banach's fixed-point theorem, $\mathcal{H}$ has a unique fixed point $u\in E$; that is $u$ satisfies (8). By (25), $u=\mathcal{H}(u)\in C\left([-T,T],H^1\left(\mathbb{R}^d\right)\right)$. By the definition 2.2, we con- clude that $u$ is a strong $H^1$-solution of (1) on $[-T,T]$. Note that $T(d,\alpha,R)$ is decreasing on $R$, then the estimate (9) holds for $c=2C_1(d,\alpha)$ by letting $R=\|u_0\|_{H^1\left(\mathbb{R}^d\right)}$ in (29).

For uniqueness, assume that $u,v$ are two strong $H^1$-solution of (1) on $[-T,T]$ with the same initial value $u_0$. Then, we have

$$u(t)-v(t)=-i\int_0^t \mathrm{e}^{-i(t-s)L}\left[f\left(u(s)\right)-f\left(v(s)\right)\right]\mathrm{d}s. \tag{31}$$

For simplicity, we set

$$w_l(t)=-i\int_0^t \mathrm{e}^{-i(t-s)L}\left[f_l\left(u(s)\right)-f_l\left(v(s)\right)\right]\mathrm{d}s,$$

for $l=1,2$, and $w=u-v$. For any interval $J\subset(-T,T)$, by (18) and Strichartz estimates (16), then we obtain

$$\|w_1\|_{L^{\infty}\left(J,L^2\left(\mathbb{R}^d\right)\right)}+\|w_1\|_{L^{q}\left(J,L^r\left(\mathbb{R}^d\right)\right)}\lesssim_{d,\alpha}\left\|f_1(u)-f_1(v)\right\|_{L^1\left(J,L^2\left(\mathbb{R}^d\right)\right)}\lesssim_{d,\alpha}\|w\|_{L^1\left(J,L^2\left(\mathbb{R}^d\right)\right)}. \tag{32}$$

Similarly, for $w_2$ we have

$$\begin{aligned}&\|w_2\|_{L^{\infty}\left(J,L^2\left(\mathbb{R}^d\right)\right)}+\|w_2\|_{L^{q}\left(J,L^r\left(\mathbb{R}^d\right)\right)}\lesssim_{d,\alpha}\left\|f_2(u)-f_2(v)\right\|_{L^{q'}\left(J,L^{r'}\left(\mathbb{R}^d\right)\right)}\\&\lesssim_{d,\alpha}\left(\|u\|^{\alpha}_{L^{\infty}\left(I,H^1\left(\mathbb{R}^d\right)\right)}+\|v\|^{\alpha}_{L^{\infty}\left(I,H^1\left(\mathbb{R}^d\right)\right)}\right)\|w\|_{L^{q'}\left(J,L^r\left(\mathbb{R}^d\right)\right)}.\end{aligned} \tag{33}$$

Note that $w=w_1+w_2$. Then, it follows from that

$$\|w\|_{L^{\infty}\left(J,L^2\left(\mathbb{R}^d\right)\right)}+\|w\|_{L^{q}\left(J,L^r\left(\mathbb{R}^d\right)\right)}\le C_2(1+B)\left(\|w\|_{L^1\left(J,L^2\left(\mathbb{R}^d\right)\right)}+\|w\|_{L^{q'}\left(J,L^r\left(\mathbb{R}^d\right)\right)}\right), \tag{34}$$

where the constant $B=\|u\|^{\alpha}_{L^{\infty}\left(I,H^1\left(\mathbb{R}^d\right)\right)}+\|v\|^{\alpha}_{L^{\infty}\left(I,H^1\left(\mathbb{R}^d\right)\right)}$ and the constant $C_2$ is independent of $J$ by above inequalities. Note that $q'<q$, we conclude that $w=0$ by the lemma 4.2.2 in [2]. So $u=v$.

Step 2. Proof of (ii). Suppose that $u_0^{(k)}\to u_0$ in $B_R$ as $k\to\infty$. By the part (i), we denote $u_k$ and $u$ by the unique solution of (1) corresponding to the initial value $u_0^{(k)}$ and $u$, respectively. We will show that $u_k \to u$ in $C\left(\left[-T,T\right],H^1\left(\mathbb{R}^d\right)\right)$ as $k \to \infty$. Note that

$$u_k\left(t\right)-u\left(t\right)=\mathrm{e}^{-itL}\left(u_0^{(k)}-u_0\right)+\mathcal{H}\left(u_k\right)-\mathcal{H}\left(u\right), \tag{35}$$

and the estimate (29) which implies that (27) holds for $v=u_k$. Note that the choosing of the time $T$ in (28), it follows from (27) with (30) that

$$d\left(u_k,u\right)\lesssim_{d,\alpha}\left\|u_0^{(k)}-u_0\right\|_{L^2\left(\mathbb{R}^d\right)}+\frac{1}{2}d\left(u_k,u\right). \tag{36}$$

Hence, we have

$$\left\|u_k-u\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|u_k-u\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\lesssim_{d,\alpha}\left\|u_0^{(k)}-u_0\right\|_{L^2\left(\mathbb{R}^d\right)}. \tag{37}$$

Next, we need to estimate $\left\|\nabla\left(u_k-u\right)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}$. Note that $\nabla$ commutes with $\mathrm{e}^{-itL}$, and so

$$\nabla u\left(t\right)=\mathrm{e}^{-itL}\nabla u_0-i\int_0^t\mathrm{e}^{-i(t-s)L}\nabla f\left(u\left(s\right)\right)\mathrm{d}s. \tag{38}$$

A similar identity holds for $u_k$. We use the assumption $f\in C^1\left(\mathbb{R}^2,\mathbb{R}^2\right)$, which implies that $\nabla f\left(u\right)=f'\left(u\right)\nabla u$, where $f'\left(u\right)$ is a $2\times 2$ real matrix. Therefore, we may write

$$\begin{aligned}\nabla\left(u_k-u\right)\left(t\right)=\mathrm{e}^{-itL}\nabla\left(u_0^{(k)}-u_0\right)-i\int_0^t\mathrm{e}^{-i(t-s)L}f'\left(u_k\right)\nabla\left(u_k-u\right)\mathrm{d}s\\ -i\int_0^t\mathrm{e}^{-i(t-s)L}\left(f'\left(u_k\right)-f'\left(u\right)\right)\nabla u\mathrm{d}s.\end{aligned} \tag{39}$$

Note that $f_1$ and $f_2$ are also $C^1$, so that $f'=f_1'+f_2'$, and from (17) we deduce that $\left|f_1'(z)\right|\le C_3$ and $\left|f_2'(z)\right|\lesssim_\alpha |z|^\alpha$ for any $z\in\mathbb{C}$ and some constant $C_3$. Therefore, arguing as in Step 1, we obtain the estimate

$$\begin{aligned}&\left\|\nabla\left(u_k-u\right)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\nabla\left(u_k-u\right)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\\ &\lesssim_{d,\alpha}\left[\left\|\nabla\left(u_0^{(k)}-u_0\right)\right\|_{L^2\left(\mathbb{R}^d\right)}+T\left\|\nabla\left(u_k-u\right)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\right.\\ &\quad+T^{\frac{q-q'}{qq'}}\left\|u_k\right\|^\alpha_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\left\|\nabla\left(u_k-u\right)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\\ &\quad+\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\\ &\quad\left.+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\right].\end{aligned} \tag{40}$$

By choosing $T=T\left(d,\alpha,R\right)$ as (28) and noting that $u_k\in B_R$, from (40) we obtain that

$$\begin{aligned}&\left\|\nabla\left(u_k-u\right)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\nabla\left(u_k-u\right)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\\ &\lesssim_{d,\alpha}\left[\left\|\nabla\left(u_0^{(k)}-u_0\right)\right\|_{L^2\left(\mathbb{R}^d\right)}+\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}\right.\\ &\quad\left.+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\right].\end{aligned} \tag{41}$$

There, if we prove that

$$\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\to 0, \tag{42}$$

as $k\to\infty$, then we have

$$\left\|\nabla\left(u_k-u\right)\right\|_{L^\infty\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\nabla\left(u_k-u\right)\right\|_{L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)}\to 0, \tag{43}$$

as $k\to\infty$, which, combined with (37), yields the desired convergence. we prove (42) by contradiction, and we assume that there exists $\varepsilon_0>0$, and a subsequence, which we still denote by $\{u_k\}_{k\ge 1}$ such that

$$\left\|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right\|_{L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right)}+\left\|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right\|_{L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right)}\ge\varepsilon_0. \tag{44}$$

By using (37) and possibly extracting a subsequence, we may assume that $u_k\to u$ a.e. on $(-T,T)\times\mathbb{R}^d$ and that there exists $v\in L^q\left((-T,T),L^r\left(\mathbb{R}^d\right)\right)$ such that $|u_k|\le v$ a.e. on $(-T,T)\times\mathbb{R}^d$. In particular, both $\left(f_1'(u_k)-f_1'(u)\right)\nabla u$ and $\left(f_2'(u_k)-f_2'(u)\right)\nabla u$ converge to 0 a.e. on $(-T,T)\times\mathbb{R}^d$. Since

$$\left|\left(f_1'(u_k)-f_1'(u)\right)\nabla u\right|\le 2C_3\left|\nabla u\right|\in L^1\left((-T,T),L^2\left(\mathbb{R}^d\right)\right),$$

and

$$\left|\left(f_2'(u_k)-f_2'(u)\right)\nabla u\right|\lesssim_\alpha\left(|u_k|^\alpha+|u|^\alpha\right)|\nabla u|\lesssim_\alpha\left(v^\alpha+|u|^\alpha\right)|\nabla u|\in L^{q'}\left((-T,T),L^{r'}\left(\mathbb{R}^d\right)\right),$$

we obtain from the dominated convergence a contradiction with (44).

Step 3. Proof of (iii). Consider $u_0\in H^1\left(\mathbb{R}^d\right)$ and let

$$T_{\max}(u_0)=\sup\left\{T>0:\text{there exists a solution of }(1)\text{ on }[0,T]\right\},$$

$$T_{\min}(u_0)=\sup\left\{T>0:\text{there exists a solution of }(1)\text{ on }[-T,0]\right\}.$$

It follows from part (i) there exists a solution

$$u\in C\left(\left(-T_{\min},T_{\max}\right),H^1\left(\mathbb{R}^d\right)\right),$$

of (1).

Step 4. Proof of (iv). Suppose now that $T_{\max}<\infty$, and assume that there exist $M<\infty$ and a sequence $t_j\nearrow T_{\max}$ such that $\left\|u(t_j)\right\|_{H^1\left(\mathbb{R}^d\right)}\le M$. Let $k$ be such that $t_k+T(d,\alpha,M)>T_{\max}(u_0)$. By part (i), from the initial data $u(t_k)$, one can extend $u$ up to $t_k+T(d,\alpha,M)$, which contradicts maximality. Therefore,

$$\left\|u(t)\right\|_{H^1\left(\mathbb{R}^d\right)}\to\infty,\text{ as }t\nearrow T_{\max}.$$

One shows by the same argument that if $T_{\min}<\infty$, then

$$\left\|u(t)\right\|_{H^1\left(\mathbb{R}^d\right)}\to\infty,\text{ as }t\searrow -T_{\min x}.$$

This completes the proof. □

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