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The Bounds for Eigenvalues of Normalized Laplacian Matrices and Signless Laplacian Matrices

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Abstract

In this paper, we found the bounds of the extreme eigenvalues of normalized Laplacian matrices and signless Laplacian matrices by using their traces. In addition, we found the bounds for *k*-th eigenvalues of normalized Laplacian matrix and signless Laplacian matrix.

Keywords

Normalized Laplacian Matrix, Signless Laplacian Matrix, Bounds of Eigenvalue

1. Introduction

Let $G(V,E)$ be a simple graph with the vertex set $V=\{v_1,v_2,\cdots,v_n\}$ and edge set of E. For $v_i \in V$, the degree of v_i, the set of neighbours of v_i are denoted by d_i and N_i, respectively. If v_i and v_j are adjacent, we denote $v_i \sim v_j$ of short use $i \sim j$.

The adjacency matrix, Laplacian matrix and diagonal matrix of vertex degree of a G graph are denoted by $A(G)$, $L(G)$, $D(G)$, respectively. Clearly

$$L(G)=D(G)-A(G)$$

The normalized Laplacian matrix of G is defined as $\mathscr{L}(G)=D^{-1/2}(G)L(G)D^{-1/2}(G)$ *i.e.*, $\mathscr{L}(G)=\left[\ell_{ij}\right]_{n\times n}$ where

$$\ell_{ij}=\begin{cases} 1 & ; \quad \text{if } i=j \\ \dfrac{-1}{\sqrt{d_i d_j}} & ; \quad \text{if } i \sim j \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

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The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$ *i.e.*, $Q(G) = \left[q_{ij}\right]_{n\times n}$ where

$$q_{ij} = \begin{cases} d_i & ; \quad \text{if } i = j \\ 1 & ; \quad \text{if } i \sim j \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Since $\mathscr{L}(G)$ normalized Laplacian matrix and $Q(G)$ signless Laplacian matrix are real symetric matrices, their eigenvalues are real. We denote the eigenvalues of $\mathscr{L}(G)$ and $Q(G)$ by

$$\lambda_1(\mathscr{L}(G)) \geq \cdots \geq \lambda_n(\mathscr{L}(G))$$

and

$$\lambda_1(Q(G)) \geq \cdots \geq \lambda_n(Q(G))$$

respectively.

Now we give some bounds for normalized Laplacian matrix and signless Laplacian matrix.

1. **Oliveira and de Lima's bound [1]:** For a simple connected graph G with n vertices and m edges, $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$

$$\lambda_1(Q(G)) \leq \max_i \left\{ \frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2} \right\} \tag{1}$$

where $m_i = \frac{1}{d_i}\sum_{i\sim j} d_j$.

2. **Another Oliveira and de Lima's bound [1]:**

$$\lambda_1(Q(G)) \leq \max_i \left\{ d_i + \sqrt{d_i m_i} \right\} \tag{2}$$

where $m_i = \frac{1}{d_i}\sum_{i\sim j} d_j$.

3. **Li, Liu *et al*. bound's [2] [3]:**

$$\lambda_1(Q(G)) \leq \frac{\Delta + \delta - 1\sqrt{(\Delta + \delta - 1)^2 + 8(2m - (n-1)\delta)}}{2} \tag{3}$$

4. **Rojo and Soto's bound [4]:** If λ_1 is the largest eigenvalue of $\mathscr{L}$ then

$$\left|\lambda_1(\mathscr{L}(G))\right| \leq 2 - \min_{i<j}\left(\frac{|N_i \cap N_j|}{\max\{d_i, d_j\}} \right) \tag{4}$$

where the minimum is taken over all pairs (i, j), $(1 \leq i < j \leq n)$.

In this paper, we found extreme eigenvalues of normalized Laplacian matrix and signless Laplacian matrix of a G graph with using theirs traces.

To obtain bounds for eigenvalues of $\mathscr{L}(G)$ and $Q(G)$ we need the followings lemmas and theorems.

Lemma 1. Let W and $\lambda = (\lambda_j)$ be nonzero column vectors, $e = (1,1,\cdots,1)^{\mathrm{T}}$, $C = I_n - \frac{ee^{\mathrm{T}}}{n}$ $m = \frac{\lambda^{\mathrm{T}}e}{n}$, $s^2 = \frac{\lambda^{\mathrm{T}}C\lambda}{n}$ and I_n is an identity matrix. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,

$$-s\sqrt{nW^{\mathrm{T}}CW} \leq W^{\mathrm{T}}\lambda - mW^{\mathrm{T}}e = W^{\mathrm{T}}C\lambda \leq s\sqrt{nW^{\mathrm{T}}CW}$$

$$\sum_j (\lambda_j - \lambda_n)^2 = n\left[s^2 + (m - \lambda_n)^2\right]$$

$$\sum_j \left(\lambda_1 - \lambda_j\right)^2 = n\left[s^2 + \left(\lambda_1 - m\right)^2\right]$$

$$\lambda_n \le m - \frac{s}{\sqrt{n-1}} \le m + \frac{s}{\sqrt{n-1}} \le \lambda_1$$

Theorem 1 [5]. Let A be a $n \times n$ complex matrix. Conjugate transpose of A denoted by A^*. Let $B = AA^*$ whose eigenvalues are $\lambda_1(B) \ge \lambda_2(B) \ge \cdots \ge \lambda_n(B)$. Then

$$m - s\sqrt{n-1} \le \lambda_n^2(B) \le m - \frac{s}{\sqrt{n-1}}$$

and

$$m + \frac{s}{\sqrt{n-1}} \le \lambda_1^2(B) \le m + s\sqrt{n-1}$$

where $m = \frac{trB}{n}$ and $s^2 = \frac{trB^2}{n} - m.$

2. Main Results for Normalized Laplacian Matrix

Theorem 2. Let G be a simple graph and $\mathscr{L}(G)$ be a normalized Laplacian matrix of G. If the eigenvalues of $\mathscr{L}(G)$ are $\lambda_1(\mathscr{L}(G)) \ge \lambda_2(\mathscr{L}(G)) \ge \cdots \ge \lambda_n(\mathscr{L}(G))$, then

$$\lambda_n \mathscr{L}(G) \le \sqrt{\left(1 + \frac{2}{n}\sum_{i\sim j, i<j}\frac{1}{d_i d_j}\right) + \sqrt{\frac{tr\left[L(G)\right]^4 - nm^2}{n(n-1)}}} \tag{5}$$

$$\lambda_1 \mathscr{L}(G) \ge \sqrt{\left(1 + \frac{2}{n}\sum_{i\sim j, i<j}\frac{1}{d_i d_j}\right) + \sqrt{\frac{tr\left[L(G)\right]^4 - nm^2}{n(n-1)}}} \tag{6}$$

$$\lambda_1 \mathscr{L}(G) \le \sqrt{1 + \frac{2}{n}\sum_{i\sim j, i<j}\frac{1}{d_i d_j} + \sqrt{\left(\frac{tr\left[L(G)\right]^4}{n} - m^2\right)(n-1)}} \tag{7}$$

Proof. Clearly

$$tr\left[\mathscr{L}(G)\right]^2 = n + 2\sum_{i\sim j, i<j}\frac{1}{d_i d_j}$$

and

$$tr\left[\mathscr{L}(G)\right]^4 = \sum_{i=1}^{n}\left(1 + \sum_{i\sim j}\frac{1}{d_i d_j}\right)^2 + 2\sum_{i<j}\left(\sum_{k\in N_i\cap N_j}\frac{1}{d_k\sqrt{d_i d_j}} - \sum_{i\sim j}\frac{2}{\sqrt{d_i d_j}}\right)^2$$

Since $\mathscr{L}(G)$ real symmetric matrix, we found the result from Theorem 1.

Example 1. Let $G = (V, E)$ with $V = \{1,2,3,4,5,6\}$ and $E = \{(1,2),(1,5),(2,3),(2,4),(2,6),(3,4),(3,5),(4,5),(5,6)\}$

$\lambda(\mathscr{L}(G))$	(4)	(6) (lower bound)	(7) (upper bound)
1.86	2	1.34	1.93

3. Main Results for Signless Laplacian Matrix

Theorem 3. Let G be a simple graph and $Q(G)$ be a signless Laplacian matrix of G. If the eigenvalues of

$Q(G)$ are $\lambda_1(Q(G)) \geq \lambda_2(Q(G)) \geq \cdots \geq \lambda_n(Q(G))$, then

$$\lambda_n(Q(G)) \leq \sqrt{\left(1+\frac{2}{n}\sum_{i\sim j,i<j}\frac{1}{d_i d_j}\right)+\sqrt{\frac{tr\left[Q(G)\right]^4-nm^2}{n(n-1)}}} \tag{8}$$

$$\lambda_1(Q(G)) \geq \sqrt{\left(1+\frac{2}{n}\sum_{i\sim j,i<j}\frac{1}{d_i d_j}\right)+\sqrt{\frac{tr\left[Q(G)\right]^4-nm^2}{n(n-1)}}} \tag{9}$$

$$\lambda_1(Q(G)) \leq \sqrt{1+\frac{2}{n}\sum_{i\sim j,i<j}\frac{1}{d_i d_j}+\sqrt{\left(\frac{tr\left[Q(G)\right]^4}{n}-m^2\right)(n-1)}} \tag{10}$$

Proof. Clearly

$$tr\left[Q(G)\right]^2 = n+2\sum_{i\sim j,i<j}\frac{1}{d_i d_j}$$

and

$$tr\left[Q(G)\right]^4 = \sum_{i=1}^{n}\left(1+\sum_{i\sim j}\frac{1}{d_i d_j}\right)^2 + 2\sum_{i<j}\left(\sum_{k\in N_i\cap N_j}\frac{1}{d_k\sqrt{d_i d_j}}-\sum_{i\sim j}\frac{2}{\sqrt{d_i d_j}}\right)^2$$

Since $Q(G)$ was real symmetric matrix, we found the result from Theorem 1.

Example 2. Let $G=(V,E)$ with $V=\{1,2,3,4,5,6,7\}$ and $E=\{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3),(3,5),(4,5),(4,6)\}$

$\lambda(Q(G))$	(1)	(2)	(3)	(9) (lower bound)	(10) (upper bound)
7.67	9.08	9.74	9.34	4.58	7.76

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