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Quasi-Exactly Solvable Time-Dependent Hamiltonians

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Abstract

A generalized method which helps to find a time-dependent Schrödinger equation for any static potential is established. We illustrate this method with two examples. Indeed, we use this method to find the time-dependent Hamiltonian of quasi-exactly solvable Lamé equation and to construct the matrix 2 × 2 time-dependent polynomial Hamiltonian.

Keywords

Quasi-Exactly Solvable, Time-Dependent Hamiltonian

1. Introduction

Another direction of investigation of quasi-exactly solvable *Schrödinger* is the study of time-dependent Hamiltonian. Time-dependence can be set through the potential. A first step is the direction was done in [1]. This is related to the quasi-exactly solvable sextic anharmonic oscillator potentials. The *Schrödinger* equation is now considered with a time-dependent potential $V(x,t)$,

$$i\partial_t \psi(x,t) = H\psi(x,t), \tag{1}$$

where

$$H = -\frac{\partial^2}{\partial x^2} + V(x,t). \tag{2}$$

The time-dependent potentials constructed from the well-known family of quasi-exactly solvable sextic anharmonic oscillator potentials

$$V(x) = v^2x^6 + 2\mu v x^4 + \left[\mu^2 - (4n+3)v\right]x^2, \quad v > 0, \mu \in IR, n \in IN \tag{3}$$

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are of the following form [1]

$$V(x,t)=u^4(t)x^6+2\beta u^3(t)x^4+\left[\beta^2-(4n+3+2k)-\frac{3\dot{u}^2(t)-2u(t)\ddot{u}(t)}{16u^4(t)}\right]u^2(t)x^2+\frac{k(k-1)}{x^2}, \qquad (4)$$

where $x>0, t\geq 0$, n is a non-negative integer, $k\geq 0$, β is real constant and $u(t)$ is an arbitrary function of $t\geq 0$ which is positive. If $k>1$, the last term in the above potential $V(x,t)$ may be viewed as a centrifugal term in radial equation with x playing the role of radial coordinate. The domain of the definition of the potential (4) may be extended to the real line if $k=0,1$. After some algebraic manipulations, one has obtained the algebraic solutions of the Equation (1) of the form

$$\psi(x,t)=\exp\left[\sigma(x,t)-\frac{i\dot{u}(t)}{8u(t)}x^2+\frac{1}{2}\left(k+\frac{1}{2}\right)\log u(t)-4i\lambda\int_0^t u(t)\mathrm{d}s\right]\phi\left(\sqrt{u(t)}x\right), \qquad (5)$$

where the function $\sigma(x,t)$ is in terms of an arbitrary function $u(t)$,

$$\sigma(x,t)=-\frac{u^2(t)}{4}x^4-\frac{\beta u(t)}{2}x^2+k\log x. \qquad (6)$$

In this paper, we will construct time-dependent *Schrödinger* equation for any potential. It means that we will find algebraic solutions namely $\psi(x,t)$ of that equation and one can build a time-dependent potential from any non time-dependent one. Note here that the static potential considered can be either quasi-exactly solvable (QES) or simply exactly solvable [2]-[4]. It is understood that we will generalize the formalism considered in Ref. [1] where the authors have constructed a time-dependent *Schrödinger* equation for only one family of quasi-exactly solvable sextic anharmonic oscillator potentials.

2. Construction of a Time-Dependent Schrödinger Equation

The main results are summarized by the following proposition:

2.1. Proposition

Let $V(y)$ be a potential and $\phi(y)$ be a solution of the eigenvalue equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2}+V(y)\right]\phi(y)=\lambda\phi(y) \qquad (7)$$

with eigenvalue λ. Let $\omega(t)$ be a positive (and derivable) function of t. Then, the solution of the *Schrödinger* equation

$$i\frac{\partial}{\partial t}\psi(x,t)=\left[-\frac{\partial^2}{\partial x^2}+V(x,t)\right]\psi(x,t) \qquad (8)$$

with time-dependent potential

$$V(x,t)=\omega^2(t)V(\omega(t)x)+\frac{x^2}{4}\left(\dot{\Omega}-\Omega^2\right),\quad \Omega\equiv\frac{\dot{\omega}}{\omega} \qquad (9)$$

is given by

$$\psi(x,t)=\sqrt{\omega}(t)\exp\left[-\int i\lambda\omega^2(t)\mathrm{d}t-\frac{i\dot{\omega}(t)}{4\omega(t)}x^2\right]\phi(\omega(t)x). \qquad (10)$$

Proof of the Proposition

We will discuss here an original method to construct time-dependent Hamiltonians which possess algebraic eigenvectors. Let us consider the Schrödinger equation,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2}+V(y)\right]\phi(y)=\lambda\phi(y), \qquad (11)$$

with $\phi(y)$ is an eigenfunction with eigenvalue λ of the Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V(y). \tag{12}$$

Note here that this Hamiltonian H (or the potential $V(y)$) doesn't depend on time t explicitly, it means that t doesn't enter neither in the eigenvalue λ, nor in the eigenfunction $\phi(y)$. Let us pose

$$\begin{gathered} y = \omega(t)x, \\ -\frac{\mathrm{d}^2}{\mathrm{d}y^2} = -\frac{1}{\omega^2(t)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}. \end{gathered} \tag{13}$$

As a consequence, the spectral Equation (11) is written as

$$\left[-\frac{\partial^2}{\partial x^2} + \omega^2(t)V(\omega(t)x)\right]\phi(\omega(t)x) = \lambda\omega^2(t)\phi(\omega(t)x). \tag{14}$$

Let us pose $\psi(x,t) = R(t,x)\phi(\omega(t)x)$ and extend the effective potential of the above equation noted $\omega^2(t)V(\omega(t)x)$ by adding a new term $\Delta(x,t)$ and consider a full Schrödinger equation of the form

$$\left[-\frac{\partial^2}{\partial x^2} + \omega^2(t)V(\omega(t)x) + \Delta(x,t)\right]R(t,x)\phi(\omega(t)x) = i\partial_t\left(R(t,x)\phi(\omega(t)x)\right). \tag{15}$$

The next step is to determine the unknown function $R(t,x)$ so that one can deduce the time-dependent algebraic solutions $\psi(x,t)$ of the Equation (15) and relate it to (14). Obviously, the above Equation (15) can be developed as follows

$$-\frac{\partial^2 R}{\partial x^2}\phi - 2\frac{\partial R\partial\phi}{\partial x\partial x} - R\frac{\partial^2\phi}{\partial x^2} + \omega^2(t)V(\omega(t)x)R\phi + \Delta(x,t)R\phi = i\frac{\partial R}{\partial t}\phi + iR\frac{\partial\phi}{\partial t}, \tag{16}$$

which can be rewritten

$$R\left[-\frac{\partial^2}{\partial x} + \omega^2(t)V(\omega(t)x)\right]\phi + \Delta(x,t)R\phi - \frac{\partial^2 R}{\partial x^2}\phi - 2\frac{\partial R\partial\phi}{\partial x\partial x} = i\frac{\partial R}{\partial t}\phi + iR\frac{\partial\phi}{\partial t}. \tag{17}$$

Manifestly, this equation can be written in terms of ϕ (*i.e.* the first derivative terms of ϕ must be omitted (must vanish)) only if the following condition is imposed

$$\begin{aligned} &-2\frac{\partial R}{\partial x}\frac{\partial\phi(\omega(t)x)}{\partial x} = iR\frac{\partial\phi'(\omega(t)x)}{\partial t} \\ &\Rightarrow -2\frac{\partial R}{\partial x}\omega\phi' = iRx\dot{\omega}\phi' \\ &\Rightarrow -2\frac{\partial R}{\partial x}\omega = iRx\dot{\omega} \\ &\Rightarrow R(x,t) = \hat{R}(t)\exp\left(-\frac{i}{4}x^2\frac{\dot{\omega}}{\omega}\right) \end{aligned} \tag{18}$$

with this expression of the function $R(t,x)$, the Equation (17) takes the following form

$$R\left[-\frac{\partial^2}{\partial x^2} + \omega^2(t)V(\omega(t)x)\right]\phi + \Delta(x,t)R\phi - \frac{\partial^2 R}{\partial x^2}\phi = i\frac{\partial R}{\partial t}\phi. \tag{19}$$

Replacing the expression $\left[-\frac{\partial^2}{\partial x^2} + \omega^2(t)V(\omega(t)x)\right]\phi$ by its equivalent one in this above equation, *i.e.* $\lambda\omega^2\phi$ as it is given in (14), one can write

$$R\lambda\omega^2\phi(\omega(t)x) + R\Delta(t,x)\phi(\omega(t)x) - \frac{\partial^2 R}{\partial x^2}\phi(\omega(t)x) = i\frac{\partial R}{\partial t}\phi(\omega(t)x), \tag{20}$$

which can be rewritten

$$\lambda\omega^2 R + \Delta(t,x) R - \frac{\partial^2 R}{\partial x^2} = i\frac{\partial R}{\partial t}. \tag{21}$$

From this equation, the added term $\Delta(t,x)$ to the initial potential in (15) is easily expressed as

$$\Delta(t,x) = \frac{i\frac{\partial R}{\partial t} + \frac{\partial^2 R}{\partial x^2}}{R(x,t)} - \lambda w^2. \tag{22}$$

Replacing $R(t,x)$ in this equation by expression (18) and after some algebraic manipulations, one can write

$$\begin{aligned}\Delta(t,x) &= \frac{i\dot{\hat{R}}(t) + i\hat{R}(t)\left(-\frac{i}{4}x^2\dot{\Omega}\right) + \hat{R}(t)\left(-\frac{1}{4}x^2\Omega^2 - i\frac{\Omega}{2}\right)}{\hat{R}(t)} - \lambda\omega^2 \\ &= i\frac{\dot{\hat{R}}(t)}{\hat{R}(t)} + \frac{x^2}{4}\dot{\Omega} - \frac{x^2}{4}\Omega^2 - i\frac{\Omega}{2} - \lambda\omega^2,\end{aligned} \tag{23}$$

where $\Omega \equiv \frac{\dot{\omega}}{\omega}$.

One can easily remark that $\Delta(t,x)$ is real and non-dependent on the eigenvalue λ only if it is expressed as

$$\Delta(x,t) = \frac{x^2}{4}\dot{\Omega} - \frac{x^2}{4}\Omega^2. \tag{24}$$

This is possible due to the following condition

$$i\frac{\dot{\hat{R}}(t)}{\hat{R}(t)} - i\frac{\Omega}{2} - \lambda\omega^2 = 0. \tag{25}$$

Solving the above differential equation and after some algebraic manipulations, one can easily obtain the expression of the function $\hat{R}(t)$

$$\hat{R}(t) = \sqrt{\omega}(t)\exp\left(-\int i\lambda\omega^2(t)\mathrm{d}t\right). \tag{26}$$

With this expression of the function $\hat{R}(t)$, the algebraic solutions of the time-dependent Schrödinger equation

$$\left[-\frac{\partial^2}{\partial x^2} + w^2(t)V(w(t)x) + \Delta(x,t)\right]\psi(x,t) = i\partial_t\psi(x,t), \tag{27}$$

with the time-dependent potential

$$V(x,t) = w^2(t)V(w(t)x) + \frac{x^2}{4}\left(\dot{\Omega} - \Omega^2\right) \tag{28}$$

are determined as

$$\begin{aligned}\psi(x,t) &= R(t,x)\phi(\omega(t)x) = \hat{R}(t)\exp\left(-\frac{i}{4}x^2\frac{\dot{\omega}(t)}{\omega(t)}\right)\phi(\omega(t)x) \\ &= \sqrt{\omega}(t)\exp\left[-\int i\lambda\omega^2(t)\mathrm{d}t - \frac{i\dot{\omega}(t)}{4\omega(t)}x^2\right]\phi(\omega(t)x),\end{aligned} \tag{29}$$

where $\omega(t)$ is an arbitrary positive function of t and $\phi(y)$ is the eigenvector of the equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V(y)\right]\phi(y) = \lambda\phi(y). \tag{30}$$

It means that one has constructed a time-dependent potential from the potential $V(y)$ which is non time-dependent. This is the generalization of the particular case of potentials considered in Ref. [1]. This is a particular case of ours because one can replace the original potential (*i.e.* the potential which is non time-dependent) in Equation (28) by any one which leads to a time-dependent potential associated to the above solutions $\psi(x,t)$ as it is given by the Equation (29). These solutions are expressed in terms of the eigenvalues λ of the *Schrödinger* equation. The values of λ depend on a potential considered, *i.e.* when the potential is quasi-exactly solvable, only a part of the eigenvalues is found algebraically whereas when the potential considered is exactly solvable, all eigenvalues λ are calculated explicitly. So, we have constructed a generalized formula which helps to find time-dependent potentials, it means that one can deduce for a non time-dependent potential its associated time-dependent one. In the next step, we will use this method established previously, *i.e.* we will manipulate simply the Equation (28) and Equation (29) respectively to construct the time-dependent Lamé potential and the algebraic solutions of *Schrödinger* equation. We will also apply the above method to the known QES matrix polynomial operator [5] [6] and interesting remarks will be pointed out.

2.2. Example 1: Construction of Time-Dependent Lamé Potential

In this section, along the same lines of the above method, *i.e.* simply from the Equation (28), we will transform the non time-dependent potential associated to the Lamé equation into the time-dependent one. The Lamé equation is quasi-exactly solvable and the original form is as follows [7] [8]

$$-\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2}+k^2N(N+1)sn^2(y,k)\phi(y)=\lambda\phi(y), \tag{31}$$

where the Lamé potential is

$$V(y,k)=k^2N(N+1)sn^2(y,k),\quad N=0,1,2,\cdots. \tag{32}$$

λ is the eigenvalue of the Lamé Hamiltonian and $sn(y,k)$ is the Jacobi elliptic function with modulus $k(0\le k\le 1)$. This function is periodic (*i.e.* the Lamé potential is also periodic) with period $4K(k)$ which denotes the complete elliptic integral of the first type, *i.e.*

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2 z}}\mathrm{d}z. \tag{33}$$

Replacing the potential $V(\omega(t)x)$ in the Equation (28) by the above Lamé potential (32), we find the following time-dependent Lamé potential

$$V(x,t)=\omega^2(t)k^2N(N+1)sn^2(\omega(t)x,k)+\frac{x^2}{4}(\dot{\Omega}-\Omega^2). \tag{34}$$

It is easily observed that this last term in x^2 of (34) isn't periodic so that it spoils the periodicity of the above time-dependent Lamé potential. The above time-dependent Lamé potential (34) can become periodic only if the following condition is satisfied

$$\Delta=0,$$

$$\frac{x^2}{4}(\dot{\Omega}-\Omega^2)=0,$$

$$\dot{\Omega}=\Omega^2,$$

$$\frac{\mathrm{d}\Omega}{\Omega^2}=\mathrm{d}t,$$

$$\Omega=\frac{1}{t_0-t},$$

$$\frac{\dot{\omega}}{\omega}=\frac{1}{t_0-t},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln\omega)=\frac{1}{t_0-t},$$

$$\int \mathrm{d}(\ln\omega) = \int \frac{1}{t_0 - t}\mathrm{d}t,$$

$$\omega(t) = \frac{c}{t_0 - t}, \tag{35}$$

where c is a real constant.

From the expression of $\omega(t)$ (*i.e.* (35), the Lamé potential (34) can be now expressed in time t as follows

$$V(x,t) = \frac{c^2}{(t_0 - t)^2}k^2 N(N+1) sn^2(\omega(t)x, k). \tag{36}$$

From the above expressions (35) and (36), the time-dependent *Schrödinger* Equation (1) is of the following form

$$\left[-\frac{\partial^2}{\partial x^2} + \frac{c^2}{(t_0 - t)^2}k^2 N(N+1) sn^2(\omega(t)x, k)\right]\psi(x,t) = i\partial_t\psi(x,t). \tag{37}$$

Referring to the Equation (29) and Equation (35), the algebraic solutions of this *Schrödinger* equation are obtained

$$\begin{aligned}\psi(x,t) &= R(x,t)\phi(\omega(t)x)\\ \psi(x,t) &= \sqrt{\frac{c}{t-t}}\exp\left[i\lambda\frac{c^2}{(t_0 - t)^2}\mathrm{d}t - \frac{ix^2}{4(t_0 - t)}\right]\phi(\omega(t)x).\end{aligned} \tag{38}$$

Note that one can deduce from a non time-dependent potential (for which the eigenvalues λ exist) its corresponding time-dependent one by using the general formula established in Equation (28) while the algebraic solutions of the *Schrödinger* equation are found from the Equation (29).

2.3 Example 2: Extension to Matrix Time-Dependent *Schrödinger* Equation

The goal of this section is to construct a matrix time-dependent *Schrödinger* equation by the above method used to find the time-dependent potential of the non coupled Lamé equation. Let us consider the following matrix Hamiltonian [5] [6]

$$H(y) = -\frac{\mathrm{d}^2}{\mathrm{d}y^2}I_2 + M_6(y), \tag{39}$$

where the potential $M_6(y)$ is 2 × 2 Hermitian matrix of the form

$$M_6(y) = \left[4p_2^2 y^6 + 8p_1 p_2 y^4 + \left(4p_1^2 - 8mp_2 + 2(1-2\varepsilon)p_2\right)y^2\right]I_2 + \left(8p_2 y^2 + 4p_1\right)\sigma_3 - 8mp_2 k_0\sigma_1, \tag{40}$$

where σ_1, σ_3 are the Pauli matrices, I_2 is the matrix identity, p_1, p_2, k_0 are free real parameters and m is an integer. $H(y)$ can be written in the matrix form as follows

$$H(y) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{41}$$

where

$$\begin{aligned}H_{11}(y) &= -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 4p_2^2 y^6 + 8p_1 p_2 y^4 + \left[4p_1^2 - 8mp_2 + 2(1-2\varepsilon)p_2\right]y^2 + 8p_2 y^2 + 4p_1,\\ H_{12}(y) &= -8mp_2 k_0,\\ H_{21}(y) &= -8mp_2 k_0,\\ H_{22}(y) &= -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 4p_2^2 y^6 + 8p_1 p_2 y^4 + \left[4p_1^2 - 8mp_2 + 2(1-2\varepsilon)p_2\right]y^2 - 8p_2 y^2 - 4p_1.\end{aligned} \tag{42}$$

In this case, the usual non time-dependent eigenvalue *Schrödinger* equation is of the form

$$H(y)\begin{pmatrix}\phi_1(y)\\ \phi_2(y)\end{pmatrix}=\lambda\begin{pmatrix}\phi_1(y)\\ \phi_2(y)\end{pmatrix}, \tag{43}$$

where

$$\phi(y)=\begin{pmatrix}\phi_1(y)\\ \phi_2(y)\end{pmatrix}, \tag{44}$$

with $\phi(y)$ and λ are respectively the eigenfunction and the eigenvalue of the matrix Hamiltonian $H(y)$. Referring to the original method established in the section 2, one can assume

$$\begin{gathered} y=\omega(t)x, \\ -\frac{\mathrm{d}^2}{\mathrm{d}y^2}=-\frac{1}{w^2(t)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}. \end{gathered} \tag{45}$$

From this change of variable, the Equation (43) takes the following form

$$\begin{gathered} \left[-\frac{\partial^2}{\partial x^2}+\omega^2(t)M_6(\omega(t)x)\right]\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}=\lambda\omega^2(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}, \\ H(\omega(t)x)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}=\lambda\omega^2(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}, \\ \begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}\begin{pmatrix}\phi(\omega(t)x)\\ \phi(\omega(t)x)\end{pmatrix}=\lambda\omega^2(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}, \end{gathered} \tag{46}$$

where

$$\begin{aligned} H_{11}(\omega(t)x)=&-\frac{\partial^2}{\partial x^2}+4p_2^2\omega^8(t)x^6+8p_1p_2\omega^6(t)x^4+\left(4p_1^2-8mp_2+2(1-2\varepsilon)p_2\right)\omega^4(t)x^2\\ &+8p_2\omega^4(t)x^2+4p_1\omega^2(t),\\ H_{12}(\omega(t)x)=&-8mp_2\omega^2(t)k_0,\\ H_{21}(\omega(t)x)=&-8mp_2\omega^2(t)k_0,\\ H_{22}(\omega(t)x)=&-\frac{\partial^2}{\partial x^2}+4p_2^2\omega^8(t)x^6+8p_1p_2\omega^6(t)x^4+\left(4p_1^2-8mp_2+2(1-2\varepsilon)p_2\right)\omega^4(t)x^2\\ &-8p_2\omega^4(t)x^2-4p_1\omega^2(t). \end{aligned} \tag{47}$$

After the change of function as

$$\psi(t,x)=\begin{pmatrix}\psi_1(\omega(t)x)\\ \psi_2(\omega(t)x)\end{pmatrix}=R(t,x)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}, \tag{48}$$

one can write the matrix time-dependent *Schrödinger* equation such that the initial potential acquires a supplementary term $\Delta(t,x)$ as it was done in the method established previously in the Equation (15)

$$\left[-\frac{\partial^2}{\partial x^2}+\omega^2(t)M_6(\omega(t)x)+\Delta(t,x)\right]\begin{pmatrix}\psi_1(\omega(t)x)\\ \psi_2(\omega(t)x)\end{pmatrix}=i\partial_t\begin{pmatrix}\psi_1(\omega(t)x)\\ \psi_2(\omega(t)x)\end{pmatrix}, \tag{49}$$

which leads to

$$\left[-\frac{\partial^2}{\partial x^2}+\omega^2(t)M_6(\omega(t)x)+\Delta(t,x)\right]R(t,x)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}=i\partial_tR(t,x)\begin{pmatrix}\phi_1(\omega(t)x)\\ \phi_2(\omega(t)x)\end{pmatrix}. \tag{50}$$

In the next step, we will calculate the function $R(t,x)$ so that the algebraic solutions $\psi(t,x)$ of the time-dependent *Schrödinger* equation are deduced. From the above Equation (50), the following system is obtained

$$\begin{aligned}
&R\Big[-\frac{\partial^2}{\partial x^2}+4p_2^2\omega^8(t)x^6+8p_1p_2\omega^6(t)x^4+\left(4p_1^2-8mp_2+2(1-2\varepsilon)p_2\right)\omega^4(t)x^2\\
&+8p_2\omega^4(t)x^2+4p_1\omega^2(t)+\Delta(x,t)\Big]\phi_1(\omega x)-\frac{\partial R^2}{\partial x^2}\phi_1-2\frac{\partial R}{\partial x}\frac{\partial\phi_1}{\partial x}-8mp_2k_0\omega^2(t)R\phi_2(\omega x)\\
&=i\frac{\partial R}{\partial t}\phi_1+iR\frac{\partial\phi_1}{\partial t},\\
&R\Big[-\frac{\partial^2}{\partial x^2}+4p_2^2\omega^8(t)x^6+8p_1p_2\omega^6(t)x^4+\left(4p_1^2-8mp_2+2(1-2\varepsilon)p_2\right)\omega^4(t)x^2\\
&-8p_2\omega^4(t)x^2-4p_1\omega^2(t)+\Delta(x,t)\Big]\phi_2(\omega x)-\frac{\partial R^2}{\partial x^2}\phi_2-2\frac{\partial R}{\partial x}\frac{\partial\phi_2}{\partial x}-8mp_2k_0\omega^2(t)R\phi_1(\omega x)\\
&=i\frac{\partial R}{\partial t}\phi_2+iR\frac{\partial\phi_2}{\partial t}.
\end{aligned}\tag{51}$$

Obviously, the two equations of the above system (51) can be linear respectively in ϕ_1 and ϕ_2 (*i.e.* the first derivatives of ϕ_1 and ϕ_2 are omitted) only if the following system is satisfied

$$\begin{cases}-2\dfrac{\partial R}{\partial x}\dfrac{\partial\phi_1}{\partial x}=iR\dfrac{\partial\phi_1}{\partial t},\\[2mm] -2\dfrac{\partial R}{\partial x}\dfrac{\partial\phi_2}{\partial x}=R\dfrac{\partial\phi_2}{\partial t}.\end{cases}\tag{52}$$

One can solve the first equation (or the second equation) in ϕ_1 (or in ϕ_2) of this Equation (52) in order to find the expression of $R(t,x)$

$$R(t,x)=R(t)=\hat{R}(t)\exp\left(-\frac{i\dot{\omega}}{\omega}x^2\right).\tag{53}$$

From this expression of $R(t,x)$, as a consequence, the Equation (50) is written as follows

$$\begin{aligned}
&\left[-\frac{\partial^2}{\partial x^2}+\omega^2(t)M_6(\omega x)+\Delta(x,t)\right]\hat{R}(t)\exp\left(-\frac{i}{4}\frac{\dot{\omega}}{\omega}x^2\right)\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix}\\
&=i\partial_t\left[\hat{R}(t)\exp\left(-\frac{i}{4}\frac{\dot{\omega}}{\omega}x^2\right)\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix}\right].
\end{aligned}\tag{54}$$

In the next, the idea is to find the unknown function $\hat{R}(t,x)$, for this, one has to consider the derivative with respect to t in the second expression of the above equation and after some algebraic manipulations, the Equation (54) is written as fallows

$$\begin{aligned}
&\left[-\frac{\partial^2}{\partial x^2}+\omega^2(t)M_6(\omega x)+\Delta(x,t)+i\frac{\Omega}{2}+\frac{x^2}{4}\Omega^2\right]\hat{R}(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix}\\
&=\frac{x^2}{4}\dot{\Omega}R(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix}+i\dot{\hat{R}}(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix},
\end{aligned}\tag{55}$$

where $\dot{\Omega}\equiv\frac{\partial}{\partial t}\Omega,\Omega\equiv\frac{\dot{\omega}}{\omega}$ and $\dot{\hat{R}}(t)\equiv\frac{\partial}{\partial t}\hat{R}(t)$.

From the Equation (46), this equality can be considered

$$\left[-\frac{\partial^2}{\partial x^2}+\omega^2(t)M_6(\omega(t)x)\right]\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix}=\lambda\omega^2(t)\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix} \tag{56}$$

in the above Equation (55) and accordingly one can write

$$\Delta(x,t)=i\frac{\dot{\hat{R}}(t)}{R(t)}+\frac{x^2}{4}\dot{\Omega}-\frac{x^2}{4}\Omega^2-i\frac{\Omega}{2}-\lambda\omega^2. \tag{57}$$

As it has shown in the above method, this expression of $\Delta(x,t)$ leads to the Equation (24), Equation (25) and Equation (26).

Finally, from the expression of $\hat{R}(t)$ (26), one can deduce the algebraic solutions of the matrix time-dependent Schrödinger equation as follows

$$\psi(x,t)=\sqrt{\omega}\exp\left[-\int i\lambda\omega^2\mathrm{d}t-\frac{i}{4}x^2\frac{\dot{\omega}}{\omega}\right]\begin{pmatrix}\phi_1(\omega(t)x)\\ \varphi_2(\omega(t))\end{pmatrix}. \tag{58}$$

3. Conclusion

In this paper, referring to sextic anharmonic potentials considered in Ref. [1], we have established a generalized method which helps to construct time-dependent potential for any non time-dependent one.

Indeed, we have applied this method to construct the time-dependent potential of Lamé equation. Along the same lines of the method, we have constructed a time-dependent potential associated to the matrix polynomial Hamiltonian which was also studied in [5] [6] and interesting remarks have been pointed out.

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