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The Modified Heinz's Inequality

Takashi Yoshino
Mathematical Institute, Tohoku University, Sendai, Japan
Email: yoshino@math.tohoku.ac.jp

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ABSTRACT

In my paper [1], we aimed to determine the best possible range of γ such that the modified Heinz's inequality $(A^\gamma A^\alpha A^\gamma)^\beta \geq (A^\gamma B^\alpha A^\gamma)^\beta$ holds for any bounded linear operators A and B on a Hilbert space $\mathcal{H}$ such as $A \geq B \geq \tau I$ (some $\tau > 0$) and for any given α and β such as $\alpha > 0$ and $\beta > 0$. But the counter-examples prepared in [1] and also in [2] were not sufficient and, in this paper, we shall constitute the sufficient counter-examples which will satisfy all the lacking parts.

Keywords: Heinz's Inequality

By the same way as in the proof of Theorem ([1]), we have the following.

Lemma For any a, b, x, δ and ϵ such as $a > 1 > b > 0,\ 0 < x < a,\ 0 < \epsilon$ and $0 \leq \delta = O(\epsilon) < a - b$ where $O(\varepsilon)$ is a function of ϵ such that $\lim_{\epsilon \to 0} \frac{O(\epsilon)}{\epsilon}$ is bounded, let

$$A = \begin{pmatrix} a & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & b+\varepsilon+\delta \end{pmatrix} \text{ and } B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix}.$$

If $\left(A^\gamma A^\alpha A^\gamma\right)^\beta \geq \left(A^\gamma B^\alpha A^\gamma\right)^\beta$ for any real numbers α, β and γ such as $\alpha > 0$ and $\beta > 0$, then we have the following inequality (1).

Theorem ([1]) The region of γ such that the operator inequality

$$\left(A^\gamma A^\alpha A^\gamma\right)^\beta \geq \left(A^\gamma B^\alpha A^\gamma\right)^\beta$$

holds for any bounded linear operators A and B on a Hilbert space $\mathcal{H}$ such as $A \geq B \geq \tau I$ (some $\tau > 0$) and for any given α and β such as $\alpha > 0$ and $\beta > 0$ is as follows:

1) $0 < \alpha \leq 1, 0 < \beta \leq 1,\ -\infty < \gamma < +\infty$

2) $0 < \alpha \leq 1, 1 < \beta \leq 2$

$$\max\left\{-\frac{1}{2}, \frac{-\alpha\beta}{2(\beta-1)}\right\} \leq \gamma \leq \min\left\{0, \frac{1-\alpha\beta}{2(\beta-1)}\right\}$$

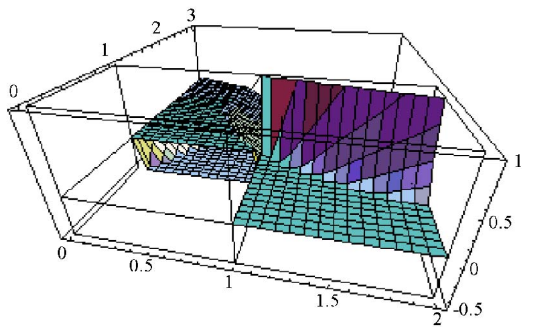
3) $0 < \alpha \leq 1, 2 < \beta \leq \frac{1}{\alpha},\ \ \gamma = 0$

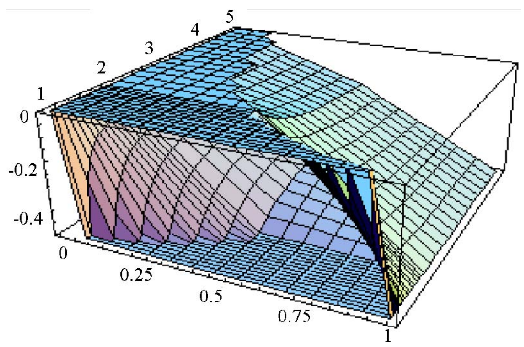
4) $0 < \alpha \leq 1, 2 < \beta$

$$\max\left\{\frac{2\alpha-1-\alpha\beta}{2(\beta-1)}, \frac{-\alpha\beta}{2(\beta-1)}\right\}$$
$$\leq \gamma \leq \min\left\{\frac{2\alpha-\alpha\beta}{2(\beta-1)}, \frac{1-\alpha\beta}{2(\beta-1)}\right\}$$

and 5) $1 < \alpha, 0 < \beta < 1,\ \max\left\{0, \frac{\alpha\beta-1}{2(1-\beta)}\right\} \leq \gamma$

$$\begin{aligned} &\frac{\beta\left(a^{\alpha\beta} - x^{\alpha\beta}\right)b^{(\alpha+2\gamma)\beta-2\gamma}}{(x^\alpha - b^\alpha)^2}\left\{\frac{\left(a^{2\gamma}-b^{2\gamma}\right)\left(x^\alpha - b^\alpha\right)}{a^{2\gamma}x^\alpha - b^{\alpha+2\gamma}} - \frac{\alpha(a-b)}{b}\lim_{\epsilon\to 0}\frac{\delta}{\epsilon}\right\} \\ &\leq \frac{a^{2\gamma(1-\beta)}}{\left(b^{\alpha+2\gamma} - a^{2\gamma}x^\alpha\right)^2}\left\{a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta}x^{\alpha\beta}\right\}. \end{aligned} \tag{1}$$





Proof Since the sufficiency of our range for the modified Heinz's inequality is already proved in [1], we have only to constitute counter-examples of A and B in the outside of our ranges.

For any a,b,x,δ and ϵ such as in Lemma, let

$$A=\begin{pmatrix} a & \sqrt{\epsilon(a-b-\delta)} \\ \sqrt{\epsilon(a-b-\delta)} & b+\epsilon+\delta \end{pmatrix} \text{ and } B=\begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix}.$$

Let $x=b\left\{\frac{(a-1)+a}{(a-1)+b}\right\}=by,\ 0<\epsilon<\frac{(a-b)(a-1)}{b}$,

and $\delta=\frac{b\epsilon}{a-1}$. Then we have $A\geq B\geq bI$ because $0<\delta<a-b,\ x=by>b$,

$$a-x=\frac{a(a-1)+ab-b(a-1)-ba}{(a-1)+b}=\frac{(a-b)(a-1)}{(a-1)+b}>0$$

and because

$$\begin{aligned}&(a-x)(\epsilon+\delta)-\epsilon(a-b-\delta)\\&=\epsilon(b-x)+(a-x)\delta+\epsilon\delta\\&=\frac{\epsilon}{a-1}\{(a-1)(b-x)+(a-x)b\}+\epsilon\delta\\&=\frac{\epsilon}{a-1}\left[b\{(a-1)+a\}-x\{(a-1)+b\}\right]+\epsilon\delta\\&=\frac{b\epsilon}{a-1}\left[\{(a-1)+a\}-y\{(a-1)+b\}\right]+\epsilon\delta=\epsilon\delta>0.\end{aligned}$$

Also if $\left(A^{\gamma}A^{\alpha}A^{\gamma}\right)^{\beta}\geq\left(A^{\gamma}B^{\alpha}A^{\gamma}\right)^{\beta}$, then, by multiplying $b^{2\alpha-\alpha\beta}\left(a^{\alpha\beta}-x^{\alpha\beta}\right)^{-1}$ to the both sides of the inequality (1) in Lemma, we have

$$\frac{\beta b^{2\gamma(\beta-1)}}{\left(y^{\alpha}-1\right)^{2}}\left\{\frac{\left(a^{2\gamma}-b^{2\gamma}\right)\left(y^{\alpha}-1\right)}{a^{2\gamma}y^{\alpha}-b^{2\gamma}}-\frac{\alpha(a-b)}{(a-1)}\right\}$$

$$\leq\frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left(b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\right)}{\left(a^{\alpha\beta}-b^{\alpha\beta}y^{\alpha\beta}\right)\left(b^{2\gamma}-a^{2\gamma}y^{\alpha}\right)^{2}}\tag{2}$$

$$=\frac{\left\{a^{2\gamma(\beta-1)}-a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\right\}\left(a^{-2\gamma\beta}b^{2\gamma\beta}-y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(a^{-2\gamma}b^{2\gamma}-y^{\alpha}\right)^{2}}\tag{3}$$

$$=\frac{a^{2\gamma}\left\{1-a^{-(\alpha+2\gamma)\beta}b^{(\alpha+2\gamma)\beta}\right\}\left(b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(b^{2\gamma}-a^{2\gamma}y^{\alpha}\right)^{2}}\tag{4}$$

$$=\frac{a^{2\gamma(1-\beta)-\alpha\beta}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left(b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(b^{2\gamma}-a^{2\gamma}y^{\alpha}\right)^{2}}.\tag{5}$$

Let $x=1,\ 0<\epsilon<(a-b)(a-1)$ and $\delta=\frac{\epsilon}{a-1}$. Then we have $A\geq B\geq bI$ because $0<\delta<a-b,\ a>x=1>b$ and because

$$\begin{aligned}&(a-x)(\epsilon+\delta)-\epsilon(a-b-\delta)=\epsilon(b-x)+(a-x)\delta+\epsilon\delta\\&=\epsilon\{(b-1)+1\}+\epsilon\delta=b\epsilon+\epsilon\delta>0.\end{aligned}$$

Also if $\left(A^{\gamma}A^{\alpha}A^{\gamma}\right)^{\beta}\geq\left(A^{\gamma}B^{\alpha}A^{\gamma}\right)^{\beta}$, then, by multiplying $\left(a^{\alpha\beta}-x^{\alpha\beta}\right)^{-1}$ to the both sides of the inequality (1) in Lemma, we have the following inequality

$$\begin{aligned}&\frac{\beta b^{(\alpha+2\gamma)\beta-2\gamma}}{\left(1-b^{\alpha}\right)^{2}}\left\{\frac{\left(b^{2\gamma}-a^{2\gamma}\right)\left(1-b^{\alpha}\right)}{b^{\alpha+2\gamma}-a^{2\gamma}}-\frac{\alpha(a-b)}{b(a-1)}\right\}\\&\leq\frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}\right\}}{\left(a^{\alpha\beta}-1\right)\left(b^{\alpha+2\gamma}-a^{2\gamma}\right)^{2}}.\end{aligned}\tag{6}$$

And, by multiplying $b^{-2(\alpha+2\gamma)(\beta-1)}$ to the both sides of the above inequality, we have

$$\begin{aligned}&\frac{\beta b^{2\gamma(1-\beta)-\alpha\beta+2\alpha-1}}{\left(1-b^{\alpha}\right)^{2}}\left\{\frac{b^{1-\alpha}\left(1-b^{-2\gamma}a^{2\gamma}\right)\left(1-b^{\alpha}\right)}{1-b^{-(\alpha+2\gamma)}a^{2\gamma}}-\frac{\alpha(a-b)}{(a-1)}\right\}\\&\leq\frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta}b^{-(\alpha+2\gamma)\beta}-1\right\}\left\{1-b^{-(\alpha+2\gamma)\beta}a^{2\gamma\beta}\right\}}{\left(a^{\alpha\beta}-1\right)\left\{1-b^{-(\alpha+2\gamma)}a^{2\gamma}\right\}^{2}}.\end{aligned}\tag{7}$$

Let $x=b^{2}$ and $\delta=0$. Then we have $A\geq B\geq b^{2}I$

because $0 < x = b^2 < b < a$ and because $(a-x)\epsilon - \epsilon(a-b) = \epsilon(b-x) > 0$.

Also if $\left(A^{\gamma}A^{\alpha}A^{\gamma}\right)^{\beta} \ge \left(A^{\gamma}B^{\alpha}A^{\gamma}\right)^{\beta}$, then, by multiplying $b^{-2(\alpha+2\gamma)(\beta-1)}$ to the both sides of the inequality (1) in Lemma, we have the following inequality

$$\frac{\beta\left(a^{\alpha\beta}-b^{2\alpha\beta}\right)b^{2\gamma(1-\beta)-\alpha\beta+2\alpha}}{\left(b^{\alpha}-1\right)^{2}}\left\{\frac{\left(b^{\alpha}-1\right)\left(b^{-2\gamma}a^{2\gamma}-1\right)}{b^{\alpha-2\gamma}a^{2\gamma}-1}\right\} \le \frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta}b^{-(\alpha+2\gamma)\beta}-1\right\}\left\{1-b^{(\alpha-2\gamma)\beta}a^{2\gamma\beta}\right\}}{\left(1-b^{\alpha-2\gamma}a^{2\gamma}\right)^{2}}. \tag{8}$$

For $\beta > 2$, let $0 < b < \left(\frac{1}{2}\right)^{\beta-1}$. Then

$0 < b < b^{\frac{1}{\beta-1}} < \frac{1}{2}$ because $0 < \frac{1}{\beta-1} < 1$.

And let $a > \dfrac{1-b^{\frac{1}{\beta-1}}-b}{1-2b^{\frac{1}{\beta-1}}}(>1)$,

$x = b^{\frac{\beta}{\beta-1}}\left\{\frac{(a-1)+a}{(a-1)+b}\right\} = b^{\frac{\beta}{\beta-1}}y = b^{\frac{1}{\beta-1}}(by)(>0)$ and $\delta = 0$. Then we have $A \ge B \ge xI$ because

$$b-x = \frac{b(a-1)+b^2-b^{\frac{\beta}{\beta-1}}(a-1)-b^{\frac{\beta}{\beta-1}}a}{(a-1)+b} = \frac{b(a-1)\left(1-b^{\frac{1}{\beta-1}}\right)+b\left(b-b^{\frac{1}{\beta-1}}a\right)}{(a-1)+b}$$
$$= \frac{b}{(a-1)+b}\left\{(a-1)\left(1-b^{\frac{1}{\beta-1}}\right)+\left(b-b^{\frac{1}{\beta-1}}a\right)\right\} = \frac{b}{(a-1)+b}\left\{\left(1-2b^{\frac{1}{\beta-1}}\right)a-\left(1-b^{\frac{1}{\beta-1}}-b\right)\right\} > 0,$$

$$a-x = \frac{a(a-1)+ab-b^{\frac{\beta}{\beta-1}}(a-1)-b^{\frac{\beta}{\beta-1}}a}{(a-1)+b} = \frac{\left(a-b^{\frac{\beta}{\beta-1}}\right)(a-1)+ab\left(1-b^{\frac{1}{\beta-1}}\right)}{(a-1)+b} > 0$$

and because $(a-x)\epsilon - \epsilon(a-b) = \epsilon(b-x) > 0$.

Also if $\left(A^{\gamma}A^{\alpha}A^{\gamma}\right)^{\beta} \ge \left(A^{\gamma}B^{\alpha}A^{\gamma}\right)^{\beta}$, then, by Lemma, we have the following inequality

$$\frac{\beta\left\{a^{\alpha\beta}-b^{\frac{1}{\beta-1}\alpha\beta}(by)^{\alpha\beta}\right\}b^{(\alpha+2\gamma)\beta-2\gamma}}{\left\{b^{\frac{1}{\beta-1}\alpha}(by)^{\alpha}-b^{\alpha}\right\}^{2}}\left\{\frac{\left\{b^{\frac{1}{\beta-1}\alpha}(by)^{\alpha}-b^{\alpha}\right\}\left(a^{2\gamma}-b^{2\gamma}\right)}{a^{2\gamma}b^{\frac{1}{\beta-1}\alpha}(by)^{\alpha}-b^{\alpha+2\gamma}}\right\}$$
$$\le \frac{a^{2\gamma(1-\beta)}}{\left\{b^{\alpha+2\gamma}-a^{2\gamma}b^{\frac{1}{\beta-1}\alpha}(by)^{\alpha}\right\}^{2}}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}b^{\frac{1}{\beta-1}\alpha\beta}(by)^{\alpha\beta}\right\}.$$

Since

$$b^{\frac{1}{\beta-1}\alpha}(by)^{\alpha}-b^{\alpha} = b^{\frac{1}{\beta-1}\alpha}\left\{(by)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha}\right\},$$

$$b^{\alpha+2\gamma}-a^{2\gamma}b^{\frac{1}{\beta-1}\alpha}(by)^{\alpha} = b^{\frac{1}{\beta-1}\alpha}\left\{b^{\frac{\beta-2}{\beta-1}\alpha+2\gamma}-a^{2\gamma}(by)^{\alpha}\right\},$$

$$b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}b^{\frac{1}{\beta-1}\alpha\beta}(by)^{\alpha\beta} = b^{\frac{1}{\beta-1}\alpha\beta}\left\{b^{\left(\frac{\beta-2}{\beta-1}\alpha+2\gamma\right)\beta}-a^{2\gamma\beta}(by)^{\alpha\beta}\right\}$$

and since $(\alpha+2\gamma)\beta-\frac{1}{\beta-1}\alpha\beta = \left\{\frac{\beta-2}{\beta-1}\alpha+2\gamma\right\}\beta$, by multiplying $b^{\frac{\alpha(2-\beta)}{\beta-1}}$ to the both sides of the above inequality, we have, for $\beta > 2, 0 < b < \left(\frac{1}{2}\right)^{\beta-1}$ and

$a > \dfrac{1-b^{\frac{1}{\beta-1}}-b}{1-2b^{\frac{1}{\beta-1}}}(>1)$,

$$\frac{\beta\left\{a^{\alpha\beta}-b^{\frac{1}{\beta-1}\alpha\beta}\left(by\right)^{\alpha\beta}\right\}b^{\left(\frac{\beta-2}{\beta-1}\alpha+2\gamma\right)\beta}}{\left\{\left(by\right)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha}\right\}^{2}}\times\left\{\frac{\left\{\left(by\right)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha}\right\}\left(a^{2\gamma}b^{-2\gamma}-1\right)}{a^{2\gamma}\left(by\right)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha+2\gamma}}\right\}$$

$$\le\frac{a^{2\gamma(1-\beta)}}{\left\{b^{\frac{\beta-2}{\beta-1}\alpha+2\gamma}-a^{2\gamma}\left(by\right)^{\alpha}\right\}^{2}}\times\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\times\left\{b^{\left(\frac{\beta-2}{\beta-1}\alpha+2\gamma\right)\beta}-a^{2\gamma\beta}\left(by\right)^{\alpha\beta}\right\}. \qquad (9)$$

Case 1 Let $0<\alpha, 1<\beta, 0<\gamma$.

Then

$$\lim_{a\to\infty}\frac{\beta b^{2\gamma(\beta-1)}}{\left(y^{\alpha}-1\right)^{2}}\left\{\frac{\left(a^{2\gamma}-b^{2\gamma}\right)\left(y^{\alpha}-1\right)}{a^{2\gamma}y^{\alpha}-b^{2\gamma}}-\frac{\alpha\left(a-b\right)}{\left(a-1\right)}\right\}=\frac{\beta b^{2\gamma(\beta-1)}}{\left(2^{\alpha}-1\right)^{2}}\left\{\frac{2^{\alpha}-1}{2^{\alpha}}-\alpha\right\}$$

because $\lim_{a\to\infty}y=\lim_{a\to\infty}\frac{(a-1)+a}{(a-1)+b}=2$ and

$$\lim_{a\to\infty}\frac{\left\{a^{2\gamma(\beta-1)}-a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\right\}\left(a^{-2\gamma\beta}b^{2\gamma\beta}-y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(a^{-2\gamma}b^{2\gamma}-y^{\alpha}\right)^{2}}=-\infty.$$

This contradicts (3).

Case 2 Let $1<\alpha, 0<\beta, \gamma<0$.

Then

$$\lim_{a\to\infty}\frac{\beta b^{2\gamma(\beta-1)}}{\left(y^{\alpha}-1\right)^{2}}\left\{\frac{\left(a^{2\gamma}-b^{2\gamma}\right)\left(y^{\alpha}-1\right)}{a^{2\gamma}y^{\alpha}-b^{2\gamma}}-\frac{\alpha\left(a-b\right)}{\left(a-1\right)}\right\}=\frac{\beta b^{2\gamma(\beta-1)}}{\left(2^{\alpha}-1\right)^{2}}\left(2^{\alpha}-1-\alpha\right)$$

because $\lim_{a\to\infty}y=2$.

If $\alpha+2\gamma\ge0$, then we have

$$\lim_{a\to\infty}\frac{a^{2\gamma}\left\{1-a^{-(\alpha+2\gamma)\beta}b^{(\alpha+2\gamma)\beta}\right\}\left(b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(b^{2\gamma}-a^{2\gamma}y^{\alpha}\right)^{2}}=0$$

and, if $\alpha+2\gamma<0$, then we have also

$$\lim_{a\to\infty}\frac{a^{2\gamma(1-\beta)-\alpha\beta}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left(b^{2\gamma\beta}-a^{2\gamma\beta}y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(b^{2\gamma}-a^{2\gamma}y^{\alpha}\right)^{2}}=\begin{cases}0, & \left(2\gamma\left(1-\beta\right)-\alpha\beta<0\right)\\ -b^{-\alpha\beta}, & \left(2\gamma\left(1-\beta\right)-\alpha\beta=0\right)\le0\\ -\infty, & \left(2\gamma\left(1-\beta\right)-\alpha\beta>0\right)\end{cases}$$

and hence, by (4) and (5), we have $2^{\alpha}-1-\alpha\le0$ and this contradicts the fact that $2^{\alpha}-1-\alpha>0$ for all $\alpha>1$.

Case 3 Let $0<\alpha, \frac{1}{\alpha}<\beta, \gamma=0$.

Then

$$\lim_{a\to\infty}\frac{\beta b^{2\gamma(\beta-1)}}{\left(y^{\alpha}-1\right)^{2}}\left\{\frac{\left(a^{2\gamma}-b^{2\gamma}\right)\left(y^{\alpha}-1\right)}{a^{2\gamma}y^{\alpha}-b^{2\gamma}}-\frac{\alpha\left(a-b\right)}{\left(a-1\right)}\right\}=\frac{-\alpha\beta}{\left(2^{\alpha}-1\right)^{2}}$$

and

$$\lim_{a\to\infty}\frac{\left\{a^{2\gamma(\beta-1)}-a^{-(\alpha\beta+2\gamma)}b^{(\alpha+2\gamma)\beta}\right\}\left(a^{-2\gamma\beta}b^{2\gamma\beta}-y^{\alpha\beta}\right)}{\left(1-a^{-\alpha\beta}b^{\alpha\beta}y^{\alpha\beta}\right)\left(a^{-2\gamma}b^{2\gamma}-y^{\alpha}\right)^{2}}=\frac{1-2^{\alpha\beta}}{\left(1-2^{\alpha}\right)^{2}}$$

because $\lim_{a\to\infty}y=2$.

By (3), we have $2^{\alpha\beta}-1-\alpha\beta\le0$ and this contradicts the fact that $2^{\alpha\beta}-1-\alpha\beta>0$ for all $\alpha\beta>1$.

Case 4 Let $1<\alpha, 0<\beta<1, 0<\gamma<\max\left\{0,\frac{\alpha\beta-1}{2(1-\beta)}\right\}$

Then $0<\gamma<\frac{\alpha\beta-1}{2(1-\beta)}$ and $2\gamma\left(1-\beta\right)<\alpha\beta-1$ and hence $\left(\alpha+2\gamma\right)\beta-2\gamma-1>0$.

Therefore we have

$$\lim_{b\to0}\frac{\beta b^{(\alpha+2\gamma)\beta-2\gamma-1}}{\left(1-b^{\alpha}\right)^{2}}\left\{\frac{b\left(b^{2\gamma}-a^{2\gamma}\right)\left(1-b^{\alpha}\right)}{b^{\alpha+2\gamma}-a^{2\gamma}}-\frac{\alpha\left(a-b\right)}{a-1}\right\}=0$$

and

$$\lim_{b\to0}\frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta}-a^{2\gamma\beta}\right\}}{\left(a^{\alpha\beta}-1\right)\left(b^{\alpha+2\gamma}-a^{2\gamma}\right)^{2}}=-\frac{a^{(\alpha+2\gamma)\beta-2\gamma}}{a^{\alpha\beta}-1}<0.$$

This contradicts (6).

Case 5 Let $0 < \alpha \le 1, 1 < \beta$,

$\gamma < \max\left\{\frac{2\alpha - 1 - \alpha\beta}{2(\beta - 1)}, \frac{-\alpha\beta}{2(\beta - 1)}\right\}$

In this case

$$\begin{cases} 2\gamma(\beta - 1) + \alpha\beta - \alpha - \max\{\alpha - 1, -\alpha\} < 0 \\ \text{and} \quad (\alpha + 2\gamma)(\beta - 1) < \max\{\alpha - 1, -\alpha\} \le 0 \\ \text{and hence} \quad \alpha + 2\gamma < 0 \quad \text{because } \beta > 1. \end{cases}$$

Then, in the case where $0 < \alpha < \frac{1}{2}$, we have $\gamma < \frac{-\alpha\beta}{2(\beta - 1)}$ and hence $2\gamma(1 - \beta) - \alpha\beta > 0$. Therefore we have

$$\lim_{a\to\infty} \frac{\beta b^{(\alpha+2\gamma)\beta - 2\gamma}}{(1 - b^\alpha)^2}\left\{\frac{(b^{2\gamma} - a^{2\gamma})(1 - b^\alpha)}{b^{\alpha+2\gamma} - a^{2\gamma}} - \frac{\alpha(a - b)}{b(a - 1)}\right\}$$
$$= \frac{\beta b^{(\alpha+2\gamma)\beta - 2\gamma}}{(1 - b^\alpha)^2}\left\{\frac{1 - b^\alpha}{b^\alpha} - \frac{\alpha}{b}\right\}$$

and

$$\lim_{a\to\infty} \frac{a^{2\gamma(1-\beta)-\alpha\beta}\left\{a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta}\right\}}{(1 - a^{-\alpha\beta})(b^{\alpha+2\gamma} - a^{2\gamma})^2}$$
$$= -\infty.$$

This contradicts (6).

And, in the case where $\frac{1}{2} \le \alpha \le 1$, we have $\gamma < \frac{2\alpha - 1 - \alpha\beta}{2(\beta - 1)}$ and hence $2\gamma(1 - \beta) - \alpha\beta + 2\alpha - 1 > 0$.

Therefore we have

$$\lim_{b\to 0} \frac{\beta b^{2\gamma(1-\beta) - \alpha\beta + 2\alpha - 1}}{(1 - b^\alpha)^2}$$
$$\times\left\{\frac{b^{1-\alpha}(1 - b^{-2\gamma}a^{2\gamma})(1 - b^\alpha)}{1 - b^{-(\alpha+2\gamma)}a^{2\gamma}} - \frac{\alpha(a - b)}{(a - 1)}\right\} = 0$$

and

$$\lim_{b\to 0} \frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta}b^{-(\alpha+2\gamma)\beta} - 1\right\}\left\{1 - b^{-(\alpha+2\gamma)\beta}a^{2\gamma\beta}\right\}}{(a^{\alpha\beta} - 1)\left\{1 - b^{-(\alpha+2\gamma)}a^{2\gamma}\right\}^2}$$
$$= -\frac{a^{2\gamma(1-\beta)}}{a^{\alpha\beta} - 1} < 0.$$

This contradicts (7).

Case 6 Let $0 < \alpha \le 1, 1 < \beta \le 2, \gamma < -\frac{1}{2}$.

Then $\alpha + 2\gamma \le 1 + 2\gamma < 0$ and

$2\gamma(1 - \beta) - \alpha\beta + 2\alpha = 2\gamma(1 - \beta) + \alpha(2 - \beta) > 0$.

And, by (8), we have

$$0 = \lim_{b\to 0} \frac{\beta(a^{\alpha\beta} - b^{2\alpha\beta})b^{2\gamma(1-\beta) - \alpha\beta + 2\alpha}}{(b^\alpha - 1)^2}$$
$$\times\left\{\frac{(b^\alpha - 1)(a^{2\gamma}b^{-2\gamma} - 1)}{a^{2\gamma}b^{\alpha - 2\gamma} - 1}\right\}$$
$$\le \lim_{b\to 0} \frac{a^{2\gamma(1-\beta)}}{(1 - a^{2\gamma}b^{\alpha-2\gamma})^2}\left\{a^{(\alpha+2\gamma)\beta}b^{-(\alpha+2\gamma)\beta} - 1\right\}$$
$$\times\left\{1 - a^{2\gamma\beta}b^{(\alpha-2\gamma)\beta}\right\}$$
$$= -a^{2\gamma(1-\beta)} < 0$$

and this is a contradiction.

Case 7 Let $0 < \alpha \le 1, 1 < \beta, \frac{1 - \alpha\beta}{2(\beta - 1)} < \gamma < 0$.

Then $(\alpha + 2\gamma)\beta > 1 + 2\gamma > 1 + \frac{1 - \alpha\beta}{\beta - 1} = \frac{\beta(1 - \alpha)}{\beta - 1} \ge 0$

and $\alpha + 2\gamma > 0$ because $\beta > 1$.

Therefore we have

$$\lim_{b\to 0} \frac{\beta b^{(\alpha+2\gamma)\beta - 2\gamma - 1}}{(1 - b^\alpha)^2}\left\{\frac{b(b^{2\gamma} - a^{2\gamma})(1 - b^\alpha)}{b^{\alpha+2\gamma} - a^{2\gamma}} - \frac{\alpha(a - b)}{a - 1}\right\} = 0$$

and

$$\lim_{b\to 0} \frac{a^{2\gamma(1-\beta)}\left\{a^{(\alpha+2\gamma)\beta} - b^{(\alpha+2\gamma)\beta}\right\}\left\{b^{(\alpha+2\gamma)\beta} - a^{2\gamma\beta}\right\}}{(a^{\alpha\beta} - 1)(b^{\alpha+2\gamma} - a^{2\gamma})^2}$$
$$= -\frac{a^{(\alpha+2\gamma)\beta - 2\gamma}}{a^{\alpha\beta} - 1} < 0.$$

This contradicts (6).

Case 8 Let $0 < \alpha \le 1, 2 < \beta, \frac{2\alpha - \alpha\beta}{2(\beta - 1)} < \gamma < 0$.

Then $\alpha + 2\gamma > \frac{\beta - 2}{\beta - 1}\alpha + 2\gamma > 0$.

Since $\lim_{a\to 1}(by) = \lim_{a\to 1} b\left\{\frac{(a - 1) + a}{(a - 1) + b}\right\} = 1$ and since $\lim_{a\to 1} b = 0$ in the case where

$\beta > 2, 0 < b < \left(\frac{1}{2}\right)^{\beta - 1}, a > \frac{1 - b^{\frac{1}{\beta-1}} - b}{1 - 2b^{\frac{1}{\beta-1}}}(> 1)$ because

$$a - 1 > \frac{b^{\frac{1}{\beta-1}} - b}{1 - 2b^{\frac{1}{\beta-1}}} = \frac{b^{\frac{1}{\beta-1}}\left(1 - b^{\frac{\beta-2}{\beta-1}}\right)}{1 - 2b^{\frac{1}{\beta-1}}} > 0,$$

we have

$$\lim_{a\to 1}\frac{\beta\left\{a^{\alpha\beta}-b^{\frac{1}{\beta-1}\alpha\beta}\left(by\right)^{\alpha\beta}\right\}b^{\left(\frac{\beta-2}{\beta-1}\alpha+2\gamma\right)\beta}}{\left\{\left(by\right)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha}\right\}^{2}}$$

$$\times\left\{\frac{\left\{\left(by\right)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha}\right\}\left(a^{2\gamma}b^{-2\gamma}-1\right)}{a^{2\gamma}\left(by\right)^{\alpha}-b^{\frac{\beta-2}{\beta-1}\alpha+2\gamma}}\right\}=0$$

and

$$\lim_{a\to 1}\frac{a^{2\gamma(1-\beta)}}{\left\{b^{\frac{\beta-2}{\beta-1}\alpha+2\gamma}-a^{2\gamma}\left(by\right)^{\alpha}\right\}^{2}}\left\{a^{(\alpha+2\gamma)\beta}-b^{(\alpha+2\gamma)\beta}\right\}$$

$$\times\left\{b^{\left(\frac{\beta-2}{\beta-1}\alpha+2\gamma\right)\beta}-a^{2\gamma\beta}\left(by\right)^{\alpha\beta}\right\}=-1.$$

This contradicts (9).

Therefore we completed the proof of the best possibility of the ranges in our theorem.

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