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Fixed Point of a Countable Family of Uniformly Totally Quasi-ϕ-Asymptotically Nonexpansive Multi-Valued Mappings in Reflexive Banach Spaces with Applications

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ABSTRACT

The purpose of this article is to discuss a modified Halpern-type iteration algorithm for a countable family of uniformly totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings and establish some strong convergence theorems under certain conditions. We utilize the theorems to study a modified Halpern-type iterative algorithm for a system of equilibrium problems. The results improve and extend the corresponding results of Chang *et al.* (Applied Mathematics and Computation, 218, 6489-6497).

Keywords: Multi-Valued Mapping; Totally Quasi-ϕ-Asymptotically Nonexpansive; Countable Family of Uniformly Totally Quasi-ϕ-Asymptotically Nonexpansive Multi-Valued Mappings; Firmly Convergence

1. Introduction

Throughout this paper, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We denote by *N* and *R* the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space X. A mapping $T: D \to D$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty bounded closed subsets of D, respectively.

Let X be a real Banach space with dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}$, where $x \in X$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The Hausdorff metric on $CB(D)$ is defined by

$H(A_1, A_2) = \max\left\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\right\}$, for $A_1, A_2 \in CB(D)$, where $d(x, A_2) = \inf\{\|x - y\|, y \in A_2\}$. The multi-valued mapping $T: D \to CB(D)$ is called nonexpansive if $H(Tx, Ty) \le \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T: D \to CB(D)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$. In the sequel, denote $S(X) = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be strictly convex if $\left\|\frac{x+y}{2}\right\| \le 1$ for all $x, y \in S(X)$ and $x \ne y$. A Banach space is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset S(X)$ and $\lim_{n\to\infty} \frac{\|x_n + y_n\|}{2} = 0$. The norm of Banach space X is said to be Gâteaux differentiable if for each $x, y \in S(X)$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists. In this case, X is said to be smooth. The norm of Banach space X is said to be Fréchet differentiable, if for each $x \in S(X)$, the limit (1.1) is attained uniformly for $y \in S(x)$ and the norm is uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $x, y \in S(X)$. In this case, X is said to be uniformly smooth.

The following basic properties for Banach space *X* and for the normalized duality mapping J can be found in Cioranescu [1].

(1) $X\left(X^*, resp.\right)$ is uniformly convex if and only if

$X^*(X, resp.)$ is uniformly smooth.

(2) If X is smooth, then J is single-valued and norm-to-weak* continuous.

(3) If X is reflexive, then J is onto.

(4) If X is strictly convex, then $Jx \cap Jy \neq \Phi$ for all $x, y \in X$.

(5) If X has a Fréchet differentiable norm, then J is norm-to-norm continuous.

(6) If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X.

(7) Each uniformly convex Banach space X has the Kadec-Klee property, *i.e.*, for any sequence $\{x_n\} \subset X$, if $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x \in X$.

In 1953, Mann [2] introduced the following iterative sequence $\{x_n\}$,

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n,$$

where the initial guess $x_1 \in D$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly [3]. Some attempts to construct iteration method guaranteeing the strong convergence have been made. For example, Halpern [4] proposed the following so-called Halpern iteration,

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n,$$

where $u, x_1 \in D$ are arbitrary given and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Another approach was proposed by Nakajo and Takahashi [5]. They generated a sequence as follows,

$$\begin{cases} x_1 \in X \text{ is arbitrary;} \\ y_n = \alpha_n u + (1-\alpha_n) T x_n \\ C_n = \{z \in D : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in D : \langle x_n - z, x_1 - x_n \rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_1 \ (n = 1, 2, \cdots) \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H. It should be noted here that the iteration above works only in Hilbert space setting. To extend this iteration to a Banach space, the concept of relatively nonexpansive mappings and quasi-ϕ-nonexpansive mappings are introduced by Aoyama *et al.* [6], Chang *et al.* [7,8], Chidume *et al.* [9], Matsushita *et al.* [10-12], Qin *et al.* [13], Song *et al.* [14], Wang *et al.* [15] and others.

Inspired by the work of Matsushita and Takahashi, in this paper, we introduce modifying Halpern-Mann iterations sequence for finding a fixed point of a countable family of uniformly totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings in reflexive Banach spaces $T_i : D \to D (i = 1, 2, 3, \cdots)$ and some strong convergence theorems are proved. The results presented in the paper improve and extend the corresponding results in [7].

2. Preliminaries

In the sequel, we assume that X is a smooth, strictly convex, and reflexive Banach space and D is a nonempty closed convex subset of X. In the sequel, we always use $\phi : X \times X \to R^+$ to denote the Lyapunov bifunction defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \ x, y \in X. \tag{2.1}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \tag{2.2}$$

$$\begin{aligned} &\phi(y, x) = \phi(y, z) + \phi(z, x) \\ &+2\langle z - y, Jx - Jz \rangle, \quad x, y, z \in X, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\phi\left(x, J^{-1}(\alpha Jy + (1-\alpha) Jz)\right) \\ &\le \alpha\phi(x, y) + (1-\alpha)\phi(x, z) \end{aligned} \tag{2.4}$$

for all $\alpha \in [0,1]$ and $x, y, z \in X$.

Following Alber [16], the generalized projection $\Pi_D : X \to D$ is defined by

$$\Pi_D(x) = \arg y \in D \inf \phi(y, x), \forall x \in X.$$

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

Remark 2.1 (*see* [17]) *Let* Π_D *be the generalized projection from a smooth, reflexive and strictly convex Banach space* X *onto a nonempty closed convex subset* D *of* X, *then* Π_D *is a closed and quasi-*ϕ*-nonexpansive from* X *onto* D.

Lemma 2.1 (*see* [16]) *Let* X *be a smooth, strictly convex and reflexive Banach space and* D *be a nonempty closed convex subset of* X. *Then the following conclusions hold,*

(*a*) $\phi(x, y) = 0$ *if and only if* $x = y$.

(*b*) $\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \le \phi(x, y), \ \forall x, y \in D$.

(*c*) *If* $x \in X$ *and* $z \in D$, *then* $z = \Pi_D x$ *if and only if* $\langle z - y, Jx - Jz \rangle \ge 0, \ \forall y \in D$.

Lemma 2.2 (*see* [7]) *Let* X *be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and* D *be a nonempty closed convex subset of* X. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in* D *such that* $x_n \to p$ *and* $\phi(x_n, y_n) \to 0$ *where* ϕ

is the function defined by (1.2), then $y_n \to p$.

Definition 2.1 *A point $p \in D$ is said to be an asymptotic fixed point of multi-valued mapping $T: D \to CB(D)$, if there exists a sequence $\{x_n\} \subset D$ such that $x_n \rightharpoonup x \in X$ and $d(x_n, T(x_n)) \to 0$. Denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.*

Definition 2.2

(1) *A multi-valued mapping $T: D \to CB(D)$ is said to be relatively nonexpansive, if $F(T) \neq \Phi$, $\hat{F}(T) = F(T)$, and $\phi(p,z) \le \phi(p,x), \forall x \in D$, $p \in F(T), z \in T(x)$.*

(2) *A multi-valued mapping $T: D \to CB(D)$ is said to be closed, if for any sequence $\{x_n\} \subset D$ with $x_n \to x \in X$ and $d(y, T(x_n)) \to 0$, then $d(y, T(x)) = 0$.*

Remark 2.2 *If H is a real Hilbert space, then $\phi(x,y) = \|x-y\|^2$ and Π_D is the metric projection P_D of H onto D.*

Next, We present an example of relatively nonexpansive multi-valued mapping.

Example 2.1 (*see* [18]) *Let X be a smooth, strictly convex and reflexive Banach space, D be a nonempty closed and convex subset of X and $f: D \times D \to R$ be a bifunction satisfying the conditions:*

($A1$) $f(x,x) = 0, \forall x \in D$;

($A2$) $f(x,y) + f(y,x) \le 0, \forall x, y \in D$;

($A3$) *for each* $x, y, z \in D$,

$$\lim_{t \to 0} f(tz + (1-t)x, y) \le f(x,y);$$

($A4$) *for each given* $x \in D$, *the function* $y \mapsto f(x,y)$ *is convex and lower semicontinuous.*

The "so-called" equilibrium problem for f is to find a $x^* \in D$ such that $f(x^*, y) \ge 0, \forall y \in D$. The set of its solutions is denoted by $EP(f)$.

Let $r > 0, x \in D$ and define a multi-valued mapping $T_r: D \to N(D)$ as follows,

$$T_r(x) = \left\{ z \in D, f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \forall y \in D \right\}, \quad \forall x \in D, \tag{2.5}$$

then (1) *T_r is single-valued, and so $\{z\} = T_r(x)$*; (2) *T_r is a relatively nonexpansive mapping, therefore, it is a closed quasi-ϕ-nonexpansive mapping*; (3) $F(T_r) = EP(f)$.

Definition 2.3

(1) *A multi-valued mapping $T: D \to CB(D)$ is said to be quasi-ϕ-nonexpansive, if $F(T) \neq \Phi$, and $\phi(p,z) \le \phi(p,x), \forall x \in D, p \in F(T), z \in Tx$.*

(2) *A multi-valued mapping $T: D \to CB(D)$ is said to be quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \Phi$ and there exists a real sequence $k_n \subset [1, +\infty), k_n \to 1$ such that*

$$\phi(p, z_n) \le k_n \phi(p,x), \forall x \in D, \quad p \in F(T), z_n \in T^n x; \tag{2.6}$$

(3) *A multi-valued mapping $T: D \to CB(D)$ is said to be totally quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \Phi$ and there exist nonnegative real sequences $\{v_n\}, \{\mu_n\}$, with $v_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta: R^+ \to R^+$ with $\zeta(0) = 0$ such that*

$$\phi(p, z_n) \le \phi(p,x) + v_n \zeta[\phi(p,x)] + \mu_n, \quad \forall x \in D, \forall n \ge 1, p \in F(T), z_n \in T^n x. \tag{2.7}$$

Remark 2.3 *From the definitions, it is obvious that a relatively nonexpansive multi-valued mapping is a quasi-ϕ-nonexpansive multi-valued mapping, and a quasi-ϕ-nonexpansive multi-valued mapping is a quasi-ϕ-asymptotically nonexpansive multi-valued mapping, and a quasi-ϕ-asymptotically nonexpansive multi-valued mapping is a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping, but the converse is not true.*

Lemma 2.3 Let X and D be as in Lemma 2.2. $T: D \to CB(D)$ be a closed and totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta: R^+ \to R^+$ with $\zeta(0) = 0$,if $v_n, \mu_n \to 0$ (as $n \to \infty$) and $\mu_1 = 0$, then $F(T)$ is a closed and convex subset of D.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$, such that $x_n \to x^*$. Since T is totally quasi-ϕ-asymptotically nonexpansive multi-valued mapping, we have

$$\phi(x_n, z) \le \phi(x_n, x^*) + v_1 \zeta[\phi(x_n, x^*)]$$

for all $z \in Tx^*$ and for all $n \in N$. Therefore,

$$\begin{aligned} \phi(x^*, z) &= \lim_{n \to \infty} \phi(x_n, z) \\ &\le \lim_{n \to \infty} \left\{ \phi(x_n, x^*) + v_1 \zeta[\phi(x_n, x^*)] \right\} \\ &= \phi(x^*, x^*) = 0. \end{aligned}$$

By Lemma 2.1(a), we obtain $z = x^*$. Hence, $Tx^* = \{x^*\}$. So, we have $x^* \in F(T)$. This implies $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0,1)$, and put $w = tp + (1-t)q$. Next we prove that $w \in F(T)$. Indeed, in view of the definition of ϕ, letting $z_n \in T^n w$, we have

$$\begin{aligned} \phi(w, z_n) &= \|w\|^2 - 2\langle w, Jz_n \rangle + \|z_n\|^2 \\ &= \|w\|^2 - 2\langle tp + (1-t)q, Jz_n \rangle + \|z_n\|^2 \\ &= \|w\|^2 + t\phi(p, z_n) + (1-t)\phi(q, z_n) \\ &\quad - t\|p\|^2 - (1-t)\|q\|^2. \end{aligned} \tag{2.8}$$

Since

$$t\phi(p,z_n)+(1-t)\phi(q,z_n)$$
$$\le t\left[\phi(p,w)+v_n\zeta\left[\phi(p,w)\right]+\mu_n\right]$$
$$+(1-t)\left[\phi(q,w)+v_n\zeta\left[\phi(q,w)\right]+\mu_n\right]$$
$$=t\left\{\|p\|^2-2\langle p,Jw\rangle+\|w\|^2+v_n\zeta\left[\phi(p,w)\right]+\mu_n\right\}$$
$$+(1-t)\left\{\|q\|^2-2\langle q,Jw\rangle+\|w\|^2+v_n\zeta\left[\phi(q,w)\right]+\mu_n\right\}$$
$$=t\|p\|^2+(1-t)\|q\|^2-\|w\|^2+tv_n\zeta\left[\phi(p,w)\right]$$
$$+(1-t)v_n\zeta\left[\phi(q,w)\right]+\mu_n. \tag{2.9}$$

Substituting (2.8) into (2.9) and simplifying it, we have

$$\phi(w,z_n)\le tv_n\zeta\left[\phi(p,w)\right]+(1-t)v_n\zeta\left[\phi(q,w)\right]$$
$$+\mu_n\to 0. \qquad (\text{as } n\to\infty)$$

By Lemma 2.2, we have $z_n\to w$. This implies that $z_{n+1}\left(\in TT^n w\right)\to w$. Since T is closed, we have $Tw=\{w\}$, *i.e.*, $w\in F(T)$. This completes the proof of Lemma 2.3. □

Definition 2.4 *A mapping* $T:D\to CB(D)$ *is said to be uniformly* L*-Lipschitz continuous, if there exists a constant* $L>0$ *such that* $\|x_n-y_n\|\le L\|x-y\|$, *where* $x,y\in D, x_n\in T^n x, y_n\in T^n y$.

Definition 2.5

(1) *A countable family of mappings* $\{T_i\}:D\to D$ *is said to be uniformly quasi-*ϕ*-nonexpansive, if* $F:=\bigcap_{i=1}^{\infty}F(T_i)\neq\varnothing$, *and*

$$\phi(p,z)\le\phi(p,x),\forall x\in D, p\in F, z\in T_i x.$$

(2) *A countable family of mappings* $\{T_i\}:D\to D$ *is said to be uniformly quasi-*ϕ*-asymptotically nonexpansive, if* $F:=\bigcap_{i=1}^{\infty}F(T_i)\neq\varnothing$, *and there exists a real sequence* $k_n\subset[1,+\infty), k_n\to 1$ *such that,*

$$\phi(p,z_n)\le k_n\phi(p,x),\forall x\in D, p\in F, z_n\in T_i^n x. \tag{2.10}$$

(3) *A countable family of mappings* $\{T_i\}:D\to D$ *is said to be totally uniformly quasi-*ϕ*-asymptotically nonexpansive multi-valued, if* $F:=\bigcap_{i=1}^{\infty}F(T_i)\neq\varnothing$ *and there exists nonnegative real sequences* $\{v_n\},\{\mu_n\}$ *with* $v_n\to 0,\mu_n\to 0$ (*as* $n\to\infty$) *and a strictly increasing and continuous function* $\zeta:\Re^+\to\Re^+$ *with* $\zeta(0)=0$ *such that*

$$\phi(p,z_n)\le\phi(p,x)+v_n\zeta\left[\phi(p,x)\right]+\mu_n,$$
$$\forall x\in D,\forall n\ge 1, p\in F, z_n\in T_i^n x. \tag{2.11}$$

Remark 2.4 *From the definitions, it is obvious that a countable family of uniformly quasi-*ϕ*-nonexpansive multi-valued mappings is a countable family of uniformly* quasi-ϕ-asymptotically nonexpansive multi-valued mappings, and a countable family of uniformly quasi-ϕ-asymptotically nonexpansive multi-valued mappings is a countable family of totally uniformly quasi-ϕ-asymptotically multi-valued mappings, but the converse is not true.

3. Main Results

Theorem 3.1 *Let* X *be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, D be a nonempty closed convex subset of X,* $T_i:D\to D, i=1,2,3,\cdots$ *be a closed and uniformly* L_i*-Lipschitz continuous and a countable family of uniformly totally quasi-*ϕ*-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences* $\{v_n\}$, $\{\mu_n\}$, $v_n,\mu_n\to 0$ (*as* $n\to\infty$) *and a strictly increasing continuous function* $\zeta:R^+\to R^+$ *with* $\zeta(0)=0$ *satisfying condition* (2.11). *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *such that* $\alpha_n\to 0$. *If* $\{x_n\}$ *is the sequence generated by*

$$\begin{cases} x_1\in X \text{ is arbitrary;} \quad D_1=D \\ y_{n,i}=J^{-1}\left[\alpha_n Jx_1+(1-\alpha_n)Jz_n\right], \quad z_n\in T_i^n x_n, i\ge 1 \\ D_{n+1}=\left\{z\in D_n:\sup_{i\ge 1}\phi(z,y_{n,i})\right. \\ \qquad\left.\le\alpha_n\phi(z,x_1)+(1-\alpha_n)\phi(z,x_n)+\xi_n\right\} \\ x_{n+1}=\Pi_{D_{n+1}}x_1\ (n=1,2,\cdots) \end{cases} \tag{3.1}$$

where $\xi_n=v_n\sup_{p\in F}\zeta\left[\phi(p,x_n)\right]+\mu_n$, $F(T_i)$ *is the fixed point set of* T_i, *and* $\Pi_{D_{n+1}}$ *is the generalized projection of* X *onto* D_{n+1}.

If $F:=\bigcap_{i=1}^{\infty}F(T_i)\neq\varnothing$ *and* F *is bounded and* $\mu_1=0$, *then* $\lim_{n\to\infty}x_n=\Pi_F x_1$.

Proof. (I) First, we prove that F and D_n $(n\ge 1)$ are closed and convex subsets in D. In fact, it follows from Lemma 2.3 that $F(T_i)(i\ge 1)$ is a closed and convex subsets in D. Therefore F is closed and convex subsets in D. Again by the assumption, $D_1=D$ is closed and convex. Suppose that D_n is closed and convex for some $n\ge 1$. In view of the definition of ϕ, we have

$$D_{n+1}=\left\{z\in D_n:\sup_{i\ge 1}\phi(z,y_{n,i})\le\alpha_n\phi(z,x_1)\right.$$
$$\left.+(1-\alpha_n)\phi(z,x_n)+\xi_n\right\}$$
$$=\bigcap_{i\ge 1}\left\{z\in D:\phi(z,y_{n,i})\le\alpha_n\phi(z,x_1)\right.$$
$$\left.+(1-\alpha_n)\phi(z,x_n)+\xi_n\right\}\cap D_n$$
$$=\bigcap_{i\ge 1}\left\{z\in D:2\alpha_n\langle z,Jx_1\rangle+2(1-\alpha_n)\langle z,Jx_n\rangle-2\langle z,Jy_{n,i}\rangle\right.$$
$$\left.\le\alpha_n\|x_1\|^2+(1-\alpha_n)\|x_n\|^2-\|y_{n,i}\|^2\right\}\cap D_n.$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) Next, we prove that $F \subset D_n$, for all $n \ge 1$.

In fact, it is obvious that $F \subset D_1$. Suppose that $F \subset D_n$. Hence for any $u \in F \subset D_n$, by (2.4), we have

$$\begin{aligned}
&\phi(u, y_{n,i}) = \phi\left(u, J^{-1}\left(\alpha_n Jx_1 + (1-\alpha_n) Jz_n\right)\right) \\
&\le \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, z_n) \\
&\le \alpha_n \phi(u, x_1) + (1-\alpha_n)\left\{\phi(u, x_n) + v_n \zeta\left[\phi(u, x_n)\right] + \mu_n\right\} \\
&\le \alpha_n \phi(u, x_1) \\
&+ (1-\alpha_n)\left\{\phi(u, x_n) + v_n \sup_{p \in F} \zeta\left[\phi(p, x_n)\right] + \mu_n\right\} \\
&= \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \xi_n, \ \forall i \ge 1.
\end{aligned} \tag{3.2}$$

Therefore we have

$$\sup_{i \ge 1} \phi(u, y_{n,i}) \le \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \xi_n. \tag{3.3}$$

This shows that $u \in F \subset D_{n+1}$ and so $F \subset D_n$. The conclusions are proved.

(III) Now we prove that $\{x_n\}$ converges strongly to some point $p^* \in D$.

In fact, since $x_n = \Pi_{D_n} x_1$, from Lemma 2.1(c), we have $\langle x_n - y, Jx_1 - Jx_n \rangle \ge 0, \quad \forall y \in D_n$. Again since $F \subset D_n$, we have $\langle x_n - u, Jx_1 - Jx_n \rangle \ge 0$, $\forall u \in F$. It follows from Lemma 2.1(b) that for each $u \in F$ and for each $n \ge 1$,

$$\begin{aligned}
&\phi(x_n, x_1) = \phi\left(\Pi_{D_n} x_1, x_1\right) \\
&\le \phi(u, x_1) - \phi(u, x_n) \le \phi(u, x_1).
\end{aligned} \tag{3.4}$$

Therefore, $\{\phi(x_n, x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{D_n} x_1$ and $x_{n+1} = \Pi_{D_{n+1}} x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$.

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Since X is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in $D = D_1$). Since D_n is closed and convex and $D_{n+1} \subset D_n$. This implies that D_n is weakly closed and $p^* \in D_n$ for each $n \ge 1$. In view of $x_{n_i} = \Pi_{D_{n_i}} x_1$, we have $\phi(x_{n_i}, x_1) \le \phi(p^*, x_1), \forall n_i \ge 1$. Since the norm $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\begin{aligned}
&\liminf_{n_i \to \infty} \phi(x_n, x_1) \\
&= \liminf_{n_i \to \infty} \left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1\rangle + \|x_1\|^2\right) \\
&\ge \|p^*\|^2 - 2\langle p^*, Jx_1\rangle + \|x_1\|^2 = \phi(p^*, x_1)
\end{aligned}$$

and so

$$\begin{aligned}
\phi(p^*, x_1) &\le \liminf_{n_i \to \infty} \phi(x_n, x_1) \\
&\le \limsup_{n_i \to \infty} \phi(x_n, x_1) = \phi(p^*, x_1).
\end{aligned}$$

This shows that $\lim_{n_i\to\infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1)$ and we have $\|x_{n_i}\| \to \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$, by virtue of Kadec-Klee property of X, we obtain that $\lim_{i\to\infty} x_{n_i} = p^*$. Since $\{\phi(x_n, x_1)\}$ is convergent, this together with $\lim_{n_i\to\infty} \phi(x_{n_i}, x_1) = \phi(p^*, x_1)$ shows that $\lim_{n_i\to\infty} \phi(x_n, x_1) = \phi(p^*, x_1)$. If there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$, then from Lemma 2.1, we have

$$\begin{aligned}
\phi(p^*, q) &= \lim_{n_i, n_j \to \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{n_i, n_j \to \infty} \phi\left(x_{n_i}, \Pi_{D_{n_j}} x_1\right) \\
&\le \lim_{n_i, n_j \to \infty} \left[\phi(x_{n_i}, x_1) - \phi\left(\Pi_{D_{n_j}} x_1, x_1\right)\right] \\
&= \lim_{n_i, n_j \to \infty} \left[\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1)\right] \\
&= \phi(p^*, x_1) - \phi(p^*, x_1) = 0,
\end{aligned}$$

i.e., $p^* = q$ and hence

$$x_n \to p^*. \tag{3.5}$$

By the way, from (3.4), it is easy to see that

$$\xi_n = v_n \sup_{p \in F} \zeta\left[\phi(p, x_n)\right] + \mu_n \to 0. \tag{3.6}$$

(IV) Now we prove that $p^* \in F$.

In fact, since $x_{n+1} \in D_{n+1}$, from (3.1), (3.4) and (3.5), we have

$$\begin{aligned}
&\sup_{i \ge 1} \phi(x_{n+1}, y_{n,i}) \\
&\le \alpha_n \phi(x_{n+1}, x_1) + (1-\alpha_n)\phi(x_{n+1}, x_n) + \xi_n \to 0.
\end{aligned} \tag{3.7}$$

Since $x_n \to p^*$, it follows from (3.6) and Lemma 2.2 that

$$y_{n,i} \to p^*(n \to \infty). \tag{3.8}$$

Since $\{x_n\}$ is bounded and $\{T_i\}$ is a countable family of uniformly totally quasi-ϕ-asymptotically nonexpansive multi-valued mappings, $z_n \in T_i^n x_n$ is bounded. In view of $\alpha_n \to 0$, from (3.1), we have

$$\lim_{n\to\infty} \|Jy_{n,i} - Jz_n\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - Jz_n\| = 0. \tag{3.9}$$

Since $Jy_{n,i} \to Jp^*$, this implies $Jz_n \to Jp^*$. From Remark 2.1, it yields that

$$z_n \rightharpoonup p^*. \tag{3.10}$$

Again since

$$\|z_n\| - \|p^*\| = \|Jz_n\| - \|Jp^*\| \le \|Jz_n - Jp^*\| \to 0, \tag{3.11}$$

this together with (3.9) and the Kadec-Klee-property of X shows that

$$z_n \to p^*. \tag{3.12}$$

On the other hand, by the assumptions that T_i is L_i-Lipschitz continuous for each $i \ge 1$, we have

$$\begin{aligned} & d\left(T_i z_n, z_n\right) \\ & \le d\left(T_i z_n, z_{n+1}\right) + \left\|z_{n+1} - x_{n+1}\right\| \\ & + \left\|x_{n+1} - x_n\right\| + \left\|x_n - z_n\right\| \\ & \le \left(L_i + 1\right)\left\|x_{n+1} - x_n\right\| + \left\|z_{n+1} - x_{n+1}\right\| + \left\|x_n - z_n\right\|. \end{aligned} \tag{3.13}$$

From (3.12) and $x_n \to p^*$, we have that $d\left(T_i z_n, z_n\right) \to 0$. In view of the closeness of T_i, it yields that $T_i\left(p^*\right) = \left\{p^*\right\}\left(\forall i \ge 1\right)$, which implies that $p^* \in F$.

(V) Finally we prove that $p^* = \Pi_F x_1$ and so $x_n \to \Pi_F x_1$.

Let $w = \Pi_F x_1$. Since $w \in F \subset D_n$, we have $\phi\left(p^*, x_1\right) \le \phi\left(w, x_1\right)$. This implies that

$$\phi\left(p^*, x_1\right) = \lim_{n\to\infty} \phi\left(x_n, x_1\right) \le \phi\left(w, x_1\right). \tag{3.14}$$

which yields that $p^* = w = \Pi_F x_1$. Therefore, $x_n \to \Pi_F x_1$. The proof of Theorem 3.1 is completed.

By Remark 2.4, the following corollaries are obtained. □

Corollary 3.1 Let X and D be as in Theorem 3.1, and a countable family of mappings $T_i : D \to D$ $\left(i = 1,2,3,\cdots\right)$ be a closed and uniformly L_i-Lipschitz continuous a relatively nonexpansive multi-valued mappings. Let $\left\{\alpha_n\right\}$ in $\left(0,1\right)$ with $\lim_{n\to\infty} \alpha_n = 0$. Let $\left\{x_n\right\}$ be the sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary;} \quad D_1 = D \\ y_{n,i} = J^{-1}\left[\alpha_n J x_1 + \left(1-\alpha_n\right) J z_n\right], \quad z_n \in T_i x_n \\ D_{n+1} = \left\{z \in D_n : \sup_{i\ge 1} \phi\left(z, y_{n,i}\right)\right. \\ \qquad \left. \le \alpha_n \phi\left(z, x_1\right) + \left(1-\alpha_n\right)\phi\left(z, x_n\right)\right\} \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad \left(n = 1,2,\cdots\right) \end{cases} \tag{3.15}$$

where $F\left(T_i\right)$ is the set of fixed points of T_i, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}, If $F := \bigcap_{i=1}^{\infty} F\left(T_i\right) \ne \varnothing$ and F is bounded, then $\left\{x_n\right\}$ converges strongly to $\Pi_F x_1$.

Corollary 3.2 Let X and D be as in Theorem 3.1, and a countable family of mappings $T_i : D \to D$ $\left(i = 1,2,3,\cdots\right)$ be a closed and uniformly L_i-Lipschitz continuous quasi-*phi*-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences $\left\{k_n\right\} \subset \left[1,+\infty\right)$ and $k_n \to 1$ satisfying condition (2.1). Let $\left\{\alpha_n\right\}$ be a sequence in $\left(0,1\right)$ and satisfy $\lim_{n\to\infty} \alpha_n = 0$. If $\left\{x_n\right\}$ is the sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary;} \quad D_1 = D \\ y_{n,i} = J^{-1}\left[\alpha_n J x_1 + \left(1-\alpha_n\right) J z_n\right], \quad z_n \in T_i^n x_n \\ D_{n+1} = \left\{z \in D_n : \sup_{i\ge 1} \phi\left(z, y_{n,i}\right)\right. \\ \qquad \left. \le \alpha_n \phi\left(z, x_1\right) + \left(1-\alpha_n\right)\phi\left(z, x_n\right) + \xi_n\right\} \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad \left(n = 1,2,\cdots\right) \end{cases} \tag{3.16}$$

where $F\left(T_i\right)$ is the set of fixed points of T_i, and $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}, and $\xi_n = \left(k_n - 1\right)\sup_{p\in F} \phi\left(p, x_n\right)$.

If $F := \bigcap_{i=1}^{\infty} F\left(T_i\right) \ne \varnothing$ and F is bounded, then $\left\{x_n\right\}$ converges strongly to $\Pi_F x_1$.

4. Application

We utilize Corollary 3.2 to study a modified Halpern iterative algorithm for a system of equilibrium problems.

Theorem 4.1 Let D, X and $\left\{\alpha_n\right\}$ be the same as in Theorem 3.1. Let $f : D \times D \to R$ be a bifunction satisfying conditions (A1)-(A4) as given in Example 2.6. Let $T_r : X \to D$ be a mapping defined by (2.5), *i.e.*,

$$\begin{aligned} & T_r\left(x\right) \\ & = \left\{x \in D, f\left(z, y\right) + \frac{1}{r}\left\langle y - z, Jz - Jx\right\rangle \ge 0, \forall y \in D\right\}, \end{aligned}$$

$\forall x \in X$.

Let $\left\{x_n\right\}$ be the sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary;} \quad D_1 = D \\ f\left(u_n, y\right) + 1r\left\langle y - u_n, J u_n - J x_n\right\rangle \ge 0, \\ \qquad \forall y \in D, r > 0, u_n \in T_r x_n \\ y_n = J^{-1}\left[\alpha_n J x_1 + \left(1-\alpha_n\right) J u_n\right] \\ D_{n+1} = \left\{z \in D_n : \phi\left(z, y_n\right)\right. \\ \qquad \left. \le \alpha_n \phi\left(z, x_1\right) + \left(1-\alpha_n\right)\phi\left(z, x_n\right)\right\} \\ x_{n+1} = \prod_{D_{n+1}} x_1 \left(n = 1,2,\cdots\right). \end{cases} \tag{4.1}$$

If $F\left(T_r\right) \ne \Phi$, then $\left\{x_n\right\}$ converges strongly to $\prod_{F(T)} x_1$ which is a common solution of the system of equilibrium problems for f.

Proof. In Example 2.6, we have pointed out that $u_n = T_r\left(x_n\right)$, $F\left(T_r\right) = EP\left(f\right)$ and T_r is a closed quasi-ϕ-nonexpansive mapping. Hence (4.1) can be rewritten as follows:

$$\begin{cases} x_1 \in X \text{ is arbitrary;} \quad D_1 = D \\ y_n = J^{-1}\left[\alpha_n J x_1 + \left(1-\alpha_n\right) J u_n\right], \quad u_n \in T_r x_n \\ D_{n+1} = \left\{z \in D_n : \phi\left(z, y_n\right)\right. \\ \qquad \left. \le \alpha_n \phi\left(z, x_1\right) + \left(1-\alpha_n\right)\phi\left(z, x_n\right)\right\} \\ x_{n+1} = \prod_{D_{n+1}} x_1 \quad \left(n = 1,2,\cdots\right). \end{cases}$$

Therefore the conclusion of Theorem 4.6 can be obtained from Corollary 3.2.

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