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Hyers-Ulam-Rassias Stability for the Heat Equation

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ABSTRACT

In this paper we apply the Fourier transform to prove the Hyers-Ulam-Rassias stability for one dimensional heat equation on an infinite rod. Further, the paper investigates the stability of heat equation in $\mathbb{R}^n$ with initial condition, in the sense of Hyers-Ulam-Rassias. We have also used Laplace transform to establish the modified Hyers-Ulam-Rassias stability of initial-boundary value problem for heat equation on a finite rod. Some illustrative examples are given.

Keywords: Hyers-Ulam-Rassias Stability; Heat Equation; Fourier Transform; Laplace Transform

1. Introduction and Preliminaries

The study of stability problems for various functional equations originated from a famous talk given by Ulam in 1940. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [1] gave a partial solution to Ulam's problem. Afterthen and during the last two decades a great number of papers have been extensively published concerning the various generalizations of Hyers result (see [2-10]).

Alsina and Ger [11] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$. They proved that if a differentiable function $y: I \to R$ satisfies $|y' - y| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g: I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \le 3\varepsilon$, for all $t \in I$. This result of alsina and Ger has been generalized by Takahasi *et al.* [12] to the case of the complex Banach space valued differential equation $y' = \lambda y$.

Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura *et al.* [13], Jung [14] and Wang *et al.* [15].

Li [16] established the stability of linear differential equation of second order in the sense of the Hyers and Ulam $y'' = \lambda y$. Li and Shen [17] proved the stability of nonhomogeneous linear differential equation of second order in the sense of the Hyers and Ulam $y'' + p(x)y' + q(x)y + r(x) = 0$, while Gavruta *et al.* [18] proved the Hyers-Ulam stability of the equation $y'' + \beta(x)y = 0$ with boundary and initial conditions. Jung [19] proved the Hyers-Ulam stability of first-order linear partial differential equations. Gordji *et al.* [20] generalized Jung's result to first order and second order Nonlinear partial differential equations. Lungu and Craciun [21] established results on the Ulam-Hyers stability and the generalized Ulam-HyersRassias stability of nonlinear hyperbolic partial differential equations.

In this paper we consider the Hyers-Ulam-Rassias stability of the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < t \le T < \infty, -\infty < x < \infty \tag{1}$$

with the initial condition

$$u(x,0) = \mu(x) \tag{2}$$

where $\mu(x) \in C(-\infty,\infty)$, and

$$u(x,t) \in C_1^2(\mathbb{R} \times (0,\infty)).$$

We also use a similar argument to establish the Hyers-Ulam-Rassias for the heat equation in higher dimension

$$u_t = a^2 \Delta u \quad 0 < t \le T < \infty, x \in \mathbb{R}^n \tag{3}$$

with the initial condition

$$u(x,0) = \mu(x) \tag{4}$$

where $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

Moreover we have proved theorems on Hyers-Ulam-Rassias-Gavruta stability for the heat equation in a finite rod.

Definition 1 We will say that the Equation (1) has the

Hyers-Ulam-Rassias stability with respect to $\varphi > 0$, if there exists $K > 0$ such that for each $\varepsilon > 0$ and for each solution $u(x,t) \in C_1^2(\mathbb{R}^n \times (0,\infty))$ of the inequality

$$\left|u_t - a^2 \Delta u\right| \le \varepsilon \varphi(x,t) \quad (5)$$

with the initial condition (2), then there exists a solution $w(x,t) \in C_1^2(\mathbb{R}^n \times (0,\infty))$ of the Equation (1), such that

$$\left|u(x,t) - w(x,t)\right| \le K \varepsilon \varphi(x,t),$$

$$\forall (x,t) \in \mathbb{R}^n \times (0,\infty),$$

where K is a constant that does not depend on ε nor on $u(x,t)$, and $\varphi(x,t) \in C(\mathbb{R}^n \times (0,\infty))$.

Definition 2 We will say that the equation (1) has the Hyers-Ulam-Rassias-Gavruta (HURG) stability with respect to $\varphi > 0$, if there exists $K > 0$ such that for each $\varepsilon > 0$ and for each solution $u(x,t) \in C_1^2(\mathbb{R}^n \times (0,\infty))$ of the inequality

$$\left|u_t - a^2 \Delta u\right| \le \varepsilon \varphi(x,t) \quad (6)$$

with the initial condition (2), then there exists a solution $w(x,t) \in C_1^2(\mathbb{R}^n \times (0,\infty))$ of the Equation (1), such that

$$\left|u(x,t) - w(x,t)\right| \le K \varepsilon \varphi(x,t),$$

$$\forall (x,t) \in \mathbb{R}^n \times (0,\infty),$$

where K is a constant that does not depend on ε nor on $u(x,t)$, and $\varphi(x,t) \in C(\mathbb{R}^n \times (0,\infty))$.

Definition 3 We will say that the solution of the initial value problem (1), (2) has the Hyers-Ulam-Rassias asymptotic stability with respect to $\varphi > 0$, if it is stable in the sense of Hyers and Ulam with respect to φ, and

$$\lim_{t\to\infty} \left(u(x,t) - w(x,t)\right) = 0$$

Definition 4 Assume the functions $f(x)$ and $g(x)$ defined on $x \in \mathbb{R}^n$ are continuously differentiable and absolutely integrable, then the Fourier transform of $f(x)$ is defined as

$$\mathcal{F}[f] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-i\xi x} \mathrm{d}x = F(\xi)$$

and the inverse Fourier transform of $G(\xi), \xi \in \mathbb{R}^n$ is

$$\mathcal{F}^{-1}[g] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} G(\xi) \mathrm{e}^{i\xi x} \mathrm{d}\xi = g(x)$$

Example 1 Let

$$f(x) = \mathrm{e}^{-\beta |x|^2}, x \in \mathbb{R}^n, \beta > 0$$

We find the Fourier transform of the function.

Since

$$f(x) = \mathrm{e}^{-\beta|x|^2} = \mathrm{e}^{-\beta\left(x_1^2 + \cdots + x_n^2\right)} = \mathrm{e}^{-\beta x_1^2} \cdots \mathrm{e}^{-\beta x_n^2}$$
$$= h(x_1) \cdot h(x_2) \cdots h(x_n)$$

Then

$$h(x_k) = \mathrm{e}^{-\beta x_k^2}, \ k = 1, \cdots, n$$

and by defintion 4 we have

$$F(\xi) = \prod_{k=1}^{n} H(\xi_k) \quad (7)$$

where

$$H(\xi_k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\beta x_k^2} \mathrm{e}^{-i\xi_k x_k} \mathrm{d}x_k \quad (8)$$

Differentiating $H(\xi_k)$ with respect to ξ_k, we get

$$H'(\xi_k) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\beta x_k^2} \mathrm{e}^{-i\xi_k x_k} (-ix_k) \mathrm{d}x_k$$

Integrating by parts gives

$$H'(\xi_k) = \frac{\xi_k}{2\beta} H(\xi_k)$$

Hence

$$H(\xi_k) = C \mathrm{e}^{-\xi_k^2 / 4\beta}$$

Putting $\xi_k = 0$ gives $C = H(0)$, and from (8) one has

$$H(0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\beta x_k^2} \mathrm{d}x_k$$

Using that $\int_{-\infty}^{\infty} \mathrm{e}^{-\beta z^2} \mathrm{d}z = \sqrt{\frac{\pi}{\beta}}$, we have

$$H(0) = \frac{1}{(2\beta)^{1/2}}$$

$$H(\xi_k) = \frac{1}{(2\beta)^{1/2}} \mathrm{e}^{-\xi_k^2 / 4\beta} \quad (9)$$

Therefore, from (7), (9) we obtain

$$F(\xi) = \frac{1}{(2\beta)^{n/2}} \mathrm{e}^{-|\xi|^2 / 4\beta}$$

Theorem 1 (See Evans [22]) Assume that $f(x)$ and $g(x)$ are continuously differentiable and absolutely integrable on $\mathbb{R}^n$. Then

1) for each α such that $D^\alpha f \in L(\mathbb{R}^n)$, $\mathcal{F}\left[D^\alpha f\right] = (i\xi)^n \mathcal{F}[f]$.

2) $\mathcal{F}[f * g] = (2\pi)^{\frac{n}{2}} \mathcal{F}[f] \mathcal{F}[g]$, where $f * g = \int_{\mathbb{R}^n} f(y) g(x-y) \mathrm{d}y$ is the convolution of $f(x)$

and $g(x)$.

2. On Hyers-Ulam-Rassias Stability for Heat Equation on an Infinite Rod

Theorem 2 If $u(x,t)\in C_1^2(\mathbb{R}\times(0,T])$ then the initial value problem (1), (2) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Let $\varepsilon>0$ and $u(x,t)$ be an approximate solution of the initial value problem (1), (2). We will show that there exists a function $w(x,t)\in C_1^2(\mathbb{R}\times(0,T])$ satisfying the Equation (1) and the initial condition (2) such that

$$\left|u(x,t)-w(x,t)\right|\le K\varepsilon\varphi(x,t)$$

If we take $\varphi(x,t)=\frac{1}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}$ then from inequality (5), we have

$$\begin{aligned}\frac{-\varepsilon}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}&\le\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}\\&\le\frac{\varepsilon}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}\end{aligned}\tag{10}$$

Applying Fourier Transform to inequality (10), we get

$$-\varepsilon\mathrm{e}^{-a^2\xi^2(t+1)}\le\frac{\mathrm{d}v(\xi,t)}{\mathrm{d}t}+a^2\xi^2v(\xi,t)\le\varepsilon\mathrm{e}^{-a^2\xi^2(t+1)}\tag{11}$$

Or, equivalently

$$\begin{aligned}-\varepsilon T\mathrm{e}^{-a^2\xi^2}&\le\mathrm{e}^{a^2\xi^2t}\frac{\mathrm{d}v(\xi,t)}{\mathrm{d}t}+a^2\xi^2v(\xi,t)\mathrm{e}^{a^2\xi^2t}\\&\le\varepsilon T\mathrm{e}^{-a^2\xi^2}\end{aligned}$$

Integrating the inequality from 0 to t we obtain

$$-\varepsilon T\mathrm{e}^{-a^2\xi^2}\le\mathrm{e}^{a^2\xi^2t}v(\xi,t)-v(\xi,0)\le\varepsilon T\mathrm{e}^{-a^2\xi^2}$$

From which it follows

$$\begin{aligned}-2\varepsilon T\mathrm{e}^{-a^2\xi^2(t+1)}&\le v(\xi,t)-\hat{\mu}(\xi)\mathrm{e}^{-a^2\xi^2t}\\&\le 2\varepsilon T\mathrm{e}^{-a^2\xi^2(t+1)}\end{aligned}\tag{12}$$

where $v(\xi,t)=\mathcal{F}[u(x,t)]$, and $\hat{\mu}(\xi)=\mathcal{F}[\mu(x)]$. In Example 1, we have established $\mathcal{F}\left[\mathrm{e}^{-\beta|x|^2}\right]=\frac{1}{(2\beta)^{n/2}}\mathrm{e}^{-|\xi|^2/4\beta}$. Putting $n = 1$, and $t=\frac{1}{4a^2\beta}$, we obtain $\mathrm{e}^{-a^2\xi^2t}=\mathcal{F}\left[\frac{1}{a\sqrt{2t}}\mathrm{e}^{-x^2/4a^2t}\right]$.

Now, Using the convolution theorem, from inequality (12) one has

$$\begin{aligned}&-\varepsilon T\mathcal{F}\left[\frac{1}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}\right]\\&\le\mathcal{F}[u(x,t)]-\frac{1}{2a\sqrt{\pi t}}\mathcal{F}\left[\mu(x)*\mathrm{e}^{-x^2/4a^2t}\right]\\&\le\varepsilon T\mathcal{F}\left[\frac{1}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}\right]\end{aligned}$$

Applying inverse Fourier transform to the last inequality and using convolution theorem we have

$$\begin{aligned}&-\frac{\varepsilon T}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}\\&\le u(x,t)-\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\mu(\lambda)\mathrm{e}^{-(x-\lambda)^2/4a^2t}\mathrm{d}\lambda\\&\le\frac{\varepsilon T}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}\end{aligned}$$

Let us take

$$w(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\mu(\lambda)\mathrm{e}^{-(x-\lambda)^2/4a^2t}\mathrm{d}\lambda.\tag{13}$$

Applying arguments shown above to initial-value problem (1), (2), one can show that (13) is an exact solution of Equation (1).

To show that $w(x,0)=\mu(x)$, we put $\mu=\frac{x-\lambda}{2a\sqrt{t}}$.

Then $\lambda=x-2a\sqrt{t}\mu$, $\mathrm{d}\lambda=-2a\sqrt{t}\mathrm{d}\mu$, so that

$$w(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mu\left(x-2a\sqrt{t}\mu\right)\mathrm{e}^{-\mu^2}\mathrm{d}\mu$$

Hence, as $t\to 0+$ we find

$$w(x,0)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mu(x)\mathrm{e}^{-\mu^2}\mathrm{d}\mu=\mu(x)$$

Therefore the initial value problem (1), (2) is stable in the sense of Hyers-Ulam-Rassias.

More generally, the following Theorem was established for the Hyers-Ulam-Rassias stability of heat equation in $\mathbb{R}^n$.

Theorem 3 If $u(x,t)\in C_1^2(\mathbb{R}^n\times(0,T])$, $0<T<\infty$, then the initial value problem (3), (4) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Let $\varepsilon>0$ and $u(x,t)$ be an approximate solution of the initial value problem (3), (4). We will show that there exists a function $w(x,t)\in C_1^2(\mathbb{R}^n\times(0,T])$ satisfying the Equation (3) and the initial condition (4) such that

$$\left|u(x,t)-w(x,t)\right|\le K\varepsilon\varphi(x,t)$$

Taking $\varphi(x,t)=\dfrac{1}{\left(2a\sqrt{\pi(t+1)}\right)^n}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}}$ then from the inequality (5), we have

$$-\frac{\varepsilon}{\left(2a\sqrt{(t+1)}\right)^n}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}} \le u_t-a^2\Delta u\le\frac{\varepsilon}{2a\sqrt{(t+1)}}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}}, \tag{14}$$

$t>0,\ x\in\mathbb{R}^n$

Applying Fourier Transform to inequality (14), we get

$$\left|\frac{\mathrm{d}v(\xi,t)}{\mathrm{d}t}+a^2|\xi|^2v(\xi,t)\right|\le\varepsilon\mathrm{e}^{-a^2|\xi|^2(t+1)}$$

Or, equivalently

$$-\varepsilon\mathrm{e}^{-a^2|\xi|^2}\le\mathrm{e}^{a^2|\xi|^2t}\frac{\mathrm{d}v(\xi,t)}{\mathrm{d}t}+a^2|\xi|^2v(\xi,t)\mathrm{e}^{a^2|\xi|^2t}\le\varepsilon\mathrm{e}^{-a^2|\xi|^2}$$

Integrating the inequality from 0 to t we obtain

$$-\varepsilon t\mathrm{e}^{-a^2|\xi|^2}\le\mathrm{e}^{a^2|\xi|^2t}v(\xi,t)-v(\xi,0)\le\varepsilon t\mathrm{e}^{-a^2|\xi|^2}$$

From which it follows

$$-\varepsilon T\mathrm{e}^{-a^2|\xi|^2(t+1)}\le v(\xi,t)-\hat{\mu}(\xi)\mathrm{e}^{-a^2|\xi|^2t}\le\varepsilon T\mathrm{e}^{-a^2|\xi|^2(t+1)} \tag{15}$$

where $v(\xi,t)=\mathcal{F}[u(x,t)]$, and $\hat{\mu}(\xi)=\mathcal{F}[\mu(x)]$.

Using Example 1, we find that

$$e^{-a^2|\xi|^2t}=\mathcal{F}\left[\frac{1}{\left(a\sqrt{2t}\right)^n}\mathrm{e}^{-|x|^2/4a^2t}\right],$$

and applying the convolution theorem, from inequality (15) one has

$$-\varepsilon T\mathcal{F}\left[\frac{1}{\left(2a\sqrt{(t+1)}\right)^n}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}}\right]\le\mathcal{F}[u(x,t)]-\frac{1}{\left(2a\sqrt{\pi t}\right)^n}\mathcal{F}\left[\mu(x)*\mathrm{e}^{-|x^2|/4a^2t}\right]\le\varepsilon T\mathcal{F}\left[\frac{1}{\left(2a\sqrt{(t+1)}\right)^n}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}}\right]$$

By applying the inverse Fourier transform to the last inequality, and then using convolution theorem we get

$$-\frac{\varepsilon T}{\left(2a\sqrt{t+1}\right)^n}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}}\le u(x,t)-\frac{1}{(2a\sqrt{\pi t})^n}\int_{\mathbb{R}^n}\mu(\lambda)\mathrm{e}^{-|x-\lambda|^2/4a^2t}\mathrm{d}\lambda\le\frac{\varepsilon T}{\left(2a\sqrt{t+1}\right)^n}\mathrm{e}^{-\frac{1}{4a^2}\frac{|x|^2}{t+1}}$$

Now, let us take

$$w(x,t)=\frac{1}{\left(2a\sqrt{\pi t}\right)^n}\int_{\mathbb{R}^n}\mu(\lambda)\mathrm{e}^{-|x-\lambda|^2/4a^2t}\mathrm{d}\lambda. \tag{16}$$

One can find that (16) is a solution of Equation (3).

To show that $w(x,0)=\mu(x)$, we put $\mu=\dfrac{x-\lambda}{2a\sqrt{t}}$.

Then $\lambda=x-2a\sqrt{t}\mu,\ \mathrm{d}\lambda=-2a\sqrt{t}\mathrm{d}\mu,$ so that

$$w(x,t)=\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n}\mu\left(x-2a\sqrt{t}\mu\right)\mathrm{e}^{-|\mu|^2}\mathrm{d}\mu$$

Hence as $t\to0+$ we obtain

$$w(x,0)=\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n}\mu(x)\mathrm{e}^{-|\mu|^2}\mathrm{d}\mu=\mu(x)$$

since $\dfrac{1}{\pi^{n/2}}\displaystyle\int_{\mathbb{R}^n}\mathrm{e}^{-|\mu|^2}\mathrm{d}\mu=1.$

Hence the initial value problem (3), (4) is stable in the sense of Hyers-Ulam-Rassias.

Theorem 4 Suppose that $u(x,t)\in C_1^2(\mathbb{R}\times(0,\infty))$ satisfies the inequality (5) with the initial condition $u(x,0)=\mu(x)$. Then the the initial-value problem (1), (2) is stable in the sense of HURG.

Proof. Indeed, if we take $\varphi(x,t)=\dfrac{\mathrm{e}^{-t}}{2a\sqrt{t+1}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}$ then from the inequality (5), we have

$$\frac{-\varepsilon\mathrm{e}^{-t}}{2a\sqrt{t+1}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}}\le\frac{\partial u}{\partial t}-a^2\frac{\partial^2u}{\partial x^2}\le\frac{\varepsilon\mathrm{e}^{-t}}{2a\sqrt{t+1}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}} \tag{17}$$

Applying Fourier Transform to inequality (17), we get

$$-\varepsilon e^{-t}\mathrm{e}^{-a^2\xi^2(t+1)}\le\frac{\mathrm{d}v(\xi,t)}{\mathrm{d}t}+a^2\xi^2v(\xi,t)\le\varepsilon e^{-t}\mathrm{e}^{-a^2\xi^2(t+1)}$$

Now, by applying the same argument used above, we obtain

$$-\frac{\varepsilon\left(1-\mathrm{e}^{-t}\right)}{2a\sqrt{t+1}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}} \le u(x,t)-\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\mu(\lambda)\mathrm{e}^{-(x-\lambda)^2/4a^2t}\mathrm{d}\lambda \le \frac{\varepsilon\left(1-\mathrm{e}^{-t}\right)}{2a\sqrt{t+1}}\mathrm{e}^{-\frac{1}{4a^2}\frac{x^2}{t+1}} \tag{18}$$

One takes

$$w(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\mu(\lambda)\mathrm{e}^{-(x-\lambda)^2/4a^2t}\mathrm{d}\lambda.$$

as a solution of initial-value problem (1), (2).

Therefore the initial value problem (1), (2) is stable in the sense of HURG.

Corollary 1 Suppose that $u(x,t)\in C_1^2(\mathbb{R}\times(0,\infty))$ satisfies the inequality (5) with the initial condition (2). Then the the initial-value problem (1), (2) is asymptotically stable in the sense of Hyers-Ulam-Rassias.

Proof. It follows from Theorem 4, and letting $t\to\infty$, in (18), we infer that $\lim_{t\to\infty}\left(u(x,t)-w(x,t)\right)=0.$

Remark Using similar arguments it can be shown that the initial-value problem (3), (4) is asymptotically stable in the sense of HURG.

Example 2 We find the solution of the Cauchy problem

$$4u_t=\Delta u \quad t>0 \quad x\in\mathbb{R}^n \tag{19}$$

$$u(x,0)=\mathrm{e}^{-|x|^2/2},\ x\in\mathbb{R}^n \tag{20}$$

Applying the same argument used in the proof of the Theorem 4 to the inequality

$$-\frac{\varepsilon}{\left(\pi(t+1)\right)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{t+1}} \le u_t-\frac{1}{4}\Delta u \le \frac{\varepsilon}{\left(\pi(t+1)\right)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{t+1}}$$

we get

$$\left|u(x,t)-\frac{1}{\left(\sqrt{\pi t}\right)^n}\int_{\mathbb{R}^n}\mathrm{e}^{-|x|^2/2}\mathrm{e}^{-|x-\lambda|^2/t}\mathrm{d}\lambda\right| \le \frac{\varepsilon\mathrm{e}^{-t}}{\left(\pi(t+1)\right)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{t+1}} \tag{21}$$

One can show that the function

$$w(x,t)=\frac{1}{(\pi t)^{n/2}}\int_{\mathbb{R}^n}\mathrm{e}^{-|x|^2/2}\mathrm{e}^{-|x-\lambda|^2/t}\mathrm{d}\lambda. \tag{22}$$

is a solution of the problem (19), (20).

Or, equivalently

$$w(x,t)=\prod_{k=1}^{n}\frac{1}{(\pi t)^{1/2}}\int_{-\infty}^{\infty}\mathrm{e}^{-x_k^2/2}\mathrm{e}^{-(x_k-\lambda_k)^2/t}\mathrm{d}t$$

Now, using the change of variables $z_k=\left(\frac{1+2t}{2t}\right)^{n/2}\left(\lambda_k-\frac{x_k}{1+2t}\right)$ in the integral

$$I(x_k)=\frac{1}{(\pi t)^{1/2}}\int_{-\infty}^{\infty}\mathrm{e}^{-x_k^2/2}\mathrm{e}^{-(x_k-\lambda_k)^2/t}\mathrm{d}t$$

we obtain the integral

$$I(x_k)=\frac{1}{(1+2t)^{1/2}}\int_{-\infty}^{\infty}\mathrm{e}^{-x_k^2/2(1+2t)}\mathrm{d}\lambda_k,\ t>0$$

Therefore we have

$$w(x,t)=\frac{\mathrm{e}^{-|x|^2/2(1+2t)}}{(1+2t)^{n/2}} \tag{23}$$

It is clear that $w(x,0)=\mathrm{e}^{-|x|^2/2}$.

Hence, from (21) and (23) we get

$$\left|u(x,t)-\frac{\mathrm{e}^{-|x|^2/2(1+2t)}}{(1+2t)^{n/2}}\right| \le \frac{\varepsilon\mathrm{e}^{-t}}{\left(\pi(t+1)\right)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{t+1}}$$

Hence the initial value problem (19), (20) is stable in the sense of HURG. Moreover, since $\lim_{t\to\infty}\left(u(x,t)-\frac{\mathrm{e}^{-|x|^2/2(1+2t)}}{(1+2t)^{n/2}}\right)=0$, then problem (19), (20) is asymptotically stable in the sense of HURG.

3. A Modified Hyers-Ulam-Rassias Stability for Problem of Heat Propagation in a Finite Rod

In this section we show how Laplace transform method can be used to esatblish the Hyers-Ulam-Rassias-Gavruta (HURG) stability of solution for heat equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \quad t>0,\ 0<x<l \tag{24}$$

with the initial condition

$$u(x,0)=\mu(x),\ 0\le x\le l \tag{25}$$

and the boundary conditions

$$u(0,t)=\upsilon_1(t),\ u_x(0,t)=\upsilon_2(t),\ t\ge 0 \tag{26}$$

where $\mu(x)\in C(-\infty,\infty)$, and

$$u(x,t)\in C_1^2(\mathbb{R}\times(0,\infty)).$$

We introduce the notation

$$\mathcal{L}\left[u(x,t)\right]=U(x,p),$$

where $\mathcal{L}\left[u(x,t)\right]=\int_0^\infty u(x,t)\mathrm{e}^{-pt}\mathrm{d}t.$

Theorem 5 If $u(x,t)\in C_1^2\left(\mathbb{R}\times(0,\infty)\right)$, then the initial-boundary value problem (24-26) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Given $\varepsilon>0$, Suppose $u(x,t)$ is an approximate solution of the initial value problem (24)-(26). We show that there exists an exact solution $w(x,t)\in C_1^2\left(\mathbb{R}\times(0,\infty)\right)$ satisfying the Equation (24) such that

$$\left|u(x,t)-w(x,t)\right|\le K\varepsilon$$

where k is a constant that does not explicitly depend on ε nor on $u(x,t)$.

From the definition of Hyers-Ulam stability we have

$$-\varepsilon\alpha\left(t-\frac{pl^2}{a^2}\right)\le\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}\le\varepsilon\alpha\left(t-\frac{l^2}{a^2}\right)\qquad(27)$$

where $\alpha(t-c)=0$, for $t<c$ and $\alpha(t-c)=1$, for $t>c$, $c\ge 0$.

By applying the Laplace transform to (26), (27) we obtain

$$\left|\mathcal{L}\left[\frac{\partial u}{\partial t}\right]-a^2\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right]\right|\le\varepsilon\mathcal{L}\left[\alpha\left(t-\frac{l^2}{a^2}\right)\right]\qquad(28)$$

and

$$\mathcal{L}\left[u_x(0,t)\right]=U_x(0,p)=N_1(P),$$
$$\mathcal{L}\left[u(0,t)\right]=U(0,p)=N_2(P),$$

Assuming the operation of differentiation with respect to x is interchangeable with integration with respect to t in Laplace transform, we will get

$$\begin{aligned}\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right]&=\int_0^\infty\frac{\partial^2 u}{\partial x^2}\mathrm{e}^{-pt}\mathrm{d}t=\frac{\partial^2}{\partial x^2}\left(\int_0^\infty u(x,t)\mathrm{e}^{-pt}\mathrm{d}t\right)\\&=\frac{\mathrm{d}^2U(x,p)}{\mathrm{d}x^2}\end{aligned}\qquad(29)$$

We also have

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right]=pU(x,p)-u(x,0)\qquad(30)$$

From the inequality (28), and using (29), (30) it follows that

$$\left|\frac{\mathrm{d}^2U}{\mathrm{d}x^2}-\frac{p}{a^2}U+\frac{1}{a^2}\mu(x)\right|\le\frac{\varepsilon}{pa^2}\exp\left(-\frac{pl^2}{a^2}\right)\qquad(31)$$

Integrating twice inequality (31) from 0 to x, we have

$$\begin{aligned}&-\frac{\varepsilon x^2}{pa^2}\exp\left(-\frac{pl^2}{a^2}\right)\\&\le U(x,p)-\frac{\mathrm{d}U(0,p)}{\mathrm{d}x}x-U(0,p)\\&\quad-\frac{p}{a^2}\int_0^x U(s,p)(x-s)\mathrm{d}s+\frac{1}{a^2}\int_0^x\mu(s)(x-s)\mathrm{d}s\\&\le\frac{\varepsilon x^2}{pa^2}\exp\left(-\frac{pl^2}{a^2}\right)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}U_x(0,p)&=N_1(P),\\U(0,p)&=N_2(P)\end{aligned}\qquad(32)$$

One can easily verify that the function $W(x,p)=\mathcal{L}\left[w(x,t)\right]$ which is given by

$$W(x,p)=N_1(P)x+N_2(P)+\frac{p}{a^2}\int_0^x W(s,p)(x-s)\mathrm{d}s-\frac{1}{a^2}\int_0^x\mu(s)(x-s)\mathrm{d}s$$

has to satisfy the the equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2}-\frac{p}{a^2}U+\frac{1}{a^2}\mu(x)=0$$

with boundary condition (32).

Now consider the difference $\Delta=\left|U(x,p)-W(x,p)\right|$

$$\begin{aligned}\Delta&\le\left|U(x,p)-N_1(P)x-N_2(P)-\frac{p}{a^2}\int_0^x U(s,p)(x-s)\mathrm{d}s+\frac{1}{a^2}\int_0^x\mu(s)(x-s)\mathrm{d}s\right|+\frac{p}{a^2}\int_0^x\left|\left[U(s,p)-W(s,p)\right](x-s)\right|\mathrm{d}s\\&\le\frac{\varepsilon l^2}{pa^2}\exp\left(-\frac{pl^2}{a^2}\right)+\frac{p}{a^2}\int_0^x\left|\left[U(s,p)-W(s,p)\right](x-s)\right|\mathrm{d}s\end{aligned}$$

Using Gronwall's inequality, we get the estimation

$$\left|U(x,p)-W(x,p)\right|\le\frac{\varepsilon l^2}{pa^2}\exp\left(-\frac{pl^2}{2a^2}\right)$$

Or, equivalently

$$-\mathcal{L}\left\{\alpha\left(t-\frac{l^2}{2a^2}\right)\right\}\frac{\varepsilon l^2}{a^2}\le\mathcal{L}\left\{\left[u(x,t)\right]-\left[w(x,t)\right]\right\}\le\frac{\varepsilon l^2}{a^2}\mathcal{L}\left\{\alpha\left(t-\frac{l^2}{2a^2}\right)\right\}$$

Consequently, we have

$$\max_{0\le x\le l}\left|u(x,t)-w(x,t)\right|\le\frac{\varepsilon l^2}{a^2}\alpha\left(t-\frac{l^2}{2a^2}\right)$$

Hence the initial-boundary value problem (24)-(26) is stable in the sense of HURG.

Example 3 Consider the problem

$$\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2}\qquad t>0,\ 0<x<4 \tag{33}$$

with the initial condition

$$u(x,0)=\cos x,\ 0\le x\le 4 \tag{34}$$

with the boundary conditions

$$u(0,t)=0,\ u_x(0,t)=0,\ t\ge 0 \tag{35}$$

By the definition of HURG stability we have

$$-\varepsilon\alpha(t-4)\le\frac{\partial u}{\partial t}-4\frac{\partial^2 u}{\partial x^2}\le\varepsilon\alpha(t-4) \tag{36}$$

By applying the Laplace transform to (36) we obtain

$$-\frac{\varepsilon}{p}\exp(-4p)\le\frac{\mathrm{d}^2U}{\mathrm{d}x^2}-\frac{p}{4}U+\frac{\cos x}{4}\le\frac{\varepsilon}{4p}\exp(-4p) \tag{37}$$

Integrating twice inequality (37) from 0 to x, we have

$$\left|U(x,p)-\frac{p}{4}\int_0^x U(s,p)(x-s)\mathrm{d}s-\frac{\cos x}{4}\right|\le\frac{4\varepsilon}{p}\exp(-4p)$$

with the boundary conditions

$$U(0,p)=0,\ U_x(0,p)=0$$

It is easily to verify that the function

$$W(x,p)=\frac{p}{4}\int_0^x W(s,p)(x-s)\mathrm{d}s+\frac{\cos x}{4}$$

satisfies the boundary value problem

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2}-\frac{p}{4}U+\frac{\cos x}{4}=0$$

$$U(0,p)=0,\ U_x(0,p)=0$$

Now consider the difference

$$\left|U(x,p)-W(x,p)\right|$$

$$\le\left|U(x,p)-\frac{p}{4}\int_0^x U(s,p)(x-s)\mathrm{d}s-\frac{\cos x}{4}\right|+\frac{p}{4}\int_0^x\left|\left[U(s,p)-W(s,p)\right](x-s)\right|\mathrm{d}s$$

$$\le\frac{4\varepsilon}{p}\exp(-4p)+\frac{p}{4}\int_0^x\left|\left[U(s,p)-W(s,p)\right](x-s)\right|\mathrm{d}s$$

Hence, we get the estimation

$$\left|U(x,p)-W(x,p)\right|\le\frac{4\varepsilon}{p}\exp(-2p)$$

Or, equivalently

$$-4\varepsilon\mathcal{L}\{\alpha(t-2)\}\le\mathcal{L}\{[u(x,t)]-[w(x,t)]\}\le 4\varepsilon\mathcal{L}\{\alpha(t-2)\}$$

Consequently, we have

$$\max_{0\le x\le l}\left|u(x,t)-w(x,t)\right|\le 4\varepsilon\alpha(t-2)$$

Hence the initial-boundary value problem (33)-(35) is stable in the sense of HURG.

REFERENCES

[1] D. H. Hyers, "On the Stability of the Linear Functional Equation," *Proceedings of the National Academy of Sciences of the United States of America*, Vol. 27, No. 4, 1941, pp. 222-224. doi:10.1073/pnas.27.4.222

[2] T. M. Rassias, "On the Stability of the Linear Mapping in Banach Spaces," *Proceedings of the American Mathematical Society*, Vol. 72, No. 2, 1978, pp. 297-300. doi:10.1090/S0002-9939-1978-0507327-1

[3] P. Gavruta, "A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings," *Journal of Mathematical Analysis and Applications*, Vol. 184, No. 3, 1994, pp. 431-436. doi:10.1006/jmaa.1994.1211

[4] S. M. Jung, "On the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings," *Journal of Mathematical Analysis and Applications*, Vol. 204, No. 1, 1996, pp. 221-226. doi:10.1006/jmaa.1996.0433

[5] S. M. Jung, "Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis," Hadronic Press, Palm Harbor, 2001.

[6] T. Miura, S.-E. Takahasi and H. Choda, "On the Hyers-Ulam Stability of Real Continuous Function Valued Differentiable Map," *Tokyo Journal of Mathematics*, Vol. 24, No. 2, 2001, pp. 467-476. doi:10.3836/tjm/1255958187

[7] C. G. Park, "On the Stability of the Linear Mapping in Banach Modules," *Journal of Mathematical Analysis and Applications*, Vol. 275, 2002, pp. 711-720. doi:10.1016/S0022-247X(02)00386-4

[8] C. G. Park, "Homomorphisms between Poisson JC*-Algebras," *Bulletin of the Brazilian Mathematical Society*, Vol. 36, No. 1, 2005, pp. 79-97.

doi:10.1007/s00574-005-0029-z

[9] C. G. Park, Y.-S. Cho and M. Han, "Functional Inequalities Associated with Jordan-Von Neumanntype Additive Functional Equations," *Journal of Inequalities and Applications*, Vol. 2007, 2007, Article ID: 41820.

[10] K.-W. Jun and Y.-H. Lee, "A Generalization of the Hyers-Ulam-Rassias Stability of the Pexiderized Quadratice Quations," *Journal of Mathematical Analysis and Applications*, Vol. 297, No. 1, 2004, pp. 70-86. doi:10.1016/j.jmaa.2004.04.009

[11] C. Alsina and R. Ger, "On Some Inequalities and Stability Results Related to the Exponential Function," *Journal of Inequalities and Applications*, Vol. 2, No. 4, 1998, pp. 373-380.

[12] E. Takahasi, T. Miura and S. Miyajima, "On the Hyers-Ulam Stability of the Banach Space-Valued Differential Equation $y' = \lambda y$," *Bulletin of the Korean Mathematical Society*, Vol. 39, No. 2, 2002, pp. 309-315. doi:10.4134/BKMS.2002.39.2.309

[13] T. Miura, S. Miyajima and S.-E. Takahasi, "A Characterization of Hyers-Ulam Stability of First Order Linear Differential Operators," *Journal of Mathematical Analysis and Applications*, Vol. 286, No. 1, 2007, pp. 136-146. doi:10.1016/S0022-247X(03)00458-X

[14] S. M. Jung, "Hyers-Ulam Stability of Linear Differential Equations of First Order," *Journal of Mathematical Analysis and Applications*, Vol. 311, No. 1, 2005, pp. 139-146. doi:10.1016/j.jmaa.2005.02.025

[15] G. Wang, M. Zhou and L. Sun, "Hyers-Ulam Stability of Linear Differential Equations of First Order," *Applied Mathematics Letters*, Vol. 21, No. 10, 2008, pp. 1024-1028. doi:10.1016/j.aml.2007.10.020

[16] Y. Li, "Hyers-Ulam Stability of Linear Differential Equations," *Thai Journal of Mathematics*, Vol. 8, No. 2, 2010, pp. 215-219.

[17] Y. Li and Y. Shen, "Hyers-Ulam Stability of Nonhomogeneous Linear Differential Equations of Second Order, Internat," *Journal of Mathematical Sciences*, Vol. 2009, 2009, Article ID: 576852.

[18] P. Gavruta, S. Jung and Y. Li, "Hyers-Ulam Stability for Second-Order Linear Differential Equations with Boundary Conditions," *Electron Journal of Differential Equations*, Vol. 2011, No. 80, 2011, pp. 1-7.

[19] S. M. Jung, "Hyers-Ulam Stability of Linear Partial Differential Equations of First Order," *Applied Mathematics Letters*, Vol. 22, No. 1, 2009, pp. 70-74. doi:10.1016/j.aml.2008.02.006

[20] M. E. Gordji, Y. J. Cho, M. B. Ghaemi and B. Alizadeh, "Stability of the Exact Second Order Partial Differential Equations," *Journal of Inequalities and Applications*, Vol. 2011, 2011, Article ID: 306275. doi:10.1186/1029-242X-2011-81

[21] N. Lungu and C. Craciun, "Ulam-Hyers-Rassias Stability of a Hyperbolic Partial Differential Equation," *ISRN Mathematical Analysis*, Vol. 2012, 2012, Article ID: 609754.

[22] L. C. Evans, "Partial Differential Equations," 2nd Edition, Graduate Studies in Mathematics, V19 AMS, 2010.

