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# Common Fixed Point Theorems of Multi-Valued Maps in Ultra Metric Space

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## ABSTRACT

We establish some results on coincidence and common fixed point for a two pair of multi-valued and single-valued maps in ultra metric spaces.

**Keywords:** Multi-Valued Maps; Coincidence Point; Common Fixed Point

## 1. Introduction

Roovij in [1] introduced the concept of ultra metric space. Later, C. Petalas, F. Vidalis [2] and Ljiljana Gajic [3] studied fixed point theorems of contractive type maps on a spherically complete ultra metric spaces which are generalizations of the Banach fixed point theorems. In [4] K. P. R. Rao, G. N. V. Kishore and T. Ranga Rao obtained two coincidence point theorems for three or four self maps in ultra metric space.

J. Kubiaczyk and A. N. Mostafa [5] extend the fixed point theorems from the single-valued maps to the set-valued contractive maps. Then Gajic [6] gave some generalizations of the result of [3]. Again, Rao [7] proved some common fixed point theorems for a pair of maps of Jungck type on a spherically complete ultra metric space.

In this article, we are going to establish some results on coincidence and common fixed point for two pair of multi-valued and single-valued maps in ultra metric spaces.

## 2. Basic Concept

First we introducing a notation.

Let $C(X)$ denote the class of all non empty compact subsets of $X$. For $A, B \in C(X)$, the Hausdorff metric is defined as

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

where $d(x,A) = \inf\{d(x,a) : a \in A\}$.

The following definitions will be used later.

**Definition 2.1** ([1]) Let $(X,d)$ be a metric space. If the metric $d$ satisfies strong triangle inequality

$$d(x,y) \le \max\{d(x,z), d(z,y)\}, \forall x,y,z \in X$$

Then $d$ is called an ultra metric on $X$ and $(X,d)$ is called an ultra metric space.

**Example.** Let $X \ne \theta$, $d(x,y) = \begin{cases} 0, x = y \\ 1, x \ne y \end{cases}$, then $(X,d)$ is a ultra metric space.

**Definition 2.2** ([1]) An ultra metric space is said to be spherically complete if every shrinking collection of balls in $X$ has a non empty intersection.

**Definition 2.3** An element $x \in X$ is said to be a coincidence point of $T : X \to C(X)$ and $f : X \to X$ if $fx \in Tx$. We denote

$$C(f,T) = \{x \in X \mid fx \in Tx\}$$

the set of coincidence points of $T$ and $f$.

**Definition 2.4** ([7]) Let $(X,d)$ be an ultra metric space, $f : X \to X$ and $T : X \to C(X)$. $T$ and $f$ are said to be coincidentally commuting at $z \in X$ if $fz \in Tz$ implies $fTz \subseteq Tfz$.

**Definition 2.5** ([8]) An element $x \in X$ is a common fixed point of $T,S : X \to C(X)$ and $f : X \to X$ if $x = fx \in Tx \cap Sx$.

## 3. Main Results

The following results are the main result of this paper.

**Theorem 3.1** Let $(X,d)$ be an ultra metric space. Let $T,S : X \to C(X)$ be a pair of multi-valued maps and $f,g : X \to X$ a pair of single-valued maps satisfying

(a) $fg(X)$ is spherically complete;

(b) $H(Sx,Ty) < \max\{d(fx,gy), d(fx,Sx), d(gy,Ty)\}$

for all $x,y \in X$, with $fx \ne gy$;

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(c) $fS = Sf$, $fg = gf$, $fT = Tf$, $gS = Sg$, $gT = Tg$, $ST = TS$;

(d) $S(X) \subseteq f(X), T(X) \subseteq g(X)$.

Then there exist point $u$ and $v$ in $X$, such that

$$fu \in Su, gv \in Tv, fu = gv, Su = Tv.$$

**Proof.** Let

$$B_a = \left( fga; \max\left\{ d(fga, Sga), d(fga, Tfa) \right\} \right)$$

denote the closed sphere with centered $fga$ and radius

$$\max\left\{ d(fga, Sga), d(fga, Tfa) \right\}.$$

Let $A$ be the collection of all the spheres for all $a \in fg(X)$.

Then the relation

$$B_a \le B_b \quad \text{if} \quad B_b \subseteq B_a$$

is a partial order on $A$.

Consider a totally ordered sub family $A_1$ of $A$. Since $fg(X)$ is spherically complete, we have

$$\bigcap_{B_a \in A_1} B_a = B \ne \varnothing$$

Let $fgb \in B$ where $b \in fg(X)$ and $B_a \in A_1$. Then $fgb \in B_a$. Hence

$$d(fgb, fga) \le \max\left\{ d(fga, Sga), d(fga, Tfa) \right\} \tag{1}$$

If $a = b$ then $B_a = B_b$. Assume that $a \ne b$. Let $\forall x \in B_b$, then

$$d(x, fgb) \le \max\left\{ d(fgb, Sgb), d(fgb, Tfb) \right\}$$

Since $Sga$ is nonempty compact set, then $\exists p \in Sga$ such that

$$d(fga, Sga) = d(fga, p);$$

$Tfa$ is a nonempty compact set, then $\exists q \in Tfa$ such that $d(fga, Tfa) = d(fga, q)$.

$$\begin{aligned}
&\max\left\{ d(fgb, Sgb), d(fgb, Tfb) \right\} = \max\left\{ \inf_{c \in Sgb} d(fgb, c), \inf_{d \in Tfb} d(fgb, d) \right\} \\
&\le \max\left\{ d(fgb, fga), d(fga, q), \inf_{c \in Sgb} d(q, c), d(fgb, fga), d(fga, p), \inf_{d \in Tfb} d(p, d) \right\} \\
&\le \max\left\{ d(fgb, fga), d(fga, Tfa), H(Tfa, Sgb), d(fgb, fga), d(fga, Sga), H(Sga, Tfb) \right\} \\
&< \max\left\{ d(fgb, fga), d(fga, Tfa), d(fga, Sga), \max\left\{ d(fgb, fga), d(gfa, Tfa), d(fgb, Sgb) \right\}, \right. \\
&\qquad \left. \max\left\{ d(fga, fgb), d(fga, Sga), d(gfb, Tfb) \right\} \right\} \\
&\le \max\left\{ d(fga, Tfa), d(fga, Sga) \right\}
\end{aligned}$$

from (a) (b) and Equation (1)

Now

$$\begin{aligned}
d(x, fga) &\le \max\left\{ d(x, fgb), d(fgb, fga) \right\} \\
&\le \max\left\{ d(fga, Sga), d(fga, Tfa) \right\}
\end{aligned}$$

So $x \in B_a$, we have just proved that $B_b \subseteq B_a$ for every $B_a \in A_1$. Thus $B_b$ is an upper bound in $A$ for the family $A_1$ and hence by Zorn's Lemma, there is a maximal element in $A$, say $B_z, z \in fg(X)$. There exists $w \in X$ such that $z = fgw$.

Suppose

$$f(gfgw) \notin S(gfgw), g(ffgw) \notin T(ffgw).$$

Since $Sgfgw, Tffgw$ are nonempty compact sets, then $\exists k \in Sgfgw, t \in Tffgw$ such that

$$d(fgfgw, Sgfgw) = d(fgfgw, k), \tag{2}$$

$$d(fgfgw, Tffgw) = d(fgfgw, t) \tag{3}$$

From (b), (c) and Equation (2), we have

$$\begin{aligned}
&d(Sgfgw, TSfgw) = \inf_{e \in TSfgw} d(gSfgw, e) \le \max\left\{ d(Sgfgw, fgfgw), d(fgfgw, k), \inf_{e \in TSfgw} d(k, e) \right\} \\
&\le \max\left\{ d(Sgfgw, fgfgw), H(Sgfgw, TSfgw) \right\} \\
&< \max\left\{ d(Sgfgw, fgfgw), \max\left\{ d(fgfgw, fSfgw), d(fgfgw, Sgfgw), d(gSfgw, TSfgw) \right\} \right\} \\
&= d(fgfgw, Sgfgw)
\end{aligned} \tag{4}$$

$$\begin{aligned}
&d(Tffgw, STfgw) = \inf_{h \in STfgw} d(fTfgw, h) \le \max\left\{ d(Tffgw, fgfgw), d(fgfgw, t), \inf_{h \in STfgw} d(t, h) \right\} \\
&\le \max\left\{ d(Tffgw, fgfgw), H(Tffgw, STfgw) \right\} \\
&< \max\left\{ d(Tffgw, fgfgw), \left\{ \max d(fTfgw, gffgw), d(fTfgw, STfgw), d(gffgw, Tffgw) \right\} \right\} = d(Tffgw, fgfgw)
\end{aligned} \tag{5}$$

 

From (b), (c) and Equations (2)-(5)

$$\begin{aligned} d(fggSw,SggSw) &= \inf_{m\in SggSw} d(fggSw,m) \\ &\le \max\left\{d(fggSw,TSfgw),d(TSfgw,Tffgw),d(Tffgw,fgfgw),d(fgfgw,t),\inf_{m\in SggSw} d(t,m)\right\} \\ &\le \max\left\{d(fgfgw,Sgfgw),d(fgfgw,Tffgw),H(Tffgw,SggSw)\right\} \\ &< \max\left\{d(fgfgw,Sgfgw),d(fgfgw,Tffgw),\max\left\{d(fggSw,gffgw),d(fggSw,SggSw),d(gffgw,Tffgw)\right\}\right\} \\ &\le \max\left\{d(fgfgw,Sgfgw),d(fgfgw,Tffgw)\right\} \end{aligned} \tag{6}$$

$$\begin{aligned} d(fgfTw,TffTw) &= \inf_{n\in TffTw} d(fgfTw,n) \\ &\le \max\left\{d(fgfTw,STfgw),d(STfgw,Sgfgw),d(Sgfgw,fgfgw),d(fgfgw,k),\inf_{n\in TffTw} d(k,n)\right\} \\ &\le \max\left\{d(fgfgw,Tffgw),d(fgfgw,Sgfgw),H(Sgfgw,TffTw)\right\} \\ &< \max\left\{d(fgfgw,Tffgw),d(fgfgw,Sgfgw),\max\left\{d(fgfgw,gffTw),d(fgfgw,Sgfgw),d(gffTw,TffTw)\right\}\right\} \\ &\le \max\left\{d(fgfgw,Tffgw),d(fgfgw,Sgfgw)\right\} \end{aligned} \tag{7}$$

From Equation (4) and Equation (6) we have

$$\max\left\{d(Sgfgw,TSfgw),d(fggSw,SggSw)\right\} < \max\left\{d(fgfgw,Sgfgw),d(fgfgw,Tffgw)\right\} \tag{8}$$

From Equation (5) and Equation (7) we have

$$\max\left\{d(STfgw,Tffgw),d(fgfTw,TffTw)\right\} < \max\left\{d(fgfgw,Tffgw),d(fgfgw,Sgfgw)\right\} \tag{9}$$

If

$$\max\left\{d(fgfgw,Sgfgw),d(fgfgw,Tffgw)\right\} = d(fgfgw,Sgfgw)$$

Then from Equation (8), $fgfgw \notin B_{gSw} \Rightarrow fgz \notin B_{gSw}$. Hence $B_z \not\subset B_{gSw}$. It is a contradiction to the maximality of $B_z$ in $A$, since $gSw \subseteq gf(X) = fg(X)$

If

$$\max\left\{d(fgfgw,Sgfgw),d(fgfgw,Tffgw)\right\} = d(fgfgw,Tffgw)$$

Then from Equation (9), $fgfgw \notin B_{fTw} \Rightarrow fgz \notin B_{fTw}$. Hence $B_z \not\subset B_{fTw}$. It is a contradiction to the maximality of $B_z$ in $A$, since $fTw \subseteq fg(X)$.

So

$$f(gfgw) \in S(gfgw), g(ffgw) \in T(ffgw) \tag{10}$$

In addition, $f(gfgw) = g(ffgw)$.

Using (b), (c) and Equation (10), we obtain

$$\begin{aligned} H(Sgfgw,Tffgw) &< \max\left\{d(fgfgw,gffgw), d(fgfgw,Sgfgw),d(gffgw,Tffgw)\right\} \\ &= 0 \end{aligned}$$

Hence $S(gfgw) = T(ffgw)$.

Then the proof is completed.

**Theorem 3.2** Let $(X,d)$ be an ultra metric space. Let $T,S : X \to C(X)$ be a pair of multi-valued maps and $f : X \to X$ be a single-valued maps satisfying

(a) $f(X)$ is spherically complete;

(b) $H(Sx,Ty) < \max\left\{d(fx,fy),d(fx,Sx),d(fy,Ty)\right\}$

for all $x, y \in X$ ,with $x \ne y$;

(c) $fS = Sf$, $fT = Tf$, $ST = TS$;

(d) $S(X) \subseteq f(X), T(X) \subseteq f(X)$.

Then $f, S$ and $T$ have a coincidence point in $X$.

Moreover, if $f$ and $S$, $f$ and $T$ are coincidentally commuting at $z \in C(f,T)$ and $ffz = fz$, then $f, S$ and $T$ have a common fixed point in $X$.

**Proof.** If $f = g$ in Theorem 2.1, we obtain that there exist points $u$ and $v$ in $X$ such that

$$fu \in Su, fv \in Tv, fu = fv, Su = Tv.$$

As $u \in C(f,S)$, $f$ and $S$ ipipare coincidentally commuting at $u$ and $ffu = fu$.

Write $w = fu$, then $w \in Su, w \in Tv$.

Then we have

$$fw = w$$

and

$$w = fw \in f(Su) \subseteq S(fu) = Sw.$$

Now, since also $u \in C(f,T)$, $f$ and $T$ are coincidentally commuting at $u$ and $ffu = fu$, so we obtain

$$w = fw \in f(Tv) \subseteq T(fv) = Tw.$$

Thus, we have proved that $w = fw \in Sw \cap Tw$, that is, $w$ is a common fixed point of $f$, $S$ and $T$.

**Corollary 3.3** Let $(X,d)$ be a spherically complete ultra metric space. Let $T,S : X \to C(X)$ be a pair of multi-valued maps satisfying

(a) $H(Sx,Ty) < \max\{d(x,y), d(x,Sx), d(y,Ty)\}$ for all $x, y \in X$ ,with $x \neq y$;

(b) $ST = TS$.

Then, there exists a point $z$ in $X$ such that $z \in Sz \cap Tz$ and $Sz = Tz$.

**Remark 1** If $S = T$ in Corollary 3.3, then we obtain the Theorem of Ljiljana Gajic [6].

**Remark 2** If in Theorem 3.1, $S = T, f = g$, we obtain Theorem 9 of K. P. R. Rao at [7].

**Remark 3** If $S$ and $T$ in Theorem 3.1 are single-valued maps, then: 1) we obtain the results of K. P. R. Rao [4]; 2) $S = T, f = g = I$, we obtain the result of Ljiljana Gajic [3]; 3) $S = T, f = g$, then, we obtain Theorem 4 of K. P. R. Rao at [7].

## 4. Conclusion

In this paper, we get coincidence point theorems and common fixed point theorems for two pair of multi-valued and single-valued maps satisfying different contractive conditions on spherically complete ultra metric space, which is generalized results of [3-7].

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## REFERENCES

[1] A. C. M. van Roovij, "Non Archimedean Functional Analysis," Marcel Dekker, New York, 1978.

[2] C. Petalas and F. Vidalis, "A Fixed Point Theorem in Non-Archimedaen Vector Spaces," *Proceedings of the American Mathematics Society*, Vol. 118, 1993, pp. 819-821. doi:10.1090/S0002-9939-1993-1132421-2

[3] L. Gajic, "On Ultra Metric Spaces," *Novi Sad Journal of Mathematics*, Vol. 31, No. 2, 2001, pp. 69-71.

[4] K. P. R. Rao, G. N. V. Kishore and T. Ranga Rao, "Some Coincidence Point Theorems in Ultra Metric Spaces," *International Journal of Mathematical Analysis*, Vol. 1, No. 18, 2007, pp. 897-902.

[5] J. Kubiaczyk and A. N. Mostafa, "A Multi-Valued Fixed Point Theorem in Non-Archimedean Vector Spaces," *Novi Sad Journal of Mathematics*, Vol. 26, No. 2, 1996, pp. 111-116.

[6] L. Gajic, "A Multivalued Fixed Point Theorem in Ultra Metric Spaces," *Matematicki Vesnik*, Vol. 54, No. 3-4, 2002, pp. 89-91.

[7] K. P. R. Rao and G. N. V. Kishore, "Common Fixed Point Theorems in Ultra Metric Spaces," *Journal of Mathematics*, Vol. 40, 2008, pp. 31-35.

[8] B. Damjanovic, B. Samet and C. Vetro, "Common Fixed Point Theorem for Multi-Valued Maps," *Acta Mathematica Scientia*, Vol. 32, No. 2, 2012, pp. 818-824. doi:10.1016/S0252-9602(12)60063-0

 