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A Geometrical Characterization of Spatially Curved Roberstion-Walker Space and Its Retractions

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ABSTRACT

Our aim in the present article is to introduce and study new types of retractions of closed flat Robertson-Walker W^4 model. Types of the deformation retract of closed flat Robertson-Walker W^4 model are obtained. The relations between the retraction and the deformation retract of curves in W^4 model are deduced. Types of minimal retractions of curves in W^4 model are also presented. Also, the isometric and topological folding in each case and the relation between the deformation retracts after and before folding have been obtained. New types of homotopy maps are deduced. New types of conditional folding are presented. Some commutative diagrams are obtained.

Keywords: Retraction; Deformation Retracts; Foldings; Flat Robertson-Walker Model

1. Introduction

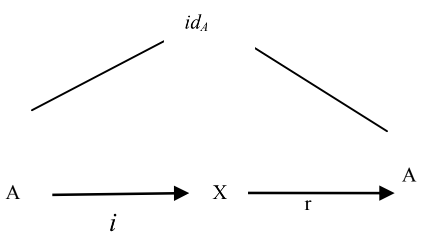
As is well known, the theory of retractions is always one of interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry El-Ahmady [1].

El-Ahmady [1-13] studied the variation of the density function on chaotic spheres in chaotic space-like Minkowski space time, folding of fuzzy hypertori and their retractions, limits of fuzzy retractions of fuzzy hyperspheres and their foldings, fuzzy folding of fuzzy horocycle, fuzzy Lobachevskian space and its folding, the deformation retract and topological folding of Buchdahi space, retraction of chaotic Ricci space, a calculation of geodesics in chaotic flat space and its folding, fuzzy deformation retract of fuzzy horospheres, on fuzzy spheres in fuzzy Minkowski space, retractions of spatially curved Robertson-Walker space, a calculation of geodesics in flat Robertson-Walker space and its folding, and retraction of chaotic black hole.

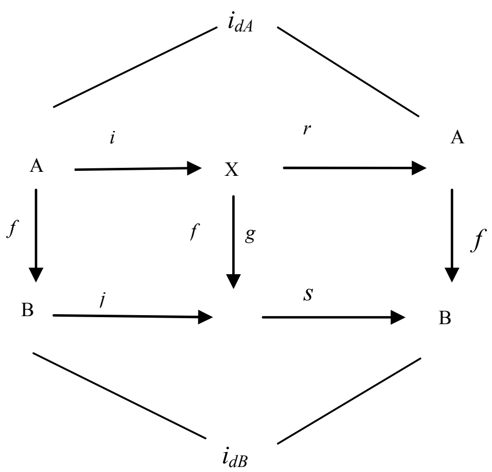
An n-dimensional topological manifold M is a Hausdorff topological space with a countable basis for the topology which is locally homeomorphic to R^n. If $h: U \to U'$ is a homeomorphism of $U \subseteq M$ onto $U' \subseteq R^n$, then h is called a chart of M and U is the associated chart domain. A collection (h_{α,U_α}) is said to be an atlas for M if $\cup_{\alpha \in A} U_\alpha = M$. Given two charts h_α, h_β such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \varnothing$, the transformation chart $h_\beta o h_\alpha^{-1}$ between open sets of R^n is defined, and if all of these charts transformation are c^∞-mappings, then the manifolds under consideration is a c^∞-manifolds. A differentiable structure on M is a differentiable atlas and a differentiable manifolds is a topological manifolds with a differentiable structure Arkowitz [14] Banchoff [15], Dubrovin [16], Kuhnel [17], Montiel [18].

Most folding problems are attractive from a pure mathematical standpoint, for the beauty of the problems themselves. The folding problems have close connections to important industrial applications Linkage folding has applications in robotics and hydraulic tube bending. Paper folding has application in sheet-metal bending, packaging, and air-bag folding Demainel [19]. Following the great Soviet geometer Pogorelov [20], also, used folding to solve difficult problems related to shell structures in civil engineering and aero space design, namely buckling instability El Naschie [21]. Isometric folding between two Riemannian manifold may be characterized as maps that send piecewise geodesic segments to a piecewise geodesic segments of the same length El-Ahmady [4]. For a topological folding the maps do not preserves lengths El-Ahmady [5,6].

A subset A of a topological space X is called a retract of X if there exists a continuous map $r: X \to A$ such that $r(a) = a$, $\forall a \in A$ where A is closed and X is open El-Ahmady [3,7]. Also, let X be a space and A a subspace. A map $r: X \to A$ such that $r(a) = a$, for all $a \in A$, is called a retraction of X onto A and A is the called a retract of X This can be re stated as follows. If $i: A \to X$ is the inclusion map, then $r: X \to A$ is a map such that $ri = id_A$. If, in addition, $ri \simeq id_X$, we call r a deformation retract and A a deformation retract of *X* Another simple-but extremely useful-idea is that of a retract. If $A, X \subset M$, then A is a retract of X if there is a commutative diagram.



If $f: A \to B$ and $g: X \to Y$, then f is a retract of g if there is a commutative diagram Arkowitz [14], Naber [22], Shick [23] and Strom [24].



2. Main Results

The flat Robertson-Walker W^4 Line element $ds^2 = -dt^2 + a^2(t)\left(dX^2 + dY^2 + dZ^2\right)$ is one example of a homogeneous isotropic cosmological spacetime geometry, but not the only one. The general Robertson-Walker W^4 Line element for a homogeneous isotropic universe has the form $ds^2 - dt^2 + a^2(t)dl^2$ where dl^2 is the line element of a homogeneous, isotropic three-dimensional space. There are only three possibilities for this. Let's now look at the closed flat Robertson-Walker W^4 model. In the present work we give first some rigorous definitions of retractions, folding and deformation retraction as well as important theorems of closed flat Robertson-Walker W^4 model. In what follows, we would like to introduce the types of retraction, folding and deformation retraction of closed flat Robertson-Walker W^4 model El-Ahmady [11,12], Hartle [25], Straumann [26] with metric

$$dl^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

The coordinate of closed flat Robertson-Walker space W^4 are

$$\begin{aligned} x_1 &= \sin\chi\sin\theta\cos\phi,\ x_3 = \sin\chi\cos\theta, \\ x_2 &= \sin\chi\sin\theta\sin\phi,\ x_4 = \cos\chi, \end{aligned} \tag{2}$$

where the range of the three polar angles (χ,θ,ϕ) is given by $0 \le \chi \le \pi, 0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$

Now, we use Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{\varphi}_i}\right) - \frac{\partial T}{\partial \varphi_i} = 0, i = 1,2,3$$

To find a geodesic which is a subset of the closed flat Robertson-Walker space W^4. Since

$$T = \frac{1}{2}\left\{\chi'^2 + \sin^2\chi\left(\theta'^2 + \sin^2\theta\phi'^2\right)\right\}$$

Then the Lagrangian equations for closed flat Robertson-Walker space W^4 are.

$$\frac{d}{ds}(\chi') - \left(\sin\chi\cos\chi\left(\theta'^2 + \sin^2\theta\phi'^2\right)\right) = 0 \tag{3}$$

$$\frac{d}{ds}\left(\sin^2\chi\theta'\right) - \left(\sin^2\chi\sin\theta\cos\theta\phi'^2\right) = 0 \tag{4}$$

$$\frac{d}{ds}\left(\sin^2\chi\sin^2\theta\phi'\right) = 0 \tag{5}$$

From Equation (5) we obtain $\sin^2\chi\sin^2\phi' =$ constant say β_1, if $\beta_1 = 0$, we obtain the following cases:

If $\theta = 0$, hence we get the coordinates of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = 0, x_2 = 0, x_3 = \sin\chi, x_4 = \cos\chi .$$

Which is the sphere s_1^1, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, it is a minimal geodesic and minimal retraction. Also, if $\theta = \frac{\pi}{6}$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \frac{1}{2}\sin\chi\cos\phi, x_2 = \frac{1}{2}\sin\chi\sin\phi,$$
$$x_3 = \frac{\sqrt{3}}{2}\sin\chi, x_4 = \cos\chi$$

Which is the hypersphere S_1^3, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, it is a minimal geodesic and minimal retraction. Again, if $\theta = \frac{\pi}{4}$ hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \frac{1}{\sqrt{2}}\sin\chi\cos\phi, x_2 = \frac{1}{\sqrt{2}}\sin\chi\sin\phi$$
$$x_3 \frac{1}{\sqrt{2}}\sin\chi, x_4 = \cos\chi$$

Which is the hypersphere S_2^3, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

it is a minimal geodesic and minimal retraction. Also, if $\theta = \frac{\pi}{3}$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \frac{\sqrt{3}}{2}\sin\chi\cos\phi, x_2 = \frac{\sqrt{3}}{2}\sin\chi\sin\phi,$$
$$x_3 = \frac{1}{2}\sin\chi, x_4 = \cos\chi$$

Which is the hypersphere S_3^3, $x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. If $\theta = \frac{\pi}{2}$ hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \sin\chi\cos\phi, x_2 = \sin\chi\sin\phi,$$
$$x_3 = 0, x_4 = \cos\chi$$

Which is the hypersphere S_1^2, $x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Again, if $\theta = \pi$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = 0, x_2 = 0, x_3 = -\sin\chi, x_4 = \cos\chi .$$

This is the sphere S_2^1, $x_1^2+x_2^2+x_3^2+x_4^2=1$ it is a minimal geodesic and minimal retraction. Also, if $\theta = 270$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = -\sin\chi\cos\phi, x_2 = -\sin\chi\sin\phi,$$
$$x_3 = 0, x_4 = \cos\chi.$$

Which is the hypersphere S_2^2, $x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. If $\phi = 0$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \sin\chi\sin\theta, x_2 = 0,$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi.$$

Which is the sphere S_3^2, $x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Again, if $\phi = \frac{\pi}{6}$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \frac{\sqrt{3}}{2}\sin\chi\sin\theta, x_2 = \frac{1}{2}\sin\chi\sin\theta,$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi.$$

Which is the hypersphere $S_4^3, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Also, if $\phi = \frac{\pi}{4}$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \frac{1}{\sqrt{2}}\sin\chi\sin\theta, x_2 = \frac{1}{\sqrt{2}}\sin\chi\sin\theta,$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi$$

Which is the hypersphere $S_5^3, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. If $\phi = \frac{\pi}{3}$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \frac{1}{2}\sin\chi\sin\theta, x_2 = \frac{1}{2}\sin\chi\sin\theta$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi$$

Which is the hypersphere $S_6^3, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Again, if $\phi = \frac{\pi}{2}$ hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = 0, x_2 = \sin\chi\sin\theta,$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi$$

Which is the hypersphere $S_4^2, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Also, if $\phi = \pi$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = -\sin\chi\sin\theta, x_2 = 0,$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi$$

Which is the sphere $S_5^2, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. If $\phi = 270$ hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = 0, x_2 = -\sin\chi\sin\theta,$$
$$x_3 = \sin\chi\cos\theta, x_4 = \cos\chi$$

This is the sphere $S_6^2, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Again, if $\chi = 0$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1 .$$

Which is the point of the hypersphere $S_7^3, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. Also, if $\chi = \frac{\pi}{2}$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = \sin\theta\cos\phi, x_2 = \sin\theta\sin\phi,$$
$$x_3 = \cos\theta, x_4 = 0$$

Which is the sphere $S_7^2, x_1^2+x_2^2+x_3^2+x_4^2=1$, it is a minimal geodesic and minimal retraction. If $\chi = \pi$

hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1.$$

Which is the point of the hypersphere $S_8^3, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, it is a minimal geodesic and minimal retraction . Also, if $\chi = 270$, hence we get the coordinate of closed flat Robertson-Walker space W^4 which are given by

$$x_1 = -\sin\theta\cos\varnothing, x_2 = -\sin\theta\sin\varnothing,$$
$$x_3 = -\cos\theta, x_4 = 0$$

Which is the sphere $S_8^2, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, it is a minimal geodesic and minimal retraction .

Theorem 1. The retractions of closed flat Robertson-Walker space W^4 are minimal geodesics and geodesic spheres.

In this position, we present some cases of deformation retract of open flat Robertson-Walker space W^4. The deformation retract of open flat Robertson-Walker space W^4 is

$$\eta : \{W^4 - \beta\} \times I \to \{W^4 - \beta\},$$

where $\{W^4 - \beta\}$ be the open flat Robertson-Walker space W^4 and is the closed interval [0, 1], be present as

$$\eta(x,h)\{(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi,$$
$$\sin\chi\cos\theta, \cos\chi - \beta\} \times I \to$$
$$\{\sin\chi\sin\theta\cos\phi\sin\chi\sin\theta\sin\varnothing, \sin\chi\cos\theta, \cos\chi - \beta\}$$

The deformation retract of the open flat Robertson-Walker space W^4 into the sphere S_1^3 is

$$\eta(m,h) = (1+h)\{(\sin\chi\sin\theta\cos\varnothing,$$
$$\sin\chi\sin\theta\sin\varnothing,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\} + \tan\frac{\pi h}{4}$$
$$= \{0,0,0,1\}$$

where

$$\eta(m,0) = \{(\sin\chi\sin\theta\cos\varnothing, \sin\chi\sin\theta\sin\varnothing,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\},$$

and

$$\eta(m,1) = \{0,0,0,1\}$$

The deformation retract of the open flat Robertson-Walker space W^4 into the sphere S_2^3 is

$$\eta(m,h) = (1 - 2h + h^2)$$
$$\{(\sin\chi\sin\theta\cos\varnothing, \sin\chi\sin\theta\sin\varnothing,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\}$$
$$+ (h^3 - 2h + 2h^2)\{0,0,\sin\chi\cos\chi\}$$

The deformation retract of the open flat Robertson-Walker space W^4 into the sphere S_3^3 is

$$\eta(m,h) = \cos\frac{\pi h}{2}\{(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi,$$
$$\sin, \sin\chi\cos\theta, \cos\chi) - \beta\}$$
$$+ \sin\frac{\pi h}{2}\{\sin\chi\ \sin, 0, \sin\chi\cos\theta, 0, \cos\chi\}$$

Now, we are going to discuss the folding $\Im$ of the open flat Robertson-Walker W^4 space

Let $\Im : W^4 \to W^4$, where

$$\Im(x_1, x_2, x_3, x_4) = (x_1, |x_2|, x_3, x_4). \tag{6}$$

An isometric folding of the open flat Robertson-Walker W^4 space into itself may be defined by

$$\Im : \{(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\} \to \{(\sin\chi\sin\theta\cos\phi,$$
$$|\sin\chi\sin\theta\sin\phi|, \sin\chi\cos\theta, \cos\chi) - \beta\}$$

The deformation retract of the folded open flat Robertson-Walker space $\Im(W^4)$ into the folded geodesic $\Im(S_1^3)$ is:

$$\eta_\Im : \{(\sin\chi\sin\theta\cos\phi, |\sin\chi\sin\theta\sin\phi|,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\} \times I \to \{(\sin\chi\sin\theta\cos\phi,$$
$$|\sin\chi\sin\theta\sin\phi|, \sin\chi\cos\theta, \cos\chi) - \beta\}$$

with

$$\eta_\Im(m,h) = (1+h)\{(\sin\chi\sin\theta\cos\phi, |\sin\chi\sin\theta\sin\phi|,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\} + \tan\frac{\pi h}{4}\{0,0,0,1\}$$

The deformation retract of the folded open flat Robertson-Walker space $\Im(W^4)$ into the folded geodesic $\Im(S_1^3)$ is:

$$\eta_\Im(m,h) = (1+h)\{(\sin\chi\sin\theta\cos\varnothing|\sin\chi\sin\theta\sin\varnothing|,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\} + (h^3 - 2h + 2h^2)$$
$$\{0,0,\sin\chi,\cos\chi\}$$

The deformation retract of the folded open flat Robertson-Walker space $\Im(W^4)$ into the folded geodesic $\Im(S_1^3)$ is:

$$\eta_\Im(m,h) = (1+h)\{(\sin\chi\sin\theta\cos\phi, |\sin\chi\sin\theta\sin\phi|,$$
$$\sin\chi\cos\theta, \cos\chi) - \beta\}$$
$$+ \sin\pi h/2\{\sin\chi\sin\theta, 0, \sin\chi\ \cos\theta, \cos\chi\}$$

Then, the following theorem has been proved.

Theorem 2. Under the defined folding and any folding homeomorphic to this type of folding, the deformation retract of the folded open flat Robertson-Walker space $\mathfrak{I}\left(W^4\right)$ into the folded geodesics is the same as the deformation retract of open flat Robertson-Walker space W^4 into the geodesics.

Now, let the folding be defined by:

$$\mathfrak{I}^*: W^4 \to W^4,$$

where

$$\mathfrak{I}^*\left(x_1, x_2, x_3, x_4\right) = \left(x_1, x_2, x_3, \left|x_4\right|\right) \tag{7}$$

The isometric folded open flat Robertson-Walker space $\mathfrak{I}\left(W^4\right)$ is:

$$\bar{R} = \left\{\left(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \left|\cos\chi\right|\right) - \beta\right\}$$

The deformation retract of the folded open flat Robertson-Walker space $\mathfrak{I}^*\left(W^4\right)$ into the folded geodesic $\mathfrak{I}^*\left(S_1^3\right)$ is:

$$\eta_{\mathfrak{I}^*} := \left\{\left(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \left|\cos\chi\right|\right) - \beta\right\} \times I\left\{\left(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \left|\cos\chi\right|\right) - \beta\right\}$$

with

$$\eta_{\mathfrak{I}^*}(m, h) = (1+h)\left\{\left(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \left|\cos\chi\right|\right) - \beta\right\} + \tan\frac{\pi h}{4}\{0,0,0,1\}.$$

The deformation retract of the folded open flat Robertson-Walker space $\mathfrak{I}^*\left(W^4\right)$ into the folded geodesic $\mathfrak{I}^*\left(S_2^3\right)$ is:

$$\eta_{\mathfrak{I}^*}(m, h) = (1+h)\left\{\left(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi\sin\chi\cos\theta, \left|\cos\chi\right|\right) - \beta\right\} + \left(h^3 - 2h + 2h^2\right)\left\{0, 0, \sin\chi, \left|\cos\chi\right|\right\}$$

The deformation retract of the folded open flat Robertson-Walker space $\mathfrak{I}^*\left(W^4\right)$ into the folded geodesic $\mathfrak{I}^*\left(S_3^3\right)$ is:

$$\eta_{\mathfrak{I}^*}(m, h) = (1+h)\left\{\left(\sin\chi\sin\theta\cos\phi\sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \left|\cos\chi\right| - \beta\right\} + \sin\frac{\pi h}{2}\left\{\sin\chi\sin\theta, 0, \sin\chi\cos\theta, 0, \left|\cos\chi\right|\right\}$$

Then, the following theorem has been proved.

Theorem 3. Under the defined folding and any folding homeomorphic to this type of folding, the deformation retract of the folded open flat Robertson-Walker space $\mathfrak{I}^*\left(W^4\right)$ into the folded geodesics is different from the deformation retract of open flat Robertson-Walker space W^4 into the geodesics.

Lemma 1. The relations between the retractions and the limits of the folding of open flat Robertson-Walker space W^4 discussed from the following commutative diagrams

$$\begin{tikzcd}
W^4 - \beta \arrow[r, "r_1"] \arrow[d, "\lim_{m\to\infty} f_m"'] & S_1^3 \arrow[d, "\lim_{m\to\infty} f_{m+1}"] \\
S_1^3 \arrow[r, "r_2"] & S_1^2
\end{tikzcd}$$

Lemma 2. The end of limits of the folding of closed flat Robertson-Walker space W^4 is a 0-dimensional space.

Proof. Let

$$\mathfrak{F}_1: S^3 \to S^3,$$

$$\mathfrak{F}_2: \mathfrak{F}_1\left(S^3\right) \to \mathfrak{F}_1(S^3), \mathfrak{F}_3: \mathfrak{F}_2\left(\mathfrak{F}_1\left(S^3\right)\right) \to \mathfrak{F}_2\left(\mathfrak{F}_1\left(S^3\right)\right)\cdots$$

$$\mathfrak{F}_n: \mathfrak{F}_{n-1}\left(\mathfrak{F}_{n-2}\cdots\left(\mathfrak{F}_3\left(\mathfrak{F}_2\left(\mathfrak{F}_1\left(S^3\right)\right)\right)\right)\cdots\right) \to \mathfrak{F}_{n-1}\left(\mathfrak{F}_{n-2}\cdots\left(\mathfrak{F}_3\left(\mathfrak{F}_2\left(\mathfrak{F}_1\left(S^3\right)\right)\right)\right)\cdots\right)$$

$$\lim_{n\to\infty}\mathfrak{F}_n\left(\mathfrak{F}_{n-1}\left(\mathfrak{F}_{n-2}\cdots\left(\mathfrak{F}_3\left(\mathfrak{F}_2\left(\mathfrak{F}_1\left(S^3\right)\right)\right)\right)\right)\cdots\right) = S^2$$

Let

$$K_1: S^2 \to S^2, K_2: K_1\left(S^2\right) \to K_1\left(S^2\right),$$

$$K_3: K_2\left(K_1\left(S^2\right)\right) \to K_2\left(K_1\left(S^2\right)\right), \cdots,$$

$$K_n: K_{n-1}\left(K_{n-2}\cdots\left(K_3\left(K_2\left(K_1\left(S^2\right)\right)\right)\right)\cdots\right) \to K_{n-1}\left(K_{n-2}\cdots\left(K_3\left(K_2\left(K_1\left(S^2\right)\right)\right)\right)\cdots\right),$$

$$\lim_{n\to\infty} K_n\left(K_{n-1}\left(K_{n-2}\cdots\left(K_3\left(K_2\left(K_1\left(S^2\right)\right)\right)\right)\right)\cdots\right) = S^1$$

Consequently,

$\lim_{s\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} h_s\left(K_m\left(\mathfrak{F}_n\left(S^3\right)\right)\right) = S^0 = 0$ -dimensional sphere, it is a minimal geodesic.

Lemma 3. The relation between the retraction and the deformation retract of open flat Robertson-Walker space W^4 discussed from the following commutative diagram

$$\begin{tikzcd}
\{S_3^3-\beta\}\times I \arrow[r, "D.R"] \arrow[d, "\eta_1"'] & \{S_3^3-\beta\} \arrow[d, "\eta_2"] \\
\{S_3^3-\beta\}\times I \arrow[r, "D.R"] & S_1^3
\end{tikzcd}$$

Theorem 4. Any folding of $S_\alpha^3 \subset W^4$ into $S_\beta^3 \subset W^4$ induces folding of $B(\pi\alpha)$ into $B(\pi\beta)$ from

$$T_{P_\alpha}\left(S_\alpha^3\right) \to T_{P_\beta}\left(S_\beta^3\right).$$

Proof. Let $S_\alpha^3 \subset W^4 \to S_\beta^3 \subset W^4$ $: S_\alpha^3 \subset W^4 \to S_\beta^3 \subset W^4$, then there is an induced folding $\mathfrak{F}^* : B(\pi\alpha) \to B(\pi\beta)$ such that

$\exp^{-1} : S_\alpha^3 \subset W^4 \to B(\pi\alpha)$ and

$\exp^{-1} : S_\beta^3 \subset W^4 \to B(\pi\beta)$ such that the following diagram is commutative

$$\begin{tikzcd}
S_\alpha^3 \subset W^4 \arrow[r, "\mathfrak{F}"] \arrow[d, "exp^{-1}"'] & S_\beta^3 \subset W^4 \arrow[d, "exp^{-1}"] \\
B(\pi\alpha) \arrow[r, "\mathfrak{F}^*"] & B(\pi\beta)
\end{tikzcd}$$

i.e. $\mathfrak{F}^* \circ \exp^{-1} = \exp^{-1} \mathfrak{F}$

Theorem 5. Any retraction of $\left(S_\alpha^3 - \leftthreetimes\right) \subset W^4$ into $\left(S_\beta^3 - \leftthreetimes\right) \subset W^4, \beta \subset \alpha$ induces retraction of $B(\pi\alpha)$ into $B(\pi\beta)$.

Proof. Let r be a retraction map,

$$r\left(S_\alpha^3 - \leftthreetimes\right) \subset W^4 \to \left(S_\beta^3 - \leftthreetimes\right) \subset W^4$$

where $\left(S_\alpha^3 - \leftthreetimes\right)$ and $\left(S_\beta^3 - \leftthreetimes\right)$ are the open sphere in $W^4, \beta \subset \alpha \neq$. Also, let

$\exp^{-1} : \left(S_\alpha^3 - \leftthreetimes\right) \subset W^4 \to B(\pi\alpha - \leftthreetimes')$ and

$\exp^{-1} : \left(S_\beta^3 - \leftthreetimes\right) \subset W^4 \to B(\pi\beta - \leftthreetimes')$

such that $B(\pi\alpha - \leftthreetimes') \supset B(\pi\beta - \leftthreetimes')$. Then we have the retraction $r^* : B(\pi\alpha - \leftthreetimes') \to B(\pi\beta - \leftthreetimes')$ such that

$$r^* \exp^{-1} = \exp^{-1}$$

$$\begin{tikzcd}
(S_\alpha^3 - \leftthreetimes) \arrow[r, "r"] \arrow[d, "exp^{-1}"'] & (S_\beta^3 - \leftthreetimes) \arrow[d, "exp^{-1}"] \\
B(\pi\alpha - \leftthreetimes') \arrow[r, "r^*"] & B(\pi\beta - \leftthreetimes')
\end{tikzcd}$$

Theorem 6. Any retraction $r_1 : T_p\left\{W^4 - \beta\right\} \to T_q\left(S^3\right)$, then the map $r_2 : \left\{W^4 - \beta\right\} \to S^3$ induced by the exponential map.

Proof. Let a retraction $r_1 : T_p\left\{W^4 - \beta\right\} \to T_q\left(S^3\right)$, be a retraction of $T_p\left\{W^4 - \beta\right\}$ into $T_q\left(S^3\right)$. Also, Let

$\exp_p : T_p\left\{W^4 - \beta\right\} \to \left\{W^4 - \beta\right\}$ and

$\exp_q : T_q\left(S^3\right) \to \left(S^3\right)$. Then we have the retraction

$r_2 : \left\{W^4 - \beta\right\} \to S^3$

such that $ro\exp_p = \exp_q or_1$

$$\begin{tikzcd}
T_p\{W^4 - \beta\} \arrow[r, "r_1"] \arrow[d, "exp_p"'] & T_q\,(S_2^3) \arrow[d, "exp_q"] \\
\{W^4 - \beta\} \arrow[r, "r_2"] & S_1^3
\end{tikzcd}$$

Theorem 7. Any retraction $r : \left\{W^4 - \beta\right\} \to S_2^3$, then the map $r_1 : T_p\left\{W^4 - \beta\right\} \to T_q\left(S_1^3\right)$ induced by the inverse exponential map.

Proof. Let a retraction $r : \left\{W^4 - \beta\right\} \to \left(S_2^3\right)$, be a retraction of $\left\{W^4 - \beta\right\}$ int S_2^3. Also, Let

$\exp_p^{-1} : \left\{W^4 - \beta\right\} \to T_p\left(W^4\right)$ and

$\exp_q^{-1} : \left(S_2^3\right) \to T_q\left(S_1^3\right)$.

Then we have the retraction

$r_1 : T_p\left\{W^4 - \beta\right\} \to T_p\left(S_1^3\right)$ such that

$$r_1 o \exp_p^{-1} = \exp_q^{-1} or$$

$$\begin{tikzcd}
\{W^4 - \beta\} \arrow[r, "r"] \arrow[d, "exp_p^{-1}"'] & S_2^3 \arrow[d, "exp_q^{-1}"] \\
T_p\{W^4 - \beta\} \arrow[r, "r_1"] & T_q\{S^3\}
\end{tikzcd}$$

Theorem 8. If the retraction of the sphere S_6^2 is $f : S_6^2 \to S_1^1$, the inclusion map of S_6^2 is $i : S_6^2 \to S_1^3, S_6^2 \subset S_1^3$, and inclusion map of S^1 is $j : S_1^1 \to S_5^2$. Then there are induces retractions such that the following diagram is commutative.

$$
\begin{tikzcd}
& id_{S_6^2} \arrow[dl, dash] \arrow[dr, dash] & \\
S_6^2 \arrow[r, "i"] \arrow[d, "f"'] & S_1^3 \arrow[r, "r"] \arrow[d, "g"] & S_6^2 \arrow[d, "f"] \\
S_1^1 \arrow[r, "j"] \arrow[dr, dash] & S_5^2 \arrow[r, "s"] & S_1^1 \arrow[dl, dash] \\
& id_{S_1^1} &
\end{tikzcd}
$$

Proof. Let the retraction map of the hypersphere S_6^2 is $f: S_6^2 \to S_1^1$, the inclusion map of S_6^2 is $i: S_6^2 \to S_1^3, S_6^2 \subset S_1^3$, $j: f\left(S_6^2\right) \to S_5^2$, the retraction map of $i\left(S_6^2\right)$ is $r: i\left(S_6^2\right) \to S_6^2$, the retraction map of $r: i\left(S_6^2\right) \to S_6^2$ is given by $s: j\left(f\left(S_6^2\right)\right) \to S_1^1$, and $f: r\left(i\left(S_6^2\right)\right) \to S_1^1$. Hence, the following diagram is commutative.

Theorem 9. If the retraction of the sphere S_1^3 is $f: S_1^3 \to S_1^1$, $\exp: S_1^3 \to T\left(S_1^3\right)$, and $\exp_1: S_1^1 \to T\left(S_1^1\right)$. Then there are induces exponential inverse map such that the following diagram is commutative.

$$
\begin{tikzcd}
& id_{S_1^3} \arrow[dl, dash] \arrow[dr, dash] & \\
S_1^3 \arrow[r, "exp"] \arrow[d, "f"'] & T(S_1^3) \arrow[r, "exp^{-1}"] \arrow[d, "g"] & S_1^3 \arrow[d, "f"] \\
S_1^1 \arrow[r, "exp_1^{-1}"] \arrow[dr, dash] & T(S_1^1) \arrow[r, "exp_1^{-1}"] & S_1^1 \arrow[dl, dash] \\
& id_{S_1^1} &
\end{tikzcd}
$$

Proof. Let the retraction map of the hypersphere S_6^2 is $f: S_1^3 \to S_1^1$, $\exp: S_1^3 \to T\left(S_1^3\right), \exp_1: f\left(S_1^3\right) \to T\left(S_1^1\right)$, $\exp^{-1}: \exp\left(S_1^3\right) \to S_1^3$, $\exp_1^{-1}: \exp_1\left(f\left(S_1^3\right)\right) \to S_1^1$, and $f: \exp^{-1}\left(\exp\left(S_1^3\right)\right) \to S_1^1$. Hence, the following diagram is commutative.

3. Conclusion

The present article deals what we consider to be closed flat Robertson-Walker W^4 model. The retractions of closed flat Robertson-Walker W^4 model are presented. The deformation retract of closed flat Robertson-Walker W^4 model will be deduced. The connection between folding and deformation retract is achieved. New types of conditional folding are presented. Also, the relations between the limits of folding and retractions are discussed. Some commutative diagrams are presented.

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