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Phase-Space Areas of the Body Motion in the Solar System Deduced from the Bohr-Sommerfeld Atomic Theory and Approximate Invariance of Their Ratios for the Pairs of Planets and Satellites

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ABSTRACT

Energy-time and momentum-position phase spaces defined by the electron orbits in the hydrogen-like atom exhibit special properties of equivalence. It is demonstrated that equivalence of the same kind can be obtained for the phase-space areas defined by the orbit pairs of planets, or satellites, which compose the solar system. In the choice of the examined areas it is useful to be guided by the Bohr-Sommerfeld atomic theory.

Keywords: Ratios of the Phase-Space Areas and Their Invariance; Planets and Satellites; Bohr-Sommerfeld Atomic Theory

1. Introduction

In physics, but not only in this domain, we look very often for parameters which allow us to identify the examined objects. For example a typical parameter of this kind is the mass which helps us to identify a given particle, or a system of particles collected together, say, in an atom. Also large systems of particles, especially those composed of mainly the same atoms or molecules, have some properties which are characteristic for the whole ensemble. A typical parameter is here a definite temperature associated—at special conditions—with the change of the system phase, for example the change is from a vapour to a liquid. Sometimes a combination of several parameters is required to be characteristic for a system.

In mechanics of a circular motion of a body along a specific closed orbit there are important phase integrals

$$S = \oint p \mathrm{d}q \tag{1}$$

met in the Sommerfeld quantum conditions; see e.g. [1]. Here p is the body momentum and q is the body position on the orbit. The integral (1) is the phase-space area of the variables (p, q) circumvented by the body in course of its motion along the orbit. The body energy E—which is conserved on the orbit—can be represented as a function of S and the frequency ν of the body motion, equal to the reciprocal value of the circulation period T, becomes

$$\nu = \frac{1}{T} = \frac{\partial E}{\partial S}. \tag{2}$$

In some special cases, however, for example for a linear oscillator, there is a linear dependence between E and S, so $\frac{\partial E}{\partial S}$ is a constant independent of E or the amplitude of oscillation. In this case we have

$$E = S\nu. \tag{3}$$

A characteristic property of (3) is that the ratio

$$S = \frac{E}{\nu} \tag{4}$$

called the adiabatic invariant, remains unchanged for slow changes of the oscillator Hamiltonian [2].

But in case of the body orbital motion in the solar system the simplifications of (3) and (4) do not hold. For from the reversed Formula (2) we obtain (see e.g. [3]):

$$\frac{1}{\nu} = T = \frac{2\pi}{(GM_S)^{1/2}} a^{3/2} = \frac{\partial S}{\partial E}, \tag{5}$$

where G is the gravitational constant, a—the major semiaxis of the Kepler orbit and M_S is the central mass which for the planetary orbital motion is the mass of the Sun. Since

$$E = -\frac{GM_S M_p}{2a}, \tag{6}$$

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where M_p is the planetary mass, we obtain

$$\begin{aligned}\partial S &= \frac{2\pi a^{3/2}}{\left(GM_S\right)^{1/2}}\partial E = \frac{2\pi a^{3/2}}{\left(GM_S\right)^{1/2}}\left(\frac{\partial E}{\partial a}\right)\partial a \\ &= \frac{2\pi a^{3/2}}{\left(GM_S\right)^{1/2}}\frac{GM_S M_p}{2a^2}\partial a \\ &= \pi\left(GM_S\right)^{1/2}\frac{M_p}{a^{1/2}}\partial a.\end{aligned} \tag{7}$$

The integration performed for (7) gives, with the accuracy to a constant term,

$$S = 2\pi\left(GM_S\right)^{1/2} M_p a^{1/2}, \tag{8}$$

whereas the product of $|E|$ and T becomes

$$\begin{aligned}|E|T &= \frac{GM_S M_p}{2a}\frac{2\pi}{\left(GM_S\right)^{1/2}}a^{3/2} \\ &= \pi\left(GM_S\right)^{1/2} M_p a^{1/2} = \frac{1}{2}S.\end{aligned} \tag{9}$$

But a similar relation can be obtained for the product of the planetary momentum P and average distant R of the moving body from the Sun, if we note that for the Kepler motion of planets the eccentricity parameter e is regularly small, so

$$R \approx a, \tag{10}$$

$$\begin{aligned}P &\approx \frac{2\pi R}{T}M_p \approx \frac{2\pi a}{2\pi a^{3/2}}\left(GM_S\right)^{1/2} M_p \\ &= \frac{\left(GM_S\right)^{1/2} M_p}{a^{1/2}}.\end{aligned} \tag{11}$$

In this case

$$\pi RP \approx \pi\left(GM_S\right)^{1/2} a^{1/2} = \frac{1}{2}S, \tag{12}$$

therefore

$$|E|T \approx \pi RP \approx \frac{1}{2}S. \tag{13}$$

Evidently, the products entering (13) together with S in (8) represent the phase-space areas of (p,q).

However, any of the factors E, T, R, and P entering (13) depends on a in a different way, and a separate dependence on a applies to S. A question which may arise here is as follows: if we have a pair of the planetary objects in which one of the objects has its

$$E_1, T_1, R_1, P_1 \tag{14}$$

at some $a = a_1$, and M_{p1}, and the other object in this pair has its

$$E_2, T_2, R_2, P_2 \tag{15}$$

at some $a = a_2$, and M_{p2}, is it possible to combine the phase-space areas obtained in terms of parameters entering (14) and (15) in such a way that the resulted combination is approximately independent of a_1, a_2, M_{p1}, and M_{p2}? In effect, such a combination may become approximately constant for the whole of the planetary system.

We find an affirmative answer to that question, also for the satellitary systems, and details of the corresponding formulae are given below. It should be noted, however, that a search for the required formalism is much facilitated when similarities which exist in description of the Kepler's planetary problem and the electron motion in the framework of the Bohr-Sommerfeld atomic theory are taken into account.

2. Parameters Useful in Calculating the Phase-Space Areas of the Planetary Motion

Similarities between the planetary motion performed in the gravitational field of the Sun and the electron motion in the electrostatic field of the atomic nucleus are well known. However, any quantitative reference between parameters characteristic for both kinds of the examined motions is rather difficult to assess. This difficulty seems, in general, to be quite obvious if we note a macroscopic nature of the planetary objects and microscopic properties of an atom. Nevertheless, a geometric and mechanical similarity in the behaviour of the solar system to that of an atomic system considered in the framework of the old quantum theory is evident. This similarity may suggest a quantitative search for comparison between the macroworld of planets and the microworld connected with the electron motion in an atom. A search of this kind can materialize in definite calculations when the properties of the phase space associated with an electron circulating in the atom are compared with the phase space of the planetary, or satellitary, motion in the solar system.

In practice, the position-momentum and energy-time phase spaces can be defined both for the orbiting planets and the circulating electron. Here, for the sake of simplicity, the elliptical character of orbits can be approximately neglected equally in the macroscopic and the microscopic case. Let us begin with the action function S_n for a planet on the orbit n which satisfies the relation

$$S_n = 2\pi P_n R_n = 2|E_n|T_n, \tag{16}$$

whereas a similar action function for an electron moving on a circular trajectory corresponding to the atomic state n is

$$S_n = 2\pi p_n^e r_n^e = 2\left|E_n^e\right|T_n^e. \tag{17}$$

Symbols P_n, R_n, E_n and T_n in (16) denote respectively the momentum, orbital radius

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$$R_n = \frac{1}{2} a_n \left(\sqrt{1-e_n^2} + 1 \right), \tag{18}$$

energy and time period on the planetary orbit n having the major semiaxis a_n and the eccentricity e_n [4-6], whereas p_n^e, r_n^e, E_n^e and T_n^e in (17) are similar parameters for an electron in its quantum state, or on an orbit, labelled also by n. In our calculations we are guided by the atomic properties which are given by the formulae [7]:

$$p_n^e = m_e v_n^e = \frac{2\pi m_e Z e^2}{nh}, \tag{19}$$

$$r_n^e = \frac{n^2 h^2}{4\pi^2 m_e Z e^2}, \tag{20}$$

$$E_n^e = -\frac{2\pi^2 m_e Z^2 e^4}{n^2 h^2}, \tag{21}$$

$$T_n^e = \frac{n^3 h^3}{4\pi^2 m_e Z^2 e^4}. \tag{22}$$

Here Ze is the nucleus charge, e the electron charge, m_e the electron mass, and h is the Planck constant.

3. Change of the Phase-Space Area Due to the Orbit Change

When the orbit n is changed to another orbit, say n', the parts of the difference in the action function equal to

$$\Delta S_{n'n} = S_{n'} - S_n = \Delta S_{n'n}^{(d)} + \Delta S_{n'n}^{(nd)} \tag{23}$$

can be examined both in the planetary and the electron case. These parts can be classified as diagonal (d) changes represented by the formulae

$$\frac{1}{2} \Delta S_{n'n}^{(d)}(p,r) = \pi \left(R_{n'} - R_n \right) \left(P_{n'} - P_n \right), \tag{24}$$

$$\frac{1}{2} \Delta S_{n'n}^{(d)}(E,T) = \left(|E_{n'}| - |E_n| \right) \left(T_{n'} - T_n \right) \tag{25}$$

concerning respectively the momentum-position and energy-time phase-space areas entering S_n and $S_{n'}$ in (16). Similar non-diagonal (nd) changes entering $\Delta S_{n'n}$ are

$$\begin{aligned} &\frac{1}{2} \Delta S_{n'n}^{(nd)}(p,r) \\ &= \pi \left[\left(R_{n'} - R_n \right) P_n + \left(P_{n'} - P_n \right) R_n \right], \end{aligned} \tag{26}$$

$$\begin{aligned} &\frac{1}{2} \Delta S_{n'n}^{(nd)}(E,T) \\ &= \left(T_{n'} - T_n \right) |E_n| + \left(|E_{n'}| - |E_n| \right) T_n. \end{aligned} \tag{27}$$

In the next step, it can be noted that relations concerning components of $\Delta S_{n'n}$ for planets defined in (24)-(27) can be calculated also for the changes $\Delta S_{n'n}$ for electrons, on condition P_n, R_n, E_n and T_n entering the mentioned formulae are replaced by p_n^e, r_n^e, E_n^e and T_n^e. The electron parameters give [6]:

$$(1/2) \Delta S_{n'n}^{(d)} \left(p^e, r^e \right) \approx -\frac{(\Delta n)^2}{n} h, \tag{28}$$

$$(1/2) \Delta S_{n'n}^{(d)} \left(E^e, T^e \right) \approx -\frac{3(\Delta n)^2}{n} h, \tag{29}$$

for small $\Delta n = n' - n$, so the ratio

$$\frac{\Delta S_{n'n}^{(d)} \left(p^e, r^e \right)}{\Delta S_{n'n}^{(d)} \left(E^e, T^e \right)} = 1/3 \tag{30}$$

is practically a constant number independent of the orbit index n. In a further step

$$\frac{1}{2} \Delta S_{n'n}^{(nd)} \left(p^e, r^e \right) \approx \frac{h}{2} \left[\Delta n + 2 \frac{(\Delta n)^2}{n} \right], \tag{31}$$

$$\frac{1}{2} \Delta S_{n'n}^{(nd)} \left(E^e, T^e \right) \approx \frac{h}{2} \left[\Delta n + 6 \frac{(\Delta n)^2}{n} \right], \tag{32}$$

so the ratio of (31) and (32) is approximately also a constant:

$$\frac{\Delta S_{n'n}^{(nd)} \left(p^e, r^e \right)}{\Delta S_{n'n}^{(nd)} \left(E^e, T^e \right)} \approx 1, \tag{33}$$

which holds especially on condition when $\Delta n \ll n$.

A check of the properties of (30) and (33) for the solar system has been done on the numerical way; see [6]. An almost constant behaviour of (30) is widely confirmed, although this ratio becomes close rather to unity, especially for planets, than to 1/3 obtained in (30). On the other hand, the ratio (33) fluctuates more than (30) for different planetary, or satellitary, pairs chosen to occupy n and n'. This behaviour can be attributed to unequal second terms entering the square brackets in (31) and (32). In the most part of the examined planetary cases the ratio between (31) and (32) is about 2 instead of the result approximately equal to 1 obtained in (33).

4. Phase-Space Areas Made Free from the Orbit Index and Their Comparison

A more convenient way of comparison between the phase-space areas for an electron moving in the atom and similar areas obtained for circulating planets is when these areas are made, at least approximately, free from the orbital index. This situation, in fact, can be obtained beginning with the areas for the moving electron. For example, such an area in the momentum-position space about state n can be defined as

$$\pi \left(r_{n+1}^e - r_n^e \right) \left(p_{n+1}^e + p_n^e \right) \frac{1}{2} \approx n \frac{2}{n} \frac{h}{2} = h. \tag{34}$$

The result in (34) holds on condition n is assumed to be a large number. The area in (34) is a belt of thickness

$$r_{n+1}^{e}-r_{n}^{e}=\left[(n+1)^{2}-n^{2}\right]\frac{h^{2}}{4\pi^{2}m_{e}Ze^{2}}\approx\frac{2nh^{2}}{4\pi^{2}m_{e}Ze^{2}}\tag{35}$$

plotted in the coordinate space, and the length of the belt is proportional to

$$\pi\left(p_{n+1}^{e}+p_{n}^{e}\right)\frac{1}{2}=\frac{2\pi^{2}m_{e}Ze^{2}}{2h}\left(\frac{1}{n+1}+\frac{1}{n}\right)\approx\frac{2\pi^{2}m_{e}Ze^{2}}{nh}\tag{36}$$

in the momentum space. A similar area can be defined when the roles of p_e and r_e are reversed. In this case we obtain the phase-space area equal to

$$\pi\left|p_{n+1}^{e}-p_{n}^{e}\right|\left(r_{n+1}^{e}+r_{n}^{e}\right)\frac{1}{2}\approx\frac{1}{2}h.\tag{37}$$

Here the thickness of the belt is

$$\left|p_{n+1}^{e}-p_{n}^{e}\right|=\frac{2\pi^{2}m_{e}Ze^{2}}{h}\left|\frac{1}{n+1}-\frac{1}{n}\right|\approx\frac{2\pi m_{e}Ze^{2}}{hn^{2}}\tag{38}$$

and the belt length is proportional to

$$\pi\left(r_{n+1}^{e}+r_{n}^{e}\right)\frac{1}{2}\approx\frac{h^{2}}{4\pi m_{e}Ze^{2}}n^{2}.\tag{39}$$

The results obtained in (34) and (37) are: 1) close to h; 2) they differ only by a factor of 2.

A similar evaluation can be done for the phase space areas composed of the electron energies and the time periods of the rotational motion about the nucleus; see (21) and (22). We obtain for one electron state the areas

$$\left(T_{n+1}^{e}-T_{n}^{e}\right)\frac{\left|E_{n+1}^{e}\right|+\left|E_{n}^{e}\right|}{2}=3n^{2}h\frac{1}{2n^{2}}=\frac{3}{2}h,\tag{40}$$

$$\left\|E_{n+1}^{e}\right|-\left|E_{n}^{e}\right\|\frac{T_{n+1}^{e}+T_{n}^{e}}{2}=\frac{2}{n^{3}}\frac{n^{3}}{2}h=h,\tag{41}$$

because for a large n we find:

$$T_{n+1}^{e}-T_{n}^{e}\approx3n^{2}\frac{h^{3}}{4\pi^{2}m_{e}Z^{2}e^{4}},\tag{42}$$

$$\frac{1}{2}\left(T_{n+1}^{e}+T_{n}^{e}\right)\approx\frac{n^{3}h^{3}}{4\pi^{2}m_{e}Z^{2}e^{4}},\tag{43}$$

$$\left\|E_{n+1}^{e}\right|-\left|E_{n}^{e}\right\|\approx\frac{2}{n^{3}}\frac{2\pi^{2}m_{e}Z^{2}e^{4}}{h^{2}},\tag{44}$$

$$\frac{1}{2}\left(\left|E_{n+1}^{e}\right|+\left|E_{n}^{e}\right|\right)\approx\frac{1}{n^{2}}\frac{2\pi^{2}m_{e}Z^{2}e^{4}}{h^{2}}.\tag{45}$$

The areas calculated in (40) and (41) are: 1) close to h; 2) differ only by a factor of $3/2$. Evidently, the area (40) is a kind of the belt having its length extended in the energy space and the belt thickness is in the period-of-time space, whereas (41) represents the belt of reversed shape properties.

Because of a similar character of the electron motion in the Bohr-Sommerfeld atom and the planetary motion in the solar system, we expect similar phase-space properties for both kinds of the motion. An essential difference is here that the electron has the same mass on any orbit n, but this does not apply to the planetary orbits. Therefore a more realistic comparison should concern the case when momenta expressions entering the examined areas are replaced by velocities, and—consequently—the energies of the moving bodies are divided by the orbiting mass. For electrons we obtain on any orbit n

$$v_{n}^{e}=\frac{p_{n}^{e}}{m_{e}}\tag{46}$$

for the velocity, and

$$\frac{1}{2}\left|\overline{V}_{n}^{epot}\right|=\frac{\left|E_{n}^{e}\right|}{m_{e}}\tag{47}$$

for the reduced energy. Because of the virial theorem, the expression (47) is equal to a half of the absolute average value of the potential acting on a moving electron. In effect of the substitutions (46) and (47) we obtain the following belt areas for the electron case:

$$A=\pi\left(r_{n+1}^{e}-r_{n}^{e}\right)\left(v_{n+1}^{e}+v_{n}^{e}\right)\frac{1}{2}=\frac{h}{m_{e}},\tag{19a}$$

$$B=\pi\left(v_{n+1}^{e}-v_{n}^{e}\right)\left(r_{n+1}^{e}+r_{n}^{e}\right)\frac{1}{2}=\frac{1}{2}\frac{h}{m_{e}},\tag{22a}$$

$$C=\frac{1}{2}\left(T_{n+1}^{e}-T_{n}^{e}\right)\frac{1}{2}\left(\left|\overline{V}_{n+1}^{epot}\right|+\left|\overline{V}_{n}^{epot}\right|\right)=\frac{3}{2}\frac{h}{m_{e}},\tag{25a}$$

$$D=\frac{1}{2}\left\|\overline{V}_{n+1}^{epot}\right|-\left|\overline{V}_{n}^{epot}\right\|\frac{1}{2}\left(T_{n+1}^{e}+T_{n}^{e}\right)=\frac{h}{m_{e}}.\tag{26a}$$

Section 5 examines similar expressions calculated for the planetary and satellitary orbits.

5. Belt Areas in the Phase-Space Calculated for Planets and Satellites

In the first step let us examine the equivalence of the belt areas in the position-momentum space and areas in the energy-time period space without the mass reduction for the body momentum and energy performed below [see (56) and (57)]. In this case we have

$$A(R,P)=\pi\left(R_{n+1}-R_{n}\right)\left(P_{n+1}+P_{n}\right)\frac{1}{2}\tag{48}$$

similar to (19a),

$$B(R,P) = \pi(P_{n+1} - P_n)(R_{n+1} + R_n)\frac{1}{2} \tag{49}$$

similar to (22a),

$$C(T,E) = (T_{n+1} - T_n)(|E_{n+1}| + |E_n|)\frac{1}{2} \tag{50}$$

similar to (25a), and

$$D(T,E) = \left||E_{n+1}| - |E_n|\right|(T_{n+1} + T_n)\frac{1}{2}, \tag{51}$$

similar to (26a) calculated for an electron in the Bohr-Sommerfeld atom. The factor of 1/2 entering the formulae (48)-(51) provides us with the arithmetical mean of the corresponding parameters defining the belt lengths. The belt areas (48)-(51) are calculated in **Tables 1** and **2** together with the ratios $B(R,P)/A(R,P)$ and $D(T,E)/C(T,E)$ which roughly satisfy the relations:

$$\frac{B(R,P)}{A(R,P)} \approx \frac{D(T,E)}{C(T,E)} \approx 2 \tag{52}$$

The input data in **Tables 1** and **2** are taken from [5,6] and **Table 3**. For the same planetary and satellitary pairs another ratio equal to

Table 1. Belt areas between the neighbouring planet orbits calculated (in Js) for the momentum-position and energy-time phase spaces and their ratios; for *A*(*R,P*) and *B*(*R,P*) see Equations. (48) and (49); for *C*(*T,E*) and *D*(*T,E*) see, respectively, (50) and (51). The entering data are based on [5,6].

Planets Pair	$A(R,P)$	$B(R,P)$	$\frac{B}{A}$	$C(T,E)$	$D(T,E)$	$\frac{D}{C}$
Venus-Mercury	0.149×10^{41}	0.403×10^{41}	2.70	0.199×10^{41}	0.351×10^{41}	1.77
Earth-Venus	0.227×10^{41}	0.300×10^{40}	0.13	0.342×10^{41}	0.841×10^{40}	0.25
Mars-Earth	0.237×10^{41}	0.962×10^{41}	4.07	0.394×10^{41}	1.120×10^{41}	2.84
Jupiter-Mars	0.215×10^{44}	0.391×10^{44}	1.82	0.255×10^{44}	0.351×10^{44}	1.37
Saturn-Jupiter	0.308×10^{44}	0.669×10^{44}	2.17	0.523×10^{44}	0.883×10^{44}	1.69
Uranus-Saturn	0.138×10^{44}	0.330×10^{44}	2.40	0.245×10^{44}	0.438×10^{44}	1.79
Neptune-Uranus	0.293×10^{43}	0.396×10^{42}	0.13	0.449×10^{43}	0.196×10^{43}	0.44
Pluto-Neptune	0.116×10^{43}	0.900×10^{43}	7.77	0.198×10^{43}	0.982×10^{43}	4.97

Table 2. Belt areas between the neighbouring satellite orbits calculated (in Js) for the momentum-position and energy-time phase spaces and their ratios; for the symbols meaning see (48)-(51). The entering data are based on [5,6] and Table 3.

Satellites Pair	$A(R,P)$	$B(R,P)$	$\frac{B}{A}$	$C(T,E)$	$D(T,E)$	$\frac{D}{C}$
Deimos-Phobos	0.516×10^{27}	0.920×10^{27}	1.78	0.968×10^{27}	1.374×10^{27}	1.42
Europa-Io	0.867×10^{36}	1.507×10^{36}	1.74	0.138×10^{37}	0.202×10^{37}	1.46
Ganymede-Europa	0.144×10^{37}	0.260×10^{37}	1.81	0.209×10^{37}	0.195×10^{37}	0.94
Callisto-Ganymede	0.319×10^{37}	0.341×10^{37}	1.07	0.513×10^{37}	0.534×10^{37}	1.04
Enceladus-Mimas	0.130×10^{33}	0.188×10^{33}	1.45	0.195×10^{33}	0.126×10^{33}	0.64
Tethys-Enceladus	0.852×10^{33}	6.400×10^{33}	7.51	0.122×10^{34}	0.600×10^{34}	4.93
Dione-Tethys	0.246×10^{34}	0.205×10^{34}	0.83	0.372×10^{34}	0.086×10^{34}	0.23
Rhea-Dione	0.746×10^{34}	1.505×10^{34}	2.02	0.110×10^{35}	0.115×10^{35}	1.05
Titan-Rhea	0.844×10^{36}	2.009×10^{36}	2.38	0.108×10^{37}	0.177×10^{37}	1.64
Iapetus-Titan	0.279×10^{37}	0.561×10^{37}	2.01	0.577×10^{37}	0.859×10^{37}	1.49
Ariel-Miranda	0.764×10^{33}	0.348×10^{34}	4.55	0.107×10^{34}	0.318×10^{34}	2.98
Umbriel-Ariel	0.158×10^{34}	0.108×10^{34}	0.69	0.241×10^{34}	0.192×10^{34}	0.80
Titania-Umbriel	0.498×10^{34}	0.746×10^{34}	1.50	0.727×10^{34}	0.515×10^{34}	0.71
Oberon-Titania	0.506×10^{34}	0.559×10^{34}	1.10	0.773×10^{34}	0.825×10^{34}	1.07

Table 3. Corrected data for one of the satellites of Saturn and two satelites of Uranus; see [5] and [6]. The period T_n is given in sidereal days (1 sidereal day = 86400 s); R_n is the mean distance of the satellite from planet center (in 10^3 m); M_{sn} is the satellite mass (in kg); E_n is the satellite energy (in J); P_n is the satellite momentum (in kg·ms^{-1}).

n	Satelite of Saturn	R_n	T_n	M_{sn}	$\lvert E_n \rvert$	P_n
7	Iapetus	3561000	79.33	1.88×10^{21}	0.1001×10^{29}	0.6138×10^{25}
n	Satelite of Uranus	R_n	T_n	M_{sn}	$\lvert E_n \rvert$	P_n
4	Titania	436300	18.704	3.48×10^{21}	0.2311×10^{29}	0.1269×10^{26}
5	Oberon	583400	13.463	2.92×10^{21}	0.1450×10^{29}	0.0920×10^{26}

$$\eta = \frac{C(T,E)+D(T,E)}{A(R,P)+B(R,P)} \approx 1.5 \tag{53}$$

is calculated in **Table 4**.

In the next step, we examine the relations between the phase-space areas when the mass reduction of the body momentum and energy mentioned at the end of Section 4 is done.

The planetary mass $M_{pn} = M_n$, or the satellite mass $M_{sn} = M_n$ enter both sides of Equation (16). The reduction of this mass gives instead of (16):

$$\frac{\pi P_n R_n}{M_n} = \frac{|E_n| T_n}{M_n} \tag{54}$$

or

$$\pi v_n R_n = \frac{1}{2}\left|\bar{V}_n^{pot}\right| T_n \tag{55}$$

where

$$v_n = \frac{2\pi R_n}{T_n} = \frac{P_n}{M_n} \tag{56}$$

is the velocity of a circulating celestial body and

$$\frac{1}{2}\left|\bar{V}_n^{pot}\right| = \frac{|E_n|}{M_n} \tag{57}$$

is the absolute value of the average potential of that body. This is so because $\frac{1}{2}\left|\bar{V}_n^{pot}\right|$ in (57) multiplied by M_n and differentiated with respect to the distance R_n equals to a half of the gravitational force acting between the motion center and the orbiting body.

A substitution of (56) and (57) instead of P_n and $|E_n|$ entering (48)-(53) [see a similar operation done for the atomic case in (46), (47) and the formulae (19a), (22a), (25a) and (26a)] gives:

$$A(R,v) = \pi\left(R_{n+1}-R_n\right)\left(\frac{P_{n+1}}{M_{n+1}}+\frac{P_n}{M_n}\right)\frac{1}{2}, \tag{58}$$

$$B(R,v) = \pi\left|\frac{P_{n+1}}{M_{n+1}}-\frac{P_n}{M_n}\right|\left(R_{n+1}+R_n\right)\frac{1}{2}, \tag{59}$$

$$C\left(T,V^{pot}\right) = \left(T_{n+1}-T_n\right)\left(\frac{|E_{n+1}|}{M_{n+1}}+\frac{|E_n|}{M_n}\right)\frac{1}{2}, \tag{60}$$

$$D\left(T,V^{pot}\right) = \left|\frac{|E_{n+1}|}{M_{n+1}}-\frac{|E_n|}{M_n}\right|\left(T_{n+1}+T_n\right)\frac{1}{2}. \tag{61}$$

Expressions

$$v_n = \frac{P_n}{M_n},\ v_{n+1} = \frac{P_{n+1}}{M_{n+1}} \tag{62}$$

are the velocities of the bodies on the orbits n and $n+1$, whereas

$$\frac{1}{2}\bar{V}_n^{pot} = \frac{E_n}{M_n},\ \frac{1}{2}\bar{V}_{n+1}^{pot} = \frac{E_{n+1}}{M_{n+1}} \tag{63}$$

represent the average body potentials on these orbits.

The ratios obtained for the hydrogen-like atom on the basis of (19a), (22a), (25a) and (26a) are:

$$\begin{aligned} &A/B = 2;\ A/C = 2/3;\ A/D = 1;\ B/C = 1/3; \\ &B/D = 1/2;\ C/D = 3/2. \end{aligned} \tag{64}$$

These ratios can be compared with similar ratios obtained for planets and satellites. We find (see **Tables 5** and **6**) that $A(R,v)/B(R,v)$ is very close to 2 and the ratio $C\left(T,V^{pot}\right)/D\left(T,V^{pot}\right)$ is very close to 3/2. The remaining ratios of (64), viz. $A(R,v)/C\left(T,V^{pot}\right)$, $A(R,v)/D\left(T,V^{pot}\right)$, $B(R,v)/C\left(T,V^{pot}\right)$ and $B(R,v)/D\left(T,V^{pot}\right)$ are presented in **Tables 7** and **8**. They are plainly close to the corresponding ratios (64). This agreement is evident especially in the case of planets; see **Table 7**.

In the case of satellites the strongest deviations of the observed data from the ratios (64) are obtained for the pairs of Deimos-Phobos and Iapetus-Titan (see **Table 8**).

Another ratio, similar to that calculated in (53) for the areas (48)-(51), can be calculated for the areas (58)-(61). This is

$$\chi = \frac{C\left(T,V^{pot}\right)+D\left(T,V^{pot}\right)}{A(R,v)+B(R,v)}. \tag{65}$$

Its values for different planetary and satellitary pairs

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Table 4. Ratios between the averaged belt areas calculated for planets and satellites. Column first: the ratios of energy-time and momentum-position phase spaces calculated according to the Formula (53). Column second: the ratios of potential-time and velocity-position phase spaces calculated according to (65). The ratios η and χ approach a similar ratio calculated for the atomic electron orbits in (66).

Planets Pair	η	χ	Satellites Pair	η	χ
	Equation (53)	Equation (65)		Equation (53)	Equation (65)
Venus-Mercury	1.00	1.82	Deimos-Phobos	1.63	1.90
Earth-Venus	1.66	1.69	Europa-Io	1.43	1.72
Mars-Earth	1.26	1.72	Ganymede-Europa	1.00	1.73
Jupiter-Mars	1.00	2.10	Callisto-Ganymede	1.59	1.76
Saturn-Jupiter	1.44	1.77	Enceladus-Callisto	1.01	1.67
Uranus-Saturn	1.46	1.81	Tethys-Enceladus	1.00	1.68
Neptune-Uranus	1.94	1.73	Dione-Tethys	1.01	1.68
Pluto-Neptune	1.09	1.68	Rhea-Dione	1.00	1.70
			Titan-Rhea	1.00	1.87
			Iapetus-Titan	1.71	1.99
			Ariel-Miranda	1.00	1.70
			Umbriel-Ariel	1.63	1.71
			Titania-Umbriel	1.00	1.74
			Oberon-Titania	1.50	1.69

Table 5. Belt areas between the neighbouring planet orbits calculated (in $Jkg^{-1}s$) for the velocity-position and potential-time phase spaces; for $A(R,v)$ and $B(R,v)$ see Equations (58) and (59); for $C(T,V^{pot})$ and $D(T,V^{pot})$ see, respectively, (60) and (61). The input data are equal to those applied in calculating Table 1. The ratios A/B and C/D have their counterparts in the data of the Bohr-Sommerfeld theory given in (64). These are A/B = 2 and C/D = 3/2.

Planets Pair	$A(R,v)$	$B(R,v)$	$\frac{A}{B}$	$C(T,V^{pot})$	$D(T,V^{pot})$	$\frac{C}{D}$
Venus-Mercury	0.659×10^{16}	0.321×10^{16}	2.05	1.045×10^{16}	0.736×10^{16}	1.42
Earth-Venus	0.421×10^{16}	0.212×10^{16}	1.99	0.641×10^{16}	0.429×10^{16}	1.49
Mars-Earth	0.659×10^{16}	0.338×10^{16}	1.95	1.021×10^{16}	0.692×10^{16}	1.48
Jupiter-Mars	0.321×10^{17}	0.174×10^{17}	1.85	0.593×10^{17}	0.447×10^{17}	1.33
Saturn-Jupiter	0.231×10^{17}	0.118×10^{17}	1.95	0.366×10^{17}	0.252×10^{17}	1.45
Uranus-Saturn	0.373×10^{17}	0.192×10^{17}	1.95	0.600×10^{17}	0.420×10^{17}	1.43
Neptune-Uranus	0.313×10^{17}	0.158×10^{17}	1.98	0.483×10^{17}	0.330×10^{17}	1.46
Pluto-Neptune	0.210×10^{17}	0.122×10^{17}	1.72	0.342×10^{17}	0.218×10^{17}	1.57

are listed in **Table 4**. On the basis of calculations done in the Bohr-Sommerfeld theory of the hydrogen-like atom the values of η given by (53) and those of χ given by (65) should approach the same value

$$\eta^{at} = \chi^{at} = \frac{C+D}{A+B} = \frac{1+3/2}{1+1/2} = 5/3; \tag{66}$$

see (19a), (22a), (25a) and (26a) at the end of Section 4.

6. Discussion

The aim of the present paper was to detect, and check, some regularities which can be obtained for the fragments of the phase-space areas defined by parameters characteristic for the motion of celestial bodies in the solar system. In this search we were guided by similar phase-space properties derived for the electron motion in

Table 6. Belt areas between the neighbouring satellite orbits calculated (in $Jkg^{-1}s$) for the velocity-position and potential-time phase spaces; for $A(R,v)$ and $B(R,v)$ see Equations (58) and (59); for $C(T,V^{pot})$ and $D(T,V^{pot})$ see, respectively, (60) and (61). The input data are equal to those applied in calculating Table 2. The ratios A/B and C/D have their counterparts in the data of the Bohr-Sommerfeld theory given in (64). These are A/B = 2 and C/D = 3/2.

Satellites Pair	$A(R,v)$	$B(R,v)$	$\frac{A}{B}$	$C(T,V^{pot})$	$D(T,V^{pot})$	$\frac{C}{D}$
Deimos-Phobos	0.776×10^{11}	0.404×10^{11}	1.92	0.130×10^{12}	0.094×10^{12}	1.39
Europa-Io	0.122×10^{14}	0.062×10^{14}	0.91	0.188×10^{14}	0.128×10^{14}	1.47
Ganymede-Europa	0.154×10^{14}	0.078×10^{14}	1.97	0.239×10^{14}	0.163×10^{14}	1.47
Callisto-Ganymede	0.243×10^{14}	0.124×10^{14}	1.95	0.382×10^{14}	0.263×10^{14}	1.46
Enceladus-Mimas	0.221×10^{13}	0.115×10^{13}	1.92	0.336×10^{13}	0.223×10^{13}	1.51
Tethys-Enceladus	0.215×10^{13}	0.106×10^{13}	2.02	0.322×10^{13}	0.218×10^{13}	1.49
Dione-Tethys	0.275×10^{13}	0.142×10^{13}	1.94	0.421×10^{13}	0.280×10^{13}	1.50
Rhea-Dione	0.436×10^{13}	0.218×10^{13}	2.00	0.664×10^{13}	0.449×10^{13}	1.48
Titan-Rhea	0.153×10^{14}	0.080×10^{14}	1.92	0.255×10^{14}	0.181×10^{14}	1.41
Iapetus-Titan	0.325×10^{14}	0.173×10^{14}	1.87	0.571×10^{14}	0.420×10^{14}	1.36
Ariel-Miranda	0.117×10^{13}	0.059×10^{13}	1.97	0.179×10^{13}	0.121×10^{13}	1.48
Umbriel-Ariel	0.120×10^{13}	0.060×10^{13}	1.99	0.183×10^{13}	0.124×10^{13}	1.48
Titania-Umbriel	0.222×10^{13}	0.112×10^{13}	1.98	0.345×10^{13}	0.236×10^{13}	1.46
Oberon-Titania	0.157×10^{13}	0.079×10^{13}	1.98	0.239×10^{13}	0.160×10^{13}	1.49

Table 7. The ratios $A(R,v)/C(T,V^{pot})$, $A(R,v)/D(T,V^{pot})$, $B(R,v)/C(T,V^{pot})$ and $B(R,v)/D(T,V^{pot})$ calculated for the belt areas of the velocity-position and potential-time spaces for planets. These ratios, as well as those calculated in Table 5, approach the ratios of the phase-space areas obtained from the electron orbits in the Bohr-Sommerfeld atom; see (64).

Planets Pair	$\frac{A(R,v)}{C(T,V^{pot})}$	$\frac{A(R,v)}{D(T,V^{pot})}$	$\frac{B(R,v)}{C(T,V^{pot})}$	$\frac{B(R,v)}{D(T,V^{pot})}$
Venus-Mercury	0.63	0.90	0.31	0.44
Earth-Venus	0.66	0.98	0.33	0.49
Mars-Earth	0.65	0.95	0.33	0.49
Jupiter-Mars	0.54	0.72	0.29	0.40
Saturn-Jupiter	0.63	0.92	0.32	0.47
Uranus-Saturn	0.62	0.89	0.32	0.46
Neptune-Uranus	0.65	0.95	0.33	0.48
Pluto-Neptune	0.61	0.96	0.36	0.56

the Bohr-Sommerfeld atom. The calculations applied the data calculated from [5] and those listed in **Tables 1** and **2** in [6]. However, several printing errors were discovered in **Table 2** of the last reference for the data of Iapetus, the satellite of Saturn, and Titania and Oberon, the satellites of Uranus. The corrected data are given in **Table 3** of the present paper.

For planets the belt areas defined in the momentum-position and energy-time phase spaces exhibit their ratios $B(R,P)/A(R,P)$ and $D(T,E)/C(T,E)$ [see (48)-(51)] considerably different from unity only for Earth-Venus, Mars-Earth, Neptune-Uranus, and Pluto-Neptune pairs; see **Table 1**. Similar belts for the phase-space areas defined for the neighbouring orbits of the satellites (**Table 2**) show the ratios $B(R,P)/A(R,P)$ and $D(T,E)/C(T,E)$ still closer to unity than in the planet

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Table 8. The ratios $A(R,v)/C(T,V^{pot})$, $A(R,v)/D(T,V^{pot})$, $B(R,v)/C(T,V^{pot})$ and $B(R,v)/D(T,V^{pot})$ calculated for the belt areas of the velocity-position and potential-time spaces for satellites. These ratios, as well as those calculated in Table 6, approach the ratios of the phase-space areas obtained from the electron orbits in the Bohr-Sommerfeld atom; see (64).

Satellites Pair	$\frac{A(R,v)}{C(T,V^{pot})}$	$\frac{A(R,v)}{D(T,V^{pot})}$	$\frac{B(R,v)}{C(T,V^{pot})}$	$\frac{B(R,v)}{D(T,V^{pot})}$
Deimos-Phobos	0.60	0.83	0.31	0.43
Europa-Io	0.65	0.95	0.33	0.48
Ganymede-Europa	0.64	0.94	0.33	0.48
Callisto-Ganymede	0.64	0.92	0.32	0.47
Enceladus-Mimas	0.66	0.99	0.34	0.52
Tethys-Enceladus	0.67	0.99	0.33	0.49
Dione-Tethys	0.65	0.98	0.34	0.51
Rhea-Dione	0.66	0.97	0.33	0.49
Titan-Rhea	0.60	0.85	0.31	0.44
Iapetus-Titan	0.57	0.77	0.30	0.41
Ariel-Miranda	0.65	0.97	0.33	0.49
Umbriel-Ariel	0.67	0.99	0.34	0.50
Titania-Umbriel	0.64	0.94	0.33	0.48
Oberon-Titania	0.66	0.98	0.33	0.49

case. The only strong deviations from unity are exhibited for the satellite pairs of Thetys-Enceladus, Dione-Thetys and Ariel-Miranda.

Some properties concerning the belt areas in the phase-space and their ratios can be deduced in an analytic way. For example, an approximate equivalence between $A(R,v)$ and $C(T,V^{pot})$ [see (58) and (60)] can be attained on condition $A(R,v)$ is multiplied by the factor of 3/2. For, from (58) and (62) we obtain:

$$\begin{aligned}&\frac{3}{2}\pi\left(R_{n+1}-R_n\right)\left(v_{n+1}+v_n\right)\frac{1}{2}\\&=\frac{3}{2}\pi\Delta R\frac{2\pi R}{T}=3\pi^2\frac{R}{T}\Delta R\end{aligned} \tag{67}$$

where

$$\Delta R=R_{n+1}-R_n, \tag{68}$$

$$v=\frac{2\pi R}{T}\approx\left(v_{n+1}+v_n\right)\frac{1}{2}. \tag{69}$$

Similarly, from (60) and (63):

$$\begin{aligned}&\left(T_{n+1}-T_n\right)\left(\left|\bar{V}_{n+1}^{pot}\right|+\left|\bar{V}_n^{pot}\right|\right)\frac{1}{4}\\&=\Delta T\frac{\varkappa M_s}{2}\left(\frac{1}{2R_{n+1}}+\frac{1}{2R_n}\right)\approx\Delta T\varkappa M_s\frac{1}{2R},\end{aligned} \tag{70}$$

where $\varkappa$ is the gravitational constant. Here

$$\Delta T=T_{n+1}-T_n, \tag{71}$$

$$\frac{1}{R}\cong\frac{1}{2}\left(\frac{1}{R_{n+1}}+\frac{1}{R_n}\right). \tag{72}$$

From the Kepler's law (see e.g. [4])

$$T_n^2\cong\frac{4\pi^2}{\varkappa M_s}R_n^3 \tag{73}$$

We obtain by differentiation the relation:

$$2T\Delta T\cong\frac{4\pi^2}{\varkappa M_s}3R^2\Delta R \tag{74}$$

valid approximately for any n. A substitution of (74) into (70) gives

$$\begin{aligned}\varkappa M_s\frac{1}{2R}\Delta T&=\varkappa M_s\frac{1}{2R}\frac{4\pi^2}{\varkappa M_s}\frac{3R^2}{2T}\Delta R\\&=3\pi^2\frac{R}{T}\Delta R,\end{aligned} \tag{75}$$

which is identical to (67). Hence, the ratio between $A(R,v)$ and $C(T,V^{pot})$ [see (58) and (60)] should be approximately equal to

$$\frac{A(R,v)}{C(T,V^{pot})}\approx\frac{2}{3}, \tag{76}$$

which is the result obtained also for A/C in the atom on the basis of (19a) and (25a). In calculations we took into account the fact that the average $\bar{V}^{pot}$ entering (63) are not much different from the actual V^{pot} because of a quasi-circular character of the orbit possessed by a moving body.

Other ratios than 2/3 given in (76) can be deduced between the areas (58), (59) and (60), (61). These ratios are identical to the ratios between the belt areas (34), (37), (40) and (41) [or (19a), (22a), (25a) and (26a)] obtained for the Bohr-Sommerfeld hydrogen atom; see (64). In **Tables 7** and **8** we calculate these ratios for the planetary and satellitary systems.

A characteristic point is that the ratios calculated in **Tables 7** and **8** are only feebly dependent on the choice of the planetary or satellitary pair. In this sense these ratios have an almost constant character similar to that postulated at the end of Section 1.

In particular, the average deviations of the ratios in **Tables 7** and **8** from the ratios calculated in (64) are respectively:

$$\left|\frac{A/C-(A/C)^{\text{Table}}}{A/C}\right| = 0.06; \quad \cdots = 0.04; \tag{77}$$

$$\left|\frac{A/D-(A/D)^{\text{Table}}}{A/D}\right| = 0.09; \quad \cdots = 0.07; \tag{78}$$

$$\left|\frac{B/C-(B/C)^{\text{Table}}}{B/C}\right| = 0.03; \quad \cdots = 0.02; \tag{79}$$

$$\left|\frac{B/D-(B/D)^{\text{Table}}}{B/D}\right| = 0.05; \quad \cdots = 0.05; \tag{80}$$

The first numbers calculated in (77)-(80) concern the data taken from **Table 7**, the second numbers concern the data taken from **Table 8**.

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